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GREGARIOUS KITE FACTORIZATION OF TENSOR PRODUCT OF COMPLETE GRAPHS

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Abstract

A kite factorization of a multipartite graph is said to be gregarious if every kite in the factorization has all its vertices in different partite sets. In this paper, we show that there exists a gregarious kite factorization of $K_m \times K_n$ if and only if $mn \equiv 0 \pmod 4$ and $(m-1)(n-1) \equiv 0 \pmod 2$, where $\times$ denotes the tensor product of graphs.

Keywords: tensor product, kite, decomposition, gregarious factor, factorization.

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1. Introduction

A *latin square* of order n is an $n \times n$ array such that each row and each column of the array contains each of the symbols from $\{1, 2, \ldots, n\}$ exactly once. Two latin squares L_1 and L_2 of order n are said to be *orthogonal* if for each $(x, y) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ there is exactly one cell (i, j) in which L_1 contains the symbol x and L_2 contains the symbol y. In other words, if L_1 and L_2 are superimposed, the resulting set of n^2 ordered pairs are distinct. The latin squares $L_1, L_2, \ldots, L_t$ of order n are said to be *mutually orthogonal* ($MOLS(n)$) if for $1 \le a \ne b \le t$, L_a and L_b are orthogonal. $N(n)$ denotes the maximum number of $MOLS(n)$.

Partition of G into subgraphs $G_1, G_2, \ldots, G_r$ such that $E(G_i) \cap E(G_j) = \emptyset$ for $i \ne j$, $i, j \in \{1, 2, \ldots, r\}$ and $E(G) = \bigcup_{i=1}^{r} E(G_i)$ is called *decomposition* of G; in this case we write G as $G = G_1 \oplus G_2 \oplus \cdots \oplus G_r$, where $\oplus$ denotes edge-disjoint sum of subgraphs. If $G_i \cong H$, $1 \le i \le r$, then we say that *H-decomposes* G; in notation $H \mid G$. A spanning subgraph of G such that each component of it is isomorphic to some graph H is called an *H-factor* of G. A partition of G into edge-disjoint H-factors is called an *H-factorization* of G; in notation $H \parallel G$. Let C_k, K_k and I_k, respectively denote a cycle, a complete graph and a null graph on k vertices. A k-regular spanning subgraph of G is called a *k-factor* of G. A *C_k-factor* of G is a 2-factor in which each component is a C_k. Decomposition of G into C_k-factors is called a *C_k-factorization* of G. A cycle containing all the vertices of G is called a *Hamilton cycle*. We say that G has a *Hamilton cycle decomposition* if its edge set can be partitioned into edge-disjoint Hamilton cycles. For an integer λ, λG denotes a graph with λ components each isomorphic to G.

The *tensor product* $G \times H$ and the *wreath product* $G \otimes H$ of two simple graphs G and H are defined as follows: $V(G \times H) = V(G \otimes H) = \{(u, v) \mid u \in V(G), v \in V(H)\}$. $E(G \times H) = \{(u, v)(x, y) \mid ux \in E(G)$ and $vy \in E(H)\}$ and $E(G \otimes H) = \{(u, v)(x, y) \mid u = x$ and $vy \in E(H)$, or $ux \in E(G)\}$. It is well known that tensor product is commutative and distributive over an edge-disjoint union of subgraphs, that is, if $G = G_1 \oplus G_2 \oplus \cdots \oplus G_r$, then $G \times H = (G_1 \times H) \oplus (G_2 \times H) \oplus \cdots \oplus (G_r \times H)$. A graph G having partite sets $V_1, V_2, \ldots, V_m$ with $\mid V_i \mid = n$, $1 \le i \le n$, and $E(G) = \{uv \mid u \in V_i$ and $v \in V_j, i \ne j\}$ is called *complete m-partite graph* and is denoted by $K_m(n)$. Note that $K_m(n)$ is same as the $K_m \otimes I_n$.

A *kite* is a graph which is obtained by attaching an edge to a vertex of the triangle, see Figure 1. We denote the kite with edge set $\{ab, bc, ca, cd\}$ by $(a, b, c; cd)$.

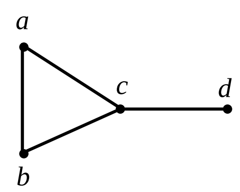

Figure 1. The kite graph.

A subgraph of a multipartite graph G is said to be *gregarious* if each of its vertices lies in different partite sets of G. A kite factorization of a multipartite graph is said to be *gregarious* if each kite in the factorization has its vertices in four different partite sets.

The study of kite-design is not new. Bermond and Schonheim [3] proved that a kite-design of order n exists if and only if $n \equiv 0, 1 \pmod 8$. Wang and Chang [18, 19] considered the existence of (K_3+e) and (K_3+e, λ)-group divisible designs of type $g^t u^1$. Wang [17] has shown that the obvious necessary conditions for the existence of resolvable (K_3+e)-group divisible design of type g^u are also sufficient. Fu *et al.* [5] have shown that there exists a gregarious kite decomposition of $K_m(n)$ if and only if $n \equiv 0, 1 \pmod 8$ for odd m or $n \geq 4$ for even m. Gionfriddo and Milici [6] considered the existence of uniformly resolvable decompositions of K_v and $K_v - I$ into paths and kites. For more results on kite designs, see [4, 7, 9, 11, 12].

In this direction, in [15] we have shown that the necessary conditions for the existence of a gregarious kite decomposition of tensor product of complete graphs are also sufficient. Further, in this paper, we show that there exists a gregarious kite factorization of $K_m \times K_n$ if and only if $mn \equiv 0 \pmod 4$ and $(m-1)(n-1) \equiv 0 \pmod 2$.
We require the following to prove our main results.

2. Preliminary Results

Theorem 1 [10]**.** *There exists a pair of mutually orthogonal latin squares ($MOLS(n)$) of order n for every $n \neq 2, 6$.*

Theorem 2 [1]**.** *If $n = p^d$ is a prime power, then $N(n) = n - 1$.*

Corrolary 3 [2]**.** *If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$, where each number p_i is a distinct prime number and $\alpha_i \geq 1, i = 1, 2, \ldots, t$, then $N(n) \geq \min\{p_i^{\alpha_i} \mid i = 1, 2, \ldots, t\}$.*

Theorem 4 [8]**.** *Let G be a graph with chromatic number $\chi(G)$. Then*
(i) $G \mid G \otimes I_n$ *if* $\chi(G) \leq N(n) + 2$ *and*
(ii) $G \parallel G \otimes I_n$ *if* $\chi(G) \leq N(n) + 1$.

Theorem 5 [17]**.** *The necessary conditions for the existence of a kite factorization of $K_m(n)$, namely, $m \geq 3$, $n(m-1) \equiv 0 \pmod 2$, $mn \equiv 0 \pmod 4$ are also sufficient.*

Theorem 6 [13]**.** $C_3 \parallel K_m$ *if and only if* $m \equiv 3 \pmod 6$.

Note 7. Let $G_1 = v_1v_2v_3v_4v_5 \cdots v_{p-1}v_pv_1$, $G_2 = v_1v_3v_5 \cdots v_pv_2v_4v_6 \cdots v_{p-3}v_{p-1}v_1$ and $G_3 = v_1v_5v_9 \cdots v_{p-1}v_3v_7v_{11} \cdots v_{p-3}v_1$ be three cycles of length p (p is odd). Now consider two graphs $G = G_1 \oplus G_2$ and $H = G_1 \oplus G_2 \oplus G_3$ as shown in Figures 2 and 3.

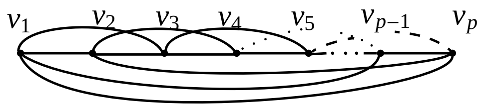

Figure 2. $G = G_1 \oplus G_2$.

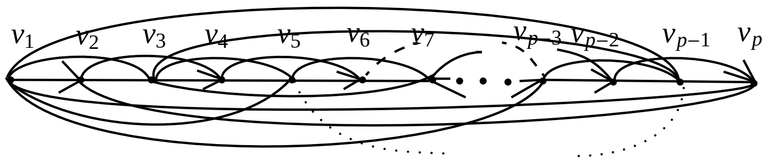

Figure 3. $H = G_1 \oplus G_2 \oplus G_3$.

Remark 8 [16]**.** Let $V(K_p) = \{v_1, v_2, \ldots, v_p\}$, p is a prime. For $1 \le i \le (p-1)/2$, let $H_i = v_1 v_{(2+(i-1))} v_{(3+[2(i-1)])} v_{(4+[3(i-1)])} v_{(5+[4(i-1)])} \cdots v_{(p+[(p-1)(i-1)])} v_1$, where the subscripts are taken modulo p with residues $1, 2, 3, \ldots, p$. Note that each H_i is a Hamilton cycle of K_p and $\{H_1, H_2, \ldots, H_{(p-1)/2}\}$ gives a Hamilton cycle decomposition of K_p, p is a prime. Further, $\{H_1, H_2, \ldots H_{(p-1)/2}\}$ can be partitioned into sets of 2 or 3 cycles such that the sum of the cycles of each set is isomorphic to G or H, respectively.

