

THE LIST EDGE COLORING AND LIST TOTAL COLORING  
OF PLANAR GRAPHS WITH MAXIMUM DEGREE  
AT LEAST 7

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**Abstract**

A graph  $G$  is edge  $k$ -choosable (respectively, total  $k$ -choosable) if, whenever we are given a list  $L(x)$  of colors with  $|L(x)| = k$  for each  $x \in E(G)$  ( $x \in E(G) \cup V(G)$ ), we can choose a color from  $L(x)$  for each element  $x$  such that no two adjacent (or incident) elements receive the same color. The list edge chromatic index  $\chi'_l(G)$  (respectively, the list total chromatic

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number  $\chi_l''(G)$  of  $G$  is the smallest integer  $k$  such that  $G$  is edge (respectively, total)  $k$ -choosable. In this paper, we focus on a planar graph  $G$ , with maximum degree  $\Delta(G) \geq 7$  and with some structural restrictions, satisfies  $\chi_l'(G) = \Delta(G)$  and  $\chi_l''(G) = \Delta(G) + 1$ .

**Keywords:** planar graph, list edge coloring, list total coloring.

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## 1. INTRODUCTION

All graphs considered in this article are finite, loopless, and without multiple edges. A proper edge  $k$ -coloring of a graph  $G$  is a coloring of the edges of  $G$  with  $k$  colors such that no two adjacent edges receive the same color. The *chromatic index*  $\chi'(G)$  is the smallest  $k$  such that  $G$  admits a proper edge  $k$ -coloring. The list edge coloring, as an extension of the proper edge coloring, has attracted a lot of attention. For any list assignment  $L : E(G) \rightarrow \mathcal{P}(N)$ , a graph  $G = (V(G), E(G))$  is *edge  $L$ -colorable* if there exists a proper edge coloring  $\phi$  of  $G$  such that  $\phi(e) \in L(e)$  for every edge  $e \in E(G)$ , and we always say that  $G$  is *edge  $L$ -colorable* and  $\phi$  is a *proper edge coloring* for  $L$ . A graph  $G = (V(G), E(G))$  is said to be *list edge  $k$ -colorable* (or *edge  $k$ -choosable*) if  $G$  is edge  $L$ -colorable for any list assignment  $L$  such that  $|L(e)| \geq k$  for any edge  $e \in E(G)$ . The *list edge chromatic index*  $\chi_l'(G)$  is the smallest  $k$  such that  $G$  is edge  $k$ -choosable. The *total coloring* (respectively, *proper (vertex) coloring*), the *total chromatic number*  $\chi''(G)$  (respectively, *chromatic number*  $\chi(G)$ ), the *list total coloring* (respectively, *list (vertex) coloring*) and the *list total chromatic number*  $\chi_l''(G)$  (respectively, *list (vertex) chromatic number*  $\chi_l(G)$ ) of a graph  $G$  are defined similarly in terms of coloring edges and vertices.

Clearly, for any graph  $G$ , we have  $\chi_l'(G) \geq \chi'(G) \geq \Delta(G)$  and  $\chi_l''(G) \geq \chi''(G) \geq \Delta(G) + 1$ , where  $\Delta(G)$  denotes the maximum degree of  $G$ . In terms of the relationship between  $\chi_l'(G)$  and  $\chi'(G)$ , there is a conjecture as follows.

**Conjecture 1.** *For any graph  $G$ ,*

- (a)  $\chi_l'(G) = \chi'(G)$ ;
- (b)  $\chi_l''(G) = \chi''(G)$ .

Vizing, Gupta, Albertson and Collins, and Bollobás and Harris (see [12] for details), independently, posed part (a) of Conjecture 1, which is well known as the list coloring conjecture or list edge coloring conjecture (LECC). As a natural extension of part (a), part (b) of Conjecture 1 was posed by Borodin *et al.* [7], it is also known as the list total coloring conjecture (LTCC).

It has been verified that Conjecture 1 holds for graphs with  $\Delta \leq 2$ , outerplanar graphs [17], etc. So far this conjecture remains open.

For edge coloring, the well-known Vizing's theorem shows that  $\chi'(G) = \Delta(G)$  or  $\chi'(G) = \Delta(G) + 1$  for every graph  $G$ . For the list edge coloring, in 1984, along with Vizing's theorem, part (a) of Conjecture 1 implies that  $\chi'_l(G) \leq \Delta(G) + 1$ . Harris showed that  $\chi'_l(G) \leq 2\Delta(G) - 2$  if  $G$  is a graph with  $\Delta(G) \geq 3$  [9], which implies  $\chi'_l(G) \leq \Delta(G) + 1$  for  $\Delta(G) = 3$ . Later, Juvan *et al.* confirmed that  $\chi'_l(G) \leq \Delta + 1$  for a graph with  $\Delta(G) = 4$  [13]. For planar graphs, it is proved that  $\chi'_l(G) \leq \Delta(G) + 1$  if  $\Delta(G) \geq 9$  by Borodin [8] and  $\Delta(G) \geq 8$  by Bonamy [5].

For total colorings, Behzad [3] and Vizing [16] posed, independently, the well-known total coloring conjecture (TCC) which says that every simple graph of maximum degree  $\Delta(G)$  admits a total  $(\Delta(G) + 2)$ -coloring. For the list total coloring, TCC and part (b) of Conjecture 1 implies that  $\chi''_l(G) \leq \Delta(G) + 2$ , which is confirmed for graphs with maximum degree  $\Delta(G) \leq 3$ , for bipartite graphs [14] and for planar graphs with  $\Delta(G) \geq 9$  [11].

Recently, the strong version of Conjecture 1 that  $\chi'_l(G) = \Delta(G)$  and  $\chi''_l(G) = \Delta(G) + 1$  has stimulated people's interest. Note that  $\chi'_