3. Gregarious Kite Factorization of $K_m \times K_n$

Lemma 9. *There exists a gregarious kite factorization of* $K_4 \times K_3$.

Proof. Let $V(K_4) = \{1, 2, 3, 4\}$ and $V(K_3) = \{1, 2, 3\}$. Then $V(K_4 \times K_3) = \bigcup_{i=1}^{4} V_i$, where $V_i = \{i_j \mid 1 \le j \le 3\}$. Now we construct a gregarious kite factorization of $K_4 \times K_3$ as follows: For $0 \le s \le 2$, let $F_s^1 = \{1_{1+s}, 2_{2+s}, 4_{3+s}; 4_{3+s}3_{1+s}\}$; $F_s^2 = \{2_{1+s}, 4_{3+s}, 3_{2+s}; 3_{2+s}1_{3+s}\}$; $F_s^3 = \{3_{1+s}, 2_{2+s}, 1_{3+s}; 1_{3+s}4_{1+s}\}$, where the subscripts are taken modulo 3 with residues 1, 2, 3. Clearly each $F_i = \bigcup_{s=0}^{2} F_s^i$, $1 \le i \le 3$, is a gregarious kite factor of $K_4 \times K_3$ and $\{F_1, F_2, F_3\}$ gives a gregarious kite factorization of $K_4 \times K_3$. ■

Lemma 10. *For* $n \equiv 3 \pmod 6$, *there exists a gregarious kite factorization of* $K_4 \times K_n$.

Proof. By Theorem 6, we have a K_3-factorization of $K_n, n = 6s+3, s \ge 1$ (since the case $s = 0$ follows from Lemma 9). Since tensor product is distributive over an edge-disjoint union of subgraphs, corresponding to each K_3-factor of K_n, we have a $(K_4 \times K_3)$-factor of $K_4 \times K_n$. Hence a K_3-factorization of K_n gives a $(K_4 \times K_3)$-factorization of $K_4 \times K_n$. By Lemma 9, we have a gregarious kite factorization of $K_4 \times K_3$. Thus combining all these we get a gregarious kite factorization of $K_4 \times K_n$. ■

Lemma 11. *For $|V(G)| = p$, $p \geq 5$ is a prime, there exists a gregarious kite factorization of $K_4 \times G$, where G is described as in Note 7.*

Proof. Let $V(K_4) = \{1, 2, 3, 4\}$ and $V(G) = \{1, 2, \dots, p\}$, $p \geq 5$. Then $V(K_4 \times G) = \bigcup_{i=1}^{4} V_i$, where $V_i = \{i_j \mid 1 \leq j \leq p\}$. Now we construct a gregarious kite factorization of $K_4 \times G$ as follows: For $0 \leq s \leq p-1$, let

$$\begin{aligned}
F_s^1 &= \{3_{1+s}, 2_{p+s}, 1_{p-1+s}; 1_{p-1+s}4_{1+s}\}; F_s^2 = \{3_{1+s}, 4_{2+s}, 2_{3+s}; 2_{3+s}1_{1+s}\};\\
F_s^3 &= \{3_{3+s}, 1_{2+s}, 4_{1+s}; 4_{1+s}2_{p-1+s}\}; F_s^4 = \{3_{1+s}, 2_{2+s}, 1_{3+s}; 1_{3+s}4_{1+s}\};\\
F_s^5 &= \{3_{1+s}, 4_{p+s}, 2_{p-1+s}; 2_{p-1+s}1_{1+s}\}; F_s^6 = \{3_{p+s}, 1_{1+s}, 4_{2+s}; 4_{2+s}2_{4+s}\}.
\end{aligned}$$

In all the above constructions the subscripts are taken modulo p with residues $1, 2, \dots, p$. Clearly each $F_i = \bigcup_{s=0}^{p-1} F_s^i$, $1 \leq i \leq 6$, is a gregarious kite factor of $K_4 \times G$ and $\{F_1, F_2, \dots, F_6\}$ gives a gregarious kite factorization of $K_4 \times G$. ■

Lemma 12. *There exists a gregarious kite factorization of $K_4 \times K_7$.*

Proof. Let $V(K_4) = \{1, 2, 3, 4\}$ and $V(K_7) = \{1, 2, \dots, 7\}$. Then $V(K_4 \times K_7) = \bigcup_{i=1}^{4} V_i$, where $V_i = \{i_j \mid 1 \leq j \leq 7\}$. Now we construct a gregarious kite factorization of $K_4 \times K_7$ as follows: For $0 \leq s \leq 6$, let

$$\begin{aligned}
F_s^1 &= \{3_{1+s}, 2_{7+s}, 1_{6+s}; 1_{6+s}4_{2+s}\}; F_s^2 = \{4_{1+s}, 2_{2+s}, 3_{7+s}; 3_{7+s}1_{4+s}\};\\
F_s^3 &= \{1_{1+s}, 3_{2+s}, 4_{7+s}; 4_{7+s}2_{4+s}\}; F_s^4 = \{3_{1+s}, 2_{2+s}, 1_{3+s}; 1_{3+s}4_{1+s}\};\\
F_s^5 &= \{3_{1+s}, 4_{5+s}, 2_{4+s}; 2_{4+s}1_{6+s}\}; F_s^6 = \{1_{1+s}, 2_{4+s}, 3_{7+s}; 3_{7+s}4_{2+s}\};\\
F_s^7 &= \{4_{1+s}, 2_{4+s}, 1_{7+s}; 1_{7+s}3_{4+s}\}; F_s^8 = \{1_{1+s}, 2_{3+s}, 4_{5+s}; 4_{5+s}3_{6+s}\};\\
F_s^9 &= \{2_{1+s}, 3_{3+s}, 4_{6+s}; 4_{6+s}1_{4+s}\}.
\end{aligned}$$

In all the above constructions the subscripts are taken modulo 7 with residues $1, 2, \dots, 7$. Clearly each $F_i = \bigcup_{s=0}^{6} F_s^i$, $1 \leq i \leq 9$, is a gregarious kite factor of $K_4 \times K_7$ and $\{F_1, F_2, \dots, F_9\}$ gives a gregarious kite factorization of $K_4 \times K_7$. ■

Lemma 13. *For $|V(H)| = p$, $p \geq 11$ is a prime, there exists a gregarious kite factorization of $K_4 \times H$, where H is described as in Note 7.*

Proof. Let $V(K_4) = \{1, 2, 3, 4\}$ and $V(H) = \{1, 2, \dots, p\}$, $p \geq 11$. Then $V(K_4 \times H) = \bigcup_{i=1}^{4} V_i$, where $V_i = \{i_j \mid 1 \leq j \leq p\}$. Now we construct a gregarious kite factorization of $K_4 \times H$ as follows: For $0 \leq s \leq p-1$, let

$$\begin{aligned}
F_s^1 &= \{4_{p-1+s}, 1_{p+s}, 2_{1+s}; 2_{1+s}3_{5+s}\}; F_s^2 = \{4_{2+s}, 3_{3+s}, 1_{1+s}; 1_{1+s}2_{5+s}\};\\
F_s^3 &= \{1_{p+s}, 4_{2+s}, 3_{1+s}; 3_{1+s}2_{5+s}\}; F_s^4 = \{1_{3+s}, 2_{p-1+s}, 3_{1+s}; 3_{1+s}4_{5+s}\};\\
F_s^5 &= \{2_{p-1+s}, 1_{p-3+s}, 4_{1+s}; 4_{1+s}3_{5+s}\}; F_s^6 = \{4_{p-1+s}, 2_{p+s}, 1_{1+s}; 1_{1+s}3_{5+s}\};\\
F_s^7 &= \{3_{2+s}, 1_{3+s}, 2_{1+s}; 2_{1+s}4_{5+s}\}; F_s^8 = \{2_{3+s}, 4_{p-1+s}, 3_{1+s}; 3_{1+s}1_{5+s}\};\\
F_s^9 &= \{2_{p+s}, 3_{p-1+s}, 4_{1+s}; 4_{1+s}1_{5+s}\}.
\end{aligned}$$

In all the above constructions the subscripts are taken modulo p with residues $1, 2, \ldots, p$. Clearly each $F_i = \bigcup_{s=0}^{p-1} F_s^i$, $1 \leq i \leq 9$, is a gregarious kite factor of $K_4 \times H$ and $\{F_1, F_2, \ldots, F_9\}$ gives a gregarious kite factorization of $K_4 \times H$. ∎

Lemma 14. *For all odd prime p, there exists a gregarious kite factorization of $K_4 \times K_p$.*

Proof. By Remark 8, we have a factorization of K_p into graphs isomorphic to G or H. A gregarious kite factorization of $K_4 \times K_p$ follows from Lemmas 9, 11, 12 and 13. ∎

Lemma 15. *For all odd prime p and $s > 1$, there exists a gregarious kite factorization of $K_4 \times K_{p^s}$.*

Proof. For $s > 1$, $K_4 \times K_{p^s} = K_4 \times \left[pK_{p^{s-1}} \oplus K_p(p^{s-1})\right] = p\left(K_4 \times K_{p^{s-1}}\right) \oplus \left[K_4 \times K_p(p^{s-1})\right]$ (since the case $s = 1$ follows from Lemma 14).

For $s = 2$, $K_4 \times K_{p^2} = p\,(K_4 \times K_p) \oplus [K_4 \times K_p(p)]$. By Lemma 14, we have a gregarious kite factorization of $p\,(K_4 \times K_p)$. By Theorem 4, we have a K_p-factorization of $K_p(p)$. Corresponding to each K_p-factor of $K_p(p)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p(p)$. Hence a K_p-factorization of $K_p(p)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p(p)$. Now the existence of a gregarious kite factorization of $(K_4 \times K_p)$ follows from Lemma 14.

For $s = 3$, $K_4 \times K_{p^3} = p\left(K_4 \times K_{p^2}\right) \oplus \left[K_4 \times K_p(p^2)\right]$. Now the gregarious kite factorization of $p\left(K_4 \times K_{p^2}\right)$ follows from the case $s = 2$. By Theorem 4, we have a K_p-factorization of $K_p(p^2)$. Corresponding to each K_p-factor of $K_p(p^2)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p(p^2)$. Hence a K_p-factorization of $K_p(p^2)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p(p^2)$. Now the existence of a gregarious kite factorization of $(K_4 \times K_p)$ follows from Lemma 14.

For $s > 1$, $K_4 \times K_{p^s} = p\left(K_4 \times K_{p^{s-1}}\right) \oplus \left[K_4 \times K_p(p^{s-1})\right]$. By the induction hypothesis on s, we have a gregarious kite factorization of $p\left(K_4 \times K_{p^{s-1}}\right)$. By Theorem 4, we have a K_p-factorization of $K_p(p^{s-1})$. Corresponding to each K_p-factor of $K_p\left(p^{s-1}\right)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p\left(p^{s-1}\right)$. Thus a K_p-factorization of $K_p\left(p^{s-1}\right)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p\left(p^{s-1}\right)$.

Now the existence of a gregarious kite factorization of $(K_4 \times K_p)$ follows from Lemma 14. Hence combining all the above we have a gregarious kite factorization of $K_4 \times K_{p^s}$, for all $s > 1$. ■

Lemma 16. *For all odd primes p, q $(p < q)$ and all integers $s, t \geq 1$, there exists a gregarious kite factorization of $K_4 \times K_{p^s q^t}$.*

Proof. For $s, t \geq 1$ and $p < q$, $K_4 \times K_{p^s q^t} = K_4 \times K_{p.p^{s-1}q^t} = K_4 \times \left[pK_{p^{s-1}q^t} \oplus K_p\left(p^{s-1}q^t\right)\right] = p\left[K_4 \times K_{p^{s-1}q^t}\right] \oplus \left[K_4 \times K_p(p^{s-1}q^t)\right]$.

Case 1. (a) For $s = 1,\ t = 1$, $K_4 \times K_{pq} = K_4 \times (pK_q \oplus K_p(q)) = p\left[K_4 \times K_q\right] \oplus \left[K_4 \times K_p(q)\right]$. By Theorem 4, we have a K_p-factorization of $K_p(q)$. Corresponding to each K_p-factor of $K_p(q)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p(q)$. Thus a K_p-factorization of $K_p(q)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p(q)$. Now the existence of a gregarious kite factorization of $K_4 \times K_p$ and $p\,(K_4 \times K_q)$ follows from Lemma 14.

(b) For $s = 1,\ t = 2$, $K_4 \times K_{pq^2} = K_4 \times \left[pK_{q^2} \oplus K_p(q^2)\right] = p\left[K_4 \times K_{q^2}\right] \oplus \left[K_4 \times K_p(q^2)\right]$. By Theorem 4, we have a K_p-factorization of $K_p(q^2)$. Corresponding to each K_p-factor of $K_p(q^2)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p(q^2)$. Thus a K_p-factorization of $K_p(q^2)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p(q^2)$. Now the existence of a gregarious kite factorization of $K_4 \times K_p$ and $p\left(K_4 \times K_{q^2}\right)$ follows from Lemmas 14 and 15, respectively.

(c) For $s = 1,\ t \geq 3$, $K_4 \times K_{pq^t} = K_4 \times \left[pK_{q^t} \oplus K_p(q^t)\right] = p\left[K_4 \times K_{q^t}\right] \oplus \left[K_4 \times K_p(q^t)\right]$. By Theorem 4, we have a K_p-factorization of $K_p(q^t)$. Corresponding to each K_p-factor of $K_p(q^t)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p(q^t)$. Thus a K_p-factorization of $K_p(q^t)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p(q^t)$. Now the existence of a gregarious kite factorization of $K_4 \times K_p$ and $p\left(K_4 \times K_{q^t}\right)$ follows from Lemmas 14 and 15, respectively.

Case 2. (a) For $s = 2,\ t = 1$, $K_4 \times K_{p^2 q} = K_4 \times K_{p.pq} = K_4 \times [pK_{pq} \oplus K_p(pq)] = p\left[K_4 \times K_{pq}\right] \oplus \left[K_4 \times K_p(pq)\right]$. By Theorem 4, we have a K_p-factorization of $K_p(pq)$. Corresponding to each K_p-factor of $K_p(pq)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p(pq)$. Thus a K_p-factorization of $K_p(pq)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p(pq)$. Now the existence of a gregarious kite factorization of $K_4 \times K_p$ and $p\left[K_4 \times K_{pq}\right]$ follows from Lemma 14 and Case 1(a), respectively.

(b) For $s = 2,\ t = 2$, $K_4 \times K_{p^2q^2} = K_4 \times K_{p.pq^2} = K_4 \times \left[pK_{pq^2} \oplus K_p(pq^2)\right] = p\left[K_4 \times K_{pq^2}\right] \oplus \left[K_4 \times K_p(pq^2)\right]$. By Theorem 4, we have a K_p-factorization of $K_p(pq^2)$. Corresponding to each K_p-factor of $K_p(pq^2)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p(pq^2)$. Thus a K_p-factorization of $K_p(pq^2)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p(pq^2)$. Now the existence of a gregarious kite factorization of $K_4 \times K_p$ and $p\left[K_4 \times K_{pq^2}\right]$ follows from Lemma 14 and Case 1(b), respectively.

(c) For $s = 2, t \geq 3$, $K_4 \times K_{p^2q^t} = K_4 \times K_{p.pq^t} = K_4 \times \left[pK_{pq^t} \oplus K_p(pq^t)\right] = p\left[K_4 \times K_{pq^t}\right] \oplus \left[K_4 \times K_p(pq^t)\right]$. By Theorem 4, we have a K_p-factorization of $K_p(pq^t)$. Corresponding to each K_p-factor of $K_p(pq^t)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p(pq^t)$. Thus a K_p-factorization of $K_p(pq^t)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p(pq^t)$. Now the existence of a gregarious kite factorization of $K_4 \times K_p$ and $p\left[K_4 \times K_{pq^t}\right]$ follows from Lemma 14 and Case 1(c), respectively.

(d) For $s, t \geq 1$, $K_4 \times K_{p^s q^t} = p\left[K_4 \times K_{p^{s-1}q^t}\right] \oplus \left[K_4 \times K_p(p^{s-1}q^t)\right]$. By induction hypothesis on s, we have a gregarious kite factorization of $p\left[K_4 \times K_{p^{s-1}q^t}\right]$. By Theorem 4, we have a K_p-factorization of $K_p\left(p^{s-1}q^t\right)$. Corresponding to each K_p-factor of $K_p\left(p^{s-1}q^t\right)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p\left(p^{s-1}q^t\right)$. Thus a K_p-factorization of $K_p\left(p^{s-1}q^t\right)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p\left(p^{s-1}q^t\right)$. Now the existence of a gregarious kite factorization of $K_4 \times K_p$ follows from Lemma 14. Hence combining all the above we have a gregarious kite factorization of $K_4 \times K_{p^s q^t}$, for all $s, t \geq 1$ and $p < q$. ■

Lemma 17. *For all odd $n > 1$, there exists a gregarious kite factorization of $K_4 \times K_n$.*

Proof. By fundamental theorem of arithmetic, any integer $n > 1$ can be uniquely written as prime powers or product of prime powers. Consider $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$, where each p_i is a distinct odd prime and $\alpha_i \geq 1, i = 1, 2, \ldots, t$. Fix $p_1^{\alpha_1} < p_2^{\alpha_2} < \cdots < p_t^{\alpha_t}$. Now,

$$\begin{aligned} K_4 \times K_n &= K_4 \times K_{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}} = K_4 \times \left[p_1^{\alpha_1} K_{p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t}} \oplus K_{p_1^{\alpha_1}}(p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t})\right] \\ &= p_1^{\alpha_1}\left[K_4 \times K_{p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t}}\right] \oplus \left[K_4 \times K_{p_1^{\alpha_1}}(p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t})\right]. \end{aligned}$$

It is enough to show that there exists a gregarious kite factorization of $K_4 \times K_{p_1^{\alpha_1}}(p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t})$ and $p_1^{\alpha_1}\left[K_4 \times K_{p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t}}\right]$.

Case 1. Consider $K_4 \times K_{p_1^{\alpha_1}}(p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t})$. By Theorem 4, we have a $K_{p_1^{\alpha_1}}$-factorization of $K_{p_1^{\alpha_1}}(p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t})$. Corresponding to each $K_{p_1^{\alpha_1}}$-factor of $K_{p_1^{\alpha_1}}(p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t})$, we have a $\left(K_4 \times K_{p_1^{\alpha_1}}\right)$-factor of $K_4 \times K_{p_1^{\alpha_1}}(p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t})$. Now the existence of a gregarious kite factorization of $K_4 \times K_{p_1^{\alpha_1}}$ follows from Lemma 15.

Case 2. Consider $p_1^{\alpha_1}\left[K_4 \times K_{p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t}}\right]$. We write

$$\begin{aligned} p_1^{\alpha_1}\left[K_4 \times K_{p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t}}\right] &= p_1^{\alpha_1}\left\{K_4 \times \left[p_2^{\alpha_2} K_{p_3^{\alpha_3} \cdots p_t^{\alpha_t}} \oplus K_{p_2^{\alpha_2}}(p_3^{\alpha_3} \cdots p_t^{\alpha_t})\right]\right\} \\ &= p_1^{\alpha_1}\left\{p_2^{\alpha_2}\left[K_4 \times K_{p_3^{\alpha_3} \cdots p_t^{\alpha_t}}\right] \oplus \left[K_4 \times K_{p_2^{\alpha_2}}(p_3^{\alpha_3} \cdots p_t^{\alpha_t})\right]\right\} \\ &= p_1^{\alpha_1} p_2^{\alpha_2}\left[K_4 \times K_{p_3^{\alpha_3} \cdots p_t^{\alpha_t}}\right] \oplus p_1^{\alpha_1}\left[K_4 \times K_{p_2^{\alpha_2}}(p_3^{\alpha_3} \cdots p_t^{\alpha_t})\right]. \end{aligned}$$

Now we have to show the existence of gregarious kite factorization of $p_1^{\alpha_1}p_2^{\alpha_2}\left[K_4 \times K_{p_3^{\alpha_3}\cdots p_t^{\alpha_t}}\right]$ and $p_1^{\alpha_1}\left[K_4 \times K_{p_2^{\alpha_2}}(p_3^{\alpha_3}\cdots p_t^{\alpha_t})\right]$. The existence of gregarious kite factorization of $p_1^{\alpha_1}\left[K_4 \times K_{p_2^{\alpha_2}}(p_3^{\alpha_3}\cdots p_t^{\alpha_t})\right]$ is similar to Case 1.

Now we can write

$$\begin{aligned} p_1^{\alpha_1}p_2^{\alpha_2}\left[K_4 \times K_{p_3^{\alpha_3}\cdots p_t^{\alpha_t}}\right] &= p_1^{\alpha_1}p_2^{\alpha_2}\left\{K_4 \times \left[p_3^{\alpha_3}K_{p_4^{\alpha_4}\cdots p_t^{\alpha_t}} \oplus K_{p_3^{\alpha_3}}(p_4^{\alpha_4}\cdots p_t^{\alpha_t})\right]\right\} \\ &= p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\left[K_4 \times K_{p_4^{\alpha_4}\cdots p_t^{\alpha_t}}\right] \\ &\oplus p_1^{\alpha_1}p_2^{\alpha_2}\left[K_4 \times K_{p_3^{\alpha_3}}(p_4^{\alpha_4}\cdots p_t^{\alpha_t})\right]. \end{aligned}$$

The existence of gregarious kite factorization of the second term $p_1^{\alpha_1}p_2^{\alpha_2}\left[K_4 \times K_{p_3^{\alpha_3}}(p_4^{\alpha_4}\cdots p_t^{\alpha_t})\right]$ is similar to Case 1.

Now we consider the first term $p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\left[K_4 \times K_{p_4^{\alpha_4}\cdots p_t^{\alpha_t}}\right]$ and repeat the above process until we end up with $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{t-1}^{\alpha_{t-1}}\left[K_4 \times K_{p_t^{\alpha_t}}\right] \oplus p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{t-2}^{\alpha_{t-2}}\left[K_4 \times K_{p_{t-1}^{\alpha_{t-1}}}(p_t^{\alpha_t})\right]$. Now the existence of a gregarious kite factorization of $K_4 \times K_{p_t^{\alpha_t}}$ and hence the first term follows from Lemma 15 and the existence of gregarious kite factorization of $K_4 \times K_{p_{t-1}^{\alpha_{t-1}}}(p_t^{\alpha_t})$ and hence the second term is similar to Case 1. Thus we have a gregarious kite factorization of $p_1^{\alpha_1}\left[K_4 \times K_{p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_t^{\alpha_t}}\right]$.

Hence from Cases 1 and 2, we have a gregarious kite factorization of $K_4 \times K_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}} = K_4 \times K_n$. ■

Lemma 18. *There exists a gregarious kite factorization of $K_8 \times K_3$.*

Proof. Let $V(K_8) = \{1, 2, \ldots, 8\}$ and $V(K_3) = \{1, 2, 3\}$. Then $V(K_8 \times K_3) = \bigcup_{i=1}^{8} V_i$, where $V_i = \{i_j \mid 1 \leq j \leq 3\}$. Now we construct a gregarious kite factorization of $K_8 \times K_3$ as follows: For $0 \leq s \leq 2$, let

$$\begin{aligned} F_s^1 &= \{(1_{1+s}, 2_{3+s}, 3_{2+s}; 3_{2+s}6_{1+s})\,(8_{1+s}, 5_{3+s}, 4_{2+s}; 4_{2+s}7_{3+s})\}; \\ F_s^2 &= \{(1_{1+s}, 5_{3+s}, 7_{2+s}; 7_{2+s}2_{3+s})\,(3_{1+s}, 6_{2+s}, 4_{3+s}; 4_{3+s}8_{1+s})\}; \\ F_s^3 &= \{(4_{1+s}, 2_{3+s}, 1_{2+s}; 1_{2+s}3_{1+s})\,(5_{1+s}, 8_{3+s}, 6_{2+s}; 6_{2+s}7_{1+s})\}; \\ F_s^4 &= \{(6_{1+s}, 2_{2+s}, 8_{3+s}; 8_{3+s}1_{1+s})\,(7_{1+s}, 5_{3+s}, 3_{2+s}; 3_{2+s}4_{3+s})\}; \\ F_s^5 &= \{(2_{1+s}, 5_{3+s}, 6_{2+s}; 6_{2+s}1_{1+s})\,(3_{1+s}, 8_{3+s}, 7_{2+s}; 7_{2+s}4_{3+s})\}; \\ F_s^6 &= \{(1_{1+s}, 8_{2+s}, 7_{3+s}; 7_{3+s}6_{2+s})\,(2_{1+s}, 4_{3+s}, 5_{2+s}; 5_{2+s}3_{3+s})\}; \\ F_s^7 &= \{(6_{1+s}, 4_{3+s}, 1_{2+s}; 1_{2+s}5_{3+s})\,(8_{1+s}, 3_{3+s}, 2_{2+s}; 2_{2+s}7_{3+s})\}. \end{aligned}$$

In all the above constructions the subscripts are taken modulo 3 with residues 1, 2, 3. Clearly each $F_i = \bigcup_{s=0}^{2} F_s^i$, $1 \leq i \leq 7$, is a gregarious kite factor of $K_8 \times K_3$ and $\{F_1, F_2, \ldots, F_7\}$ gives a gregarious kite factorization of $K_8 \times K_3$. ■

Lemma 19. *For $n \equiv 3 \pmod 6$, there exists a gregarious kite factorization of $K_8 \times K_n$.*

Proof. By Theorem 6, we have a K_3-factorization of K_n, $n = 6s+3$, $s \geq 1$ (since the case $s = 0$ follows from Lemma 18). Corresponding to each K_3-factor of K_n, we have a $(K_8 \times K_3)$-factor of $K_8 \times K_n$. Hence a K_3-factorization of K_n implies a $(K_8 \times K_3)$-factorization of $K_8 \times K_n$. By Lemma 18, we have a gregarious kite factorization of $K_8 \times K_3$. Thus combining all these we get a gregarious kite factorization of $K_8 \times K_n$, $n = 6s+3$, $s \geq 1$. ∎

Lemma 20. *For $|V(G)| = p$, $p \geq 5$ is a prime, there exists a gregarious kite factorization of $K_8 \times G$, where G is described as in Note 7.*

Proof. Let $V(K_8) = \{1, 2, \ldots, 8\}$ and $V(G) = \{1, 2, \ldots, p\}$, $p \geq 5$. Then $V(K_8 \times G) = \bigcup_{i=1}^{8} V_i$, where $V_i = \{i_j \mid 1 \leq j \leq p\}$. Now we construct a gregarious kite factorization of $K_8 \times G$ as follows: For $0 \leq s \leq p-1$, let

$$\begin{aligned}
F_s^1 &= \{(2_{1+s}, 3_{2+s}, 1_{p+s}; 1_{p+s}7_{2+s})\,(8_{1+s}, 5_{3+s}, 4_{2+s}; 4_{2+s}6_{4+s})\};\\
F_s^2 &= \{(1_{1+s}, 5_{3+s}, 7_{2+s}; 7_{2+s}2_{4+s})\,(3_{2+s}, 6_{p+s}, 4_{1+s}; 4_{1+s}8_{p-1+s})\};\\
F_s^3 &= \{(1_{3+s}, 4_{2+s}, 2_{1+s}; 2_{1+s}8_{3+s})\,(3_{2+s}, 5_{3+s}, 6_{1+s}; 6_{1+s}7_{3+s})\};\\
F_s^4 &= \{(1_{3+s}, 6_{2+s}, 8_{1+s}; 8_{1+s}2_{3+s})\,(3_{3+s}, 5_{1+s}, 7_{2+s}; 7_{2+s}4_{p+s})\};\\
F_s^5 &= \{(2_{1+s}, 5_{3+s}, 6_{2+s}; 6_{2+s}1_{4+s})\,(4_{1+s}, 7_{2+s}, 8_{3+s}; 8_{3+s}3_{1+s})\};\\
F_s^6 &= \{(1_{2+s}, 8_{3+s}, 7_{1+s}; 7_{1+s}6_{3+s})\,(5_{1+s}, 4_{2+s}, 2_{3+s}; 2_{3+s}3_{5+s})\};\\
F_s^7 &= \{(1_{2+s}, 5_{1+s}, 4_{3+s}; 4_{3+s}2_{1+s})\,(6_{2+s}, 7_{3+s}, 3_{1+s}; 3_{1+s}8_{p-1+s})\};\\
F_s^8 &= \{(3_{2+s}, 4_{3+s}, 1_{1+s}; 1_{1+s}7_{p-1+s})\,(2_{2+s}, 5_{1+s}, 8_{3+s}; 8_{3+s}6_{1+s})\};\\
F_s^9 &= \{(1_{3+s}, 2_{2+s}, 3_{1+s}; 3_{1+s}6_{3+s})\,(5_{1+s}, 8_{2+s}, 7_{3+s}; 7_{3+s}4_{5+s})\};\\
F_s^{10} &= \{(2_{3+s}, 6_{2+s}, 4_{1+s}; 4_{1+s}1_{3+s})\,(8_{2+s}, 5_{3+s}, 3_{1+s}; 3_{1+s}7_{p-1+s})\};\\
F_s^{11} &= \{(1_{3+s}, 3_{2+s}, 5_{1+s}; 5_{1+s}4_{p-1+s})\,(6_{1+s}, 8_{2+s}, 2_{3+s}; 2_{3+s}7_{5+s})\};\\
F_s^{12} &= \{(5_{2+s}, 6_{3+s}, 1_{1+s}; 1_{1+s}2_{3+s})\,(3_{1+s}, 4_{3+s}, 7_{2+s}; 7_{2+s}8_{p+s})\};\\
F_s^{13} &= \{(1_{2+s}, 8_{1+s}, 6_{3+s}; 6_{3+s}4_{5+s})\,(7_{1+s}, 5_{3+s}, 2_{2+s}; 2_{2+s}3_{p+s})\};\\
F_s^{14} &= \{(2_{1+s}, 7_{2+s}, 6_{3+s}; 6_{3+s}5_{1+s})\,(4_{1+s}, 3_{3+s}, 8_{2+s}; 8_{2+s}1_{p+s})\}.
\end{aligned}$$

In all the above constructions the subscripts are taken modulo p with residues $1, 2, \ldots, p$. Clearly each $F_i = \bigcup_{s=0}^{p-1} F_s^i$, $1 \leq i \leq 14$, is a gregarious kite factor of $K_8 \times G$ and $\{F_1, F_2, \ldots, F_{14}\}$ gives a gregarious kite factorization of $K_8 \times G$. ∎

Lemma 21. *There exists a gregarious kite factorization of $K_8 \times K_7$.*

Proof. Let $V(K_8) = \{1, 2, \ldots, 8\}$ and $V(K_7) = \{1, 2, \ldots, 7\}$. Then $V(K_8 \times K_7) = \bigcup_{i=1}^{8} V_i$, where $V_i = \{i_j \mid 1 \leq j \leq 7\}$. Now we construct a gregarious kite factorization of $K_8 \times K_7$ as follows: For $0 \le s \le 6$, let

$$\begin{aligned}
F_s^1 &= \{(6_{1+s}, 2_{5+s}, 3_{3+s}; 3_{3+s}1_{6+s})\,(8_{1+s}, 7_{3+s}, 4_{5+s}; 4_{5+s}5_{1+s})\};\\
F_s^2 &= \{(8_{3+s}, 4_{5+s}, 2_{1+s}; 2_{1+s}5_{4+s})\,(7_{1+s}, 6_{4+s}, 1_{6+s}; 1_{6+s}3_{2+s})\};\\
F_s^3 &= \{(8_{1+s}, 2_{4+s}, 3_{6+s}; 3_{6+s}5_{3+s})\,(4_{1+s}, 6_{3+s}, 7_{5+s}; 7_{5+s}1_{1+s})\};\\
F_s^4 &= \{(4_{1+s}, 1_{2+s}, 2_{6+s}; 2_{6+s}8_{2+s})\,(7_{1+s}, 3_{3+s}, 6_{5+s}; 6_{5+s}5_{1+s})\};\\
F_s^5 &= \{(2_{1+s}, 4_{4+s}, 7_{6+s}; 7_{6+s}5_{3+s})\,(8_{1+s}, 3_{3+s}, 6_{6+s}; 6_{6+s}1_{3+s})\};\\
F_s^6 &= \{(4_{1+s}, 1_{3+s}, 8_{5+s}; 8_{5+s}5_{2+s})\,(2_{1+s}, 6_{5+s}, 7_{3+s}; 7_{3+s}3_{7+s})\};\\
F_s^7 &= \{(1_{1+s}, 2_{3+s}, 7_{6+s}; 7_{6+s}8_{2+s})\,(5_{1+s}, 6_{3+s}, 4_{6+s}; 4_{6+s}3_{2+s})\};\\
F_s^8 &= \{(2_{2+s}, 3_{3+s}, 1_{1+s}; 1_{1+s}6_{5+s})\,(8_{1+s}, 4_{2+s}, 5_{3+s}; 5_{3+s}7_{7+s})\};\\
F_s^9 &= \{(7_{2+s}, 5_{3+s}, 1_{1+s}; 1_{1+s}2_{4+s})\,(4_{1+s}, 3_{2+s}, 6_{6+s}; 6_{6+s}8_{3+s})\};\\
F_s^{10} &= \{(1_{1+s}, 8_{6+s}, 6_{7+s}; 6_{7+s}4_{4+s})\,(7_{1+s}, 5_{7+s}, 3_{2+s}; 3_{2+s}2_{6+s})\};\\
F_s^{11} &= \{(8_{1+s}, 1_{5+s}, 2_{3+s}; 2_{3+s}4_{4+s})\,(6_{1+s}, 5_{3+s}, 3_{2+s}; 3_{2+s}7_{6+s})\};\\
F_s^{12} &= \{(2_{1+s}, 6_{2+s}, 5_{3+s}; 5_{3+s}1_{7+s})\,(4_{1+s}, 7_{2+s}, 8_{3+s}; 8_{3+s}3_{7+s})\};\\
F_s^{13} &= \{(7_{1+s}, 1_{2+s}, 8_{3+s}; 8_{3+s}6_{7+s})\,(5_{1+s}, 2_{3+s}, 4_{2+s}; 4_{2+s}3_{6+s})\};\\
F_s^{14} &= \{(5_{1+s}, 4_{3+s}, 1_{2+s}; 1_{2+s}8_{6+s})\,(6_{2+s}, 7_{3+s}, 3_{1+s}; 3_{1+s}2_{4+s})\};\\
F_s^{15} &= \{(1_{1+s}, 3_{2+s}, 4_{3+s}; 4_{3+s}6_{2+s})\,(5_{1+s}, 8_{3+s}, 2_{2+s}; 2_{2+s}7_{6+s})\};\\
F_s^{16} &= \{(3_{1+s}, 2_{2+s}, 1_{3+s}; 1_{3+s}4_{7+s})\,(8_{2+s}, 7_{3+s}, 5_{1+s}; 5_{1+s}6_{4+s})\};\\
F_s^{17} &= \{(6_{2+s}, 2_{3+s}, 4_{1+s}; 4_{1+s}7_{4+s})\,(3_{1+s}, 8_{2+s}, 5_{3+s}; 5_{3+s}1_{6+s})\};\\
F_s^{18} &= \{(1_{3+s}, 3_{2+s}, 5_{1+s}; 5_{1+s}4_{4+s})\,(6_{1+s}, 2_{3+s}, 8_{2+s}; 8_{2+s}7_{5+s})\};\\
F_s^{19} &= \{(1_{1+s}, 6_{3+s}, 5_{2+s}; 5_{2+s}2_{5+s})\,(7_{2+s}, 4_{3+s}, 3_{1+s}; 3_{1+s}8_{5+s})\};\\
F_s^{20} &= \{(8_{1+s}, 6_{3+s}, 1_{2+s}; 1_{2+s}4_{5+s})\,(7_{1+s}, 2_{2+s}, 5_{3+s}; 5_{3+s}3_{7+s})\};\\
F_s^{21} &= \{(2_{1+s}, 6_{3+s}, 7_{2+s}; 7_{2+s}1_{6+s})\,(4_{1+s}, 3_{3+s}, 8_{2+s}; 8_{2+s}5_{5+s})\}.
\end{aligned}$$

In all the above constructions the subscripts are taken modulo 7 with residues $1, 2, \ldots, 7$. Clearly each $F_i = \bigcup_{s=0}^{6} F_s^i$, $1 \le i \le 21$, is a gregarious kite factor of $K_8 \times K_7$ and $\{F_1, F_2, \ldots, F_{21}\}$ gives a gregarious kite factorization of $K_8 \times K_7$.

■

Lemma 22. *For $|V(H)| = p$, $p \ge 11$ is a prime, there exists a gregarious kite factorization of $K_8 \times H$, where H is described as in Note 7.*

Proof. Let $V(K_8) = \{1, 2, \ldots, 8\}$ and $V(H) = \{1, 2, \ldots, p\}$, $p \ge 11$. Then $V(K_8 \times H) = \bigcup_{i=1}^{8} V_i$, where $V_i = \{i_j \mid 1 \le j \le p\}$. Now we construct a gregarious kite factorization of $K_8 \times H$ as follows: For $0 \le s \le p-1$, let

$$\begin{aligned}
F_s^1 &= \{(7_{3+s}, 4_{5+s}, 1_{1+s}; 1_{1+s}2_{5+s})\,(5_{p-1+s}, 8_{p+s}, 6_{1+s}; 6_{1+s}3_{5+s})\};\\
F_s^2 &= \{(4_{3+s}, 2_{2+s}, 1_{1+s}; 1_{1+s}3_{5+s})\,(8_{3+s}, 6_{2+s}, 5_{1+s}; 5_{1+s}7_{5+s})\};
\end{aligned}$$

$$F_s^3 = \{(6_{p-1+s}, 4_{p+s}, 1_{1+s}; 1_{1+s}7_{5+s})\,(5_{p-1+s}, 2_{p+s}, 3_{1+s}; 3_{1+s}8_{5+s})\};$$
$$F_s^4 = \{(8_{p-3+s}, 1_{p-1+s}, 2_{1+s}; 2_{1+s}3_{5+s})\,(7_{p-1+s}, 6_{p+s}, 5_{1+s}; 5_{1+s}4_{5+s})\};$$
$$F_s^5 = \{(1_{3+s}, 2_{2+s}, 3_{1+s}; 3_{1+s}7_{5+s})\,(4_{2+s}, 5_{3+s}, 8_{1+s}; 8_{1+s}6_{5+s})\};$$
$$F_s^6 = \{(7_{2+s}, 4_{3+s}, 2_{1+s}; 2_{1+s}1_{5+s})\,(8_{p-1+s}, 5_{p+s}, 3_{1+s}; 3_{1+s}6_{5+s})\};$$
$$F_s^7 = \{(3_{2+s}, 1_{p+s}, 6_{1+s}; 6_{1+s}4_{5+s})\,(7_{1+s}, 2_{3+s}, 8_{5+s}; 8_{5+s}5_{1+s})\};$$
$$F_s^8 = \{(1_{3+s}, 7_{2+s}, 5_{1+s}; 5_{1+s}3_{5+s})\,(2_{2+s}, 6_{3+s}, 4_{1+s}; 4_{1+s}8_{5+s})\};$$
$$F_s^9 = \{(6_{p+s}, 5_{2+s}, 2_{1+s}; 2_{1+s}7_{5+s})\,(8_{p+s}, 4_{2+s}, 3_{1+s}; 3_{1+s}1_{5+s})\};$$
$$F_s^{10} = \{(8_{p+s}, 2_{p-1+s}, 1_{1+s}; 1_{1+s}5_{5+s})\,(3_{3+s}, 4_{5+s}, 7_{1+s}; 7_{1+s}6_{p-3+s})\};$$
$$F_s^{11} = \{(1_{3+s}, 6_{5+s}, 4_{1+s}; 4_{1+s}3_{5+s})\,(7_{3+s}, 8_{5+s}, 2_{1+s}; 2_{1+s}5_{5+s})\};$$
$$F_s^{12} = \{(8_{3+s}, 6_{p-1+s}, 4_{1+s}; 4_{1+s}2_{p-3+s})\,(1_{3+s}, 7_{4+s}, 5_{5+s}; 5_{5+s}3_{1+s})\};$$
$$F_s^{13} = \{(2_{3+s}, 5_{1+s}, 6_{5+s}; 6_{5+s}7_{1+s})\,(8_{2+s}, 1_{p+s}, 3_{1+s}; 3_{1+s}4_{5+s})\};$$
$$F_s^{14} = \{(7_{2+s}, 5_{p+s}, 4_{1+s}; 4_{1+s}1_{5+s})\,(6_{3+s}, 2_{p-1+s}, 3_{1+s}; 3_{1+s}8_{p-3+s})\};$$
$$F_s^{15} = \{(2_{p-1+s}, 6_{p-3+s}, 5_{1+s}; 5_{1+s}1_{5+s})\,(3_{p-1+s}, 7_{p+s}, 8_{1+s}; 8_{1+s}4_{5+s})\};$$
$$F_s^{16} = \{(1_{3+s}, 8_{p-1+s}, 7_{1+s}; 7_{1+s}5_{5+s})\,(6_{1+s}, 3_{3+s}, 2_{5+s}; 2_{5+s}4_{1+s})\};$$
$$F_s^{17} = \{(8_{2+s}, 2_{3+s}, 4_{1+s}; 4_{1+s}5_{5+s})\,(3_{2+s}, 6_{3+s}, 7_{1+s}; 7_{1+s}1_{5+s})\};$$
$$F_s^{18} = \{(1_{p+s}, 6_{p-1+s}, 8_{1+s}; 8_{1+s}5_{5+s})\,(7_{3+s}, 4_{p-1+s}, 3_{1+s}; 3_{1+s}2_{5+s})\};$$
$$F_s^{19} = \{(5_{2+s}, 3_{p+s}, 1_{1+s}; 1_{1+s}8_{5+s})\,(4_{p-1+s}, 6_{p+s}, 7_{1+s}; 7_{1+s}2_{5+s})\};$$
$$F_s^{20} = \{(4_{p-1+s}, 3_{p+s}, 5_{1+s}; 5_{1+s}2_{5+s})\,(7_{3+s}, 8_{p-1+s}, 6_{1+s}; 6_{1+s}1_{5+s})\};$$
$$F_s^{21} = \{(4_{2+s}, 5_{p+s}, 1_{1+s}; 1_{1+s}6_{5+s})\,(2_{2+s}, 8_{p+s}, 7_{1+s}; 7_{1+s}3_{5+s})\}.$$

In all the above constructions the subscripts are taken modulo p with residues $1, 2, \ldots, p$. Clearly each $F_i = \bigcup_{s=0}^{p-1} F_s^i$, $1 \le i \le 21$, is a gregarious kite factor of $K_8 \times H$ and $\{F_1, F_2, \ldots, F_{21}\}$ gives a gregarious kite factorization of $K_8 \times H$. ■

Lemma 23. *For all odd prime p, there exists a gregarious kite factorization of $K_8 \times K_p$.*

Proof. By Remark 8, K_p has a factorization into graphs isomorphic to G or H. Hence a gregarious kite factorization of $K_8 \times K_p$ follows from Lemmas 18, 20, 21 and 22. ■

Lemma 24. *For all odd prime p and $s > 1$, there exists a gregarious kite factorization of $K_8 \times K_{p^s}$.*

Proof. For $s > 1$, $K_8 \times K_{p^s} = K_8 \times \left[pK_{p^{s-1}} \oplus K_p(p^{s-1})\right] = p(K_8 \times K_{p^{s-1}}) \oplus \left[K_8 \times K_p(p^{s-1})\right]$ (since the case $s = 1$ follows from Lemma 23).

For $s = 2$, $K_8 \times K_{p^2} = p(K_8 \times K_p) \oplus \left[K_8 \times K_p(p)\right]$. By Lemma 23, we have a gregarious kite factorization of $p(K_8 \times K_p)$. By Theorem 4, we have a K_p-factorization of $K_p(p)$. Corresponding to each K_p-factor of $K_p(p)$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(p)$. Thus a K_p-factorization of $K_p(p)$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(p)$. Now the existence of a gregarious kite factorization of $(K_8 \times K_p)$ follows from Lemma 23.

For $s = 3$, $K_8 \times K_{p^3} = p(K_8 \times K_{p^2}) \oplus \left[K_8 \times K_p(p^2)\right]$. Then the gregarious kite factorization of $p(K_8 \times K_{p^2})$ follows from the case $s = 2$. By Theorem 4, we have a K_p-factorization of $K_p(p^2)$. Corresponding to each K_p-factor of $K_p(p^2)$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(p^2)$. Thus a K_p-factorization of $K_p(p^2)$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(p^2)$. Now the existence of a gregarious kite factorization of $(K_8 \times K_p)$ follows from Lemma 23.

For $s > 1$, $K_8 \times K_{p^s} = p(K_8 \times K_{p^{s-1}}) \oplus \left[K_8 \times K_p(p^{s-1})\right]$. By the induction hypothesis on s, we have a gregarious kite factorization of $p(K_8 \times K_{p^{s-1}})$. By Theorem 4, we have a K_p-factorization of $K_p(p^{s-1})$. Corresponding to each K_p-factor of $K_p(p^{s-1})$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(p^{s-1})$. Thus a K_p-factorization of $K_p(p^{s-1})$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(p^{s-1})$. Now the existence of a gregarious kite factorization of $(K_8 \times K_p)$ follows from Lemma 23. Hence combining all the above we have a gregarious kite factorization of $K_8 \times K_{p^s}$, for all $s > 1$. ■

Lemma 25. *There exists a gregarious kite factorization of $K_8 \times K_{p^s q^t}$ for all odd primes p, q $(p < q)$ and all integers $s, t \geq 1$.*

Proof. For $s, t \geq 1$ and $p < q$,

$$K_8 \times K_{p^s q^t} = K_8 \times K_{p.p^{s-1}q^t} = K_8 \times \left[pK_{p^{s-1}q^t} \oplus K_p(p^{s-1}q^t)\right]$$
$$= p\left[K_8 \times K_{p^{s-1}q^t}\right] \oplus \left[K_8 \times K_p(p^{s-1}q^t)\right].$$

Case 1. (a) For $s = 1$, $t = 1$, $K_8 \times K_{pq} = K_8 \times [pK_q \oplus K_p(q)] = p\left[K_8 \times K_q\right] \oplus [K_8 \times K_p(q)]$. By Theorem 4, we have a K_p-factorization of $K_p(q)$. Corresponding to each K_p-factor of $K_p(q)$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(q)$. Thus a K_p-factorization of $K_p(q)$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(q)$. Now the existence of a gregarious kite factorization of $K_8 \times K_p$ and $p(K_8 \times K_q)$ follows from Lemma 23.

(b) For $s = 1$, $t = 2$, $K_8 \times K_{pq^2} = K_8 \times \left[pK_{q^2} \oplus K_p(q^2)\right] = p\left[K_8 \times K_{q^2}\right] \oplus \left[K_8 \times K_p(q^2)\right]$. By Theorem 4, we have a K_p-factorization of $K_p(q^2)$. Corresponding to each K_p-factor of $K_p(q^2)$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(q^2)$. Thus a K_p-factorization of $K_p(q^2)$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(q^2)$. Now the existence of a gregarious kite factorization of $K_8 \times K_p$ and $p(K_8 \times K_{q^2})$ follows from Lemmas 23 and 24, respectively.

(c) For $s = 1$, $t \geq 3$, $K_8 \times K_{pq^t} = K_8 \times \left[pK_{q^t} \oplus K_p(q^t)\right] = p\left[K_8 \times K_{q^t}\right] \oplus \left[K_8 \times K_p(q^t)\right]$. By Theorem 4, we have a K_p-factorization of $K_p(q^t)$. Corresponding to each K_p-factor of $K_p(q^t)$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(q^t)$. Thus a K_p-factorization of $K_p(q^t)$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(q^t)$. Now the existence of a gregarious kite factorization of $K_8 \times K_p$ and $p(K_8 \times K_{q^t})$ follows from Lemmas 23 and 24, respectively.

Case 2. (a) For $s = 2$, $t = 1$, $K_8 \times K_{p^2q} = K_8 \times K_{p.pq} = K_8 \times [pK_{pq} \oplus K_p(pq)] = p\,[K_8 \times K_{pq}] \oplus [K_8 \times K_p(pq)]$. By Theorem 4, we have a K_p-factorization of $K_p(pq)$. Corresponding to each K_p-factor of $K_p(pq)$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(pq)$. Thus a K_p-factorization of $K_p(pq)$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(pq)$. Now the existence of a gregarious kite factorization of $K_8 \times K_p$ and $p\,[K_8 \times K_{pq}]$ follows from Lemma 23 and Case 1(a), respectively.

(b) For $s = 2$, $t = 2$, $K_8 \times K_{p^2q^2} = K_8 \times K_{p.pq^2} = K_8 \times \left[pK_{pq^2} \oplus K_p(pq^2)\right] = p\left[K_8 \times K_{pq^2}\right] \oplus \left[K_8 \times K_p(pq^2)\right]$. By Theorem 4, we have a K_p-factorization of $K_p(pq^2)$. Corresponding to each K_p-factor of $K_p(pq^2)$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(pq^2)$. Thus a K_p-factorization of $K_p(pq^2)$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(pq^2)$. Now the existence of a gregarious kite factorization of $K_8 \times K_p$ and $p\left[K_8 \times K_{pq^2}\right]$ follows from Lemma 23 and Case 1(b), respectively.

(c) For $s = 2$, $t \geq 3$, $K_8 \times K_{p^2q^t} = K_8 \times K_{p.pq^t} = K_8 \times \left[pK_{pq^t} \oplus K_p(pq^t)\right] = p\left[K_8 \times K_{pq^t}\right] \oplus \left[K_8 \times K_p(pq^t)\right]$. By Theorem 4, we have a K_p-factorization of $K_p(pq^t)$. Corresponding to each K_p-factor of $K_p(pq^t)$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(pq^t)$. Thus a K_p-factorization of $K_p(pq^t)$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(pq^t)$. Now the existence of a gregarious kite factorization of $K_8 \times K_p$ and $p\left[K_8 \times K_{pq^t}\right]$ follows from Lemma 23 and Case 1(c), respectively.

For $s, t \geq 1$, $K_8 \times K_{p^sq^t} = p\left[K_8 \times K_{p^{s-1}q^t}\right] \oplus \left[K_8 \times K_p(p^{s-1}q^t)\right]$. By the induction hypothesis on s, we have a gregarious kite factorization of $p\left[K_8 \times K_{p^{s-1}q^t}\right]$. By Theorem 4, we have a K_p-factorization of $K_p(p^{s-1}q^t)$. Corresponding to each K_p-factor of $K_p(p^{s-1}q^t)$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(p^{s-1}q^t)$. Thus a K_p-factorization of $K_p(p^{s-1}q^t)$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(p^{s-1}q^t)$. Now the existence of a gregarious kite factorization of $K_8 \times K_p$ follows from Lemma 23. Hence combining all the above we have a gregarious kite factorization of $K_8 \times K_{p^sq^t}$, for all $s, t \geq 1$ and $p < q$. ■

Lemma 26. *There exists a gregarious kite factorization of $K_8 \times K_n$ for all odd $n > 1$.*

Proof. By fundamental theorem of arithmetic, any integer $n > 1$ can be uniquely written as prime powers or product of prime powers.

Consider $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$, where each p_i is a distinct odd prime and $\alpha_i \geq 1, i = 1, 2, \ldots, t$. Fix $p_1^{\alpha_1} < p_2^{\alpha_2} < \cdots < p_t^{\alpha_t}$. Now,

$$\begin{aligned} K_8 \times K_n = K_8 \times K_{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}} &= K_8 \times \left[p_1^{\alpha_1} K_{p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t}} \oplus K_{p_1^{\alpha_1}}(p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t})\right] \\ &= p_1^{\alpha_1}\left[K_8 \times K_{p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t}}\right] \oplus \left[K_8 \times K_{p_1^{\alpha_1}}(p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t})\right]. \end{aligned}$$

It is enough to show that there exists a gregarious kite factorization of $K_8 \times K_{p_1^{\alpha_1}(p_2^{\alpha_2}p_3^{\alpha_3}\ldots p_t^{\alpha_t})}$ and $p_1^{\alpha_1}\left[K_8 \times K_{p_2^{\alpha_2}p_3^{\alpha_3}\ldots p_t^{\alpha_t}}\right]$.

Case 1. Consider $K_8 \times K_{p_1^{\alpha_1}(p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_t^{\alpha_t})}$. By Theorem 4, we have a $K_{p_1^{\alpha_1}}$-factorization of $K_{p_1^{\alpha_1}(p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_t^{\alpha_t})}$. Corresponding to each $K_{p_1^{\alpha_1}}$-factor of $K_{p_1^{\alpha_1}(p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_t^{\alpha_t})}$, we have a $(K_8 \times K_{p_1^{\alpha_1}})$-factor of $K_8 \times K_{p_1^{\alpha_1}(p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_t^{\alpha_t})}$. Now the existence of a gregarious kite factorization of $K_8 \times K_{p_1^{\alpha_1}}$ follows from Lemma 24.

Case 2. Consider $p_1^{\alpha_1}\left[K_8 \times K_{p_2^{\alpha_2}p_3^{\alpha_3}\ldots p_t^{\alpha_t}}\right]$. We write

$$\begin{aligned}
p_1^{\alpha_1}\left[K_8 \times K_{p_2^{\alpha_2}p_3^{\alpha_3}\ldots p_t^{\alpha_t}}\right] &= p_1^{\alpha_1}\left\{K_8 \times \left[p_2^{\alpha_2}K_{p_3^{\alpha_3}\ldots p_t^{\alpha_t}} \oplus K_{p_2^{\alpha_2}(p_3^{\alpha_3}\cdots p_t^{\alpha_t})}\right]\right\}\\
&= p_1^{\alpha_1}\left\{p_2^{\alpha_2}\left[K_8 \times K_{p_3^{\alpha_3}\ldots p_t^{\alpha_t}}\right] \oplus \left[K_8 \times K_{p_2^{\alpha_2}(p_3^{\alpha_3}\cdots p_t^{\alpha_t})}\right]\right\}\\
&= p_1^{\alpha_1}p_2^{\alpha_2}\left[K_8 \times K_{p_3^{\alpha_3}\ldots p_t^{\alpha_t}}\right] \oplus p_1^{\alpha_1}\left[K_8 \times K_{p_2^{\alpha_2}(p_3^{\alpha_3}\cdots p_t^{\alpha_t})}\right].
\end{aligned}$$

Now we have to show the existence of gregarious kite factorization of $p_1^{\alpha_1}p_2^{\alpha_2}\left[K_8 \times K_{p_3^{\alpha_3}\ldots p_t^{\alpha_t}}\right]$ and $p_1^{\alpha_1}\left[K_8 \times K_{p_2^{\alpha_2}(p_3^{\alpha_3}\cdots p_t^{\alpha_t})}\right]$. The existence of gregarious kite factorization of $p_1^{\alpha_1}\left[K_8 \times K_{p_2^{\alpha_2}(p_3^{\alpha_3}\cdots p_t^{\alpha_t})}\right]$ is similar to Case 1. Now we can write

$$\begin{aligned}
p_1^{\alpha_1}p_2^{\alpha_2}\left[K_8 \times K_{p_3^{\alpha_3}\ldots p_t^{\alpha_t}}\right] &= p_1^{\alpha_1}p_2^{\alpha_2}\left\{K_8 \times \left[p_3^{\alpha_3}K_{p_4^{\alpha_4}\ldots p_t^{\alpha_t}} \oplus K_{p_3^{\alpha_3}(p_4^{\alpha_4}\cdots p_t^{\alpha_t})}\right]\right\}\\
&= p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\left[K_8 \times K_{p_4^{\alpha_4}\ldots p_t^{\alpha_t}}\right]\\
&\oplus p_1^{\alpha_1}p_2^{\alpha_2}\left[K_8 \times K_{p_3^{\alpha_3}(p_4^{\alpha_4}\cdots p_t^{\alpha_t})}\right].
\end{aligned}$$

The existence of gregarious kite factorization of second term $p_1^{\alpha_1}p_2^{\alpha_2}\left[K_8 \times K_{p_3^{\alpha_3}(p_4^{\alpha_4}\cdots p_t^{\alpha_t})}\right]$ is similar to Case 1.

Now we consider the first term $p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\left[K_8 \times K_{p_4^{\alpha_4}\ldots p_t^{\alpha_t}}\right]$ and repeat the above process until we end up with $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{t-1}^{\alpha_{t-1}}\left[K_8 \times K_{p_t^{\alpha_t}}\right] \oplus p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{t-2}^{\alpha_{t-2}}\left[K_8 \times K_{p_{t-1}^{\alpha_{t-1}}(p_t^{\alpha_t})}\right]$. Now the existence of a gregarious kite factorization of $K_8 \times K_{p_t^{\alpha_t}}$ and hence the first term follows from Lemma 24 and the existence of gregarious kite factorization of $K_8 \times K_{p_{t-1}^{\alpha_{t-1}}(p_t^{\alpha_t})}$ and hence the second term is similar to Case 1. Thus we have a gregarious kite factorization of $p_1^{\alpha_1}\left[K_8 \times K_{p_2^{\alpha_2}p_3^{\alpha_3}\ldots p_t^{\alpha_t}}\right]$.

Hence from Cases 1 and 2, we have a gregarious kite factorization of $K_8 \times K_{p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_t^{\alpha_t}} = K_8 \times K_n$. ■

Lemma 27. *For all odd $n > 1$, there exists a gregarious kite factorization of $K \times K_n$, where K is a kite.*

Proof. Let $V(K) = \{1, 2, 3, 4\}$ and $V(K_n) = \{1, 2, \ldots, n\}$. Then $V(K \times K_n) = \bigcup_{i=1}^{4} V_i$, where $V_i = \{i_j \mid 1 \leq j \leq n\}$. Now we construct a gregarious kite factorization of $K \times K_n$ as follows: For $0 \leq s \leq n-2$, let $F_s = \bigoplus_{i=0}^{n-1} \{1_{1+i}, 2_{2+s+i}, 3_{3+2s+i}; 3_{3+2s+i}4_{4+3s+i}\}$, where the subscripts are taken modulo n with residues $1, 2, \ldots, n$. Clearly each F_s, $0 \leq s \leq n-2$ is a gregarious kite factor of $K \times K_n$ and all together gives a gregarious kite factorization of $K \times K_n$. ∎

Theorem 28. *There exists a gregarious kite factorization of $K_m \times K_n$ if and only if $m \equiv 0 \pmod 4$ and n is any odd integer greater than 1.*

Proof. *Necessity.* It follows by $4|mn$, $\{(m-1)(n-1)\}/2 \in \mathbb{N}$ (respectively, the size of a kite factor and the number of factors in a kite factorization of the graph $K_m \times K_n$).

Sufficiency. Let $m = 4s$, $s \geq 1$ and n is odd. The case $s = 1, 2$ follows from Lemmas 17 and 26, respectively. Then for $s \geq 3$, $K_{4s} \times K_n = [sK_4 \oplus K_s(4)] \times K_n = s(K_4 \times K_n) \oplus (K_s(4) \times K_n)$. Now the existence of a gregarious kite factorization of $s\,(K_4 \times K_n)$ follows from Lemma 17. By Theorem 5, we have a kite factorization of $K_s(4), s \geq 3$. Corresponding to each kite factor of $K_s(4)$, we have a $(K \times K_n)$-factor of $(K_s(4) \times K_n)$, where K is a kite. Thus a kite factorization of $K_s(4)$ implies a $(K \times K_n)$-factorization of $(K_s(4) \times K_n)$. Further, the existence of a gregarious kite factorization of $K \times K_n$ follows from Lemma 27. Hence combining all these results we have a gregarious kite factorization of $K_m \times K_n$. ∎

Conclusion. In this paper, we give a complete solution for the existence of a gregarious kite factorization of $K_m \times K_n$.

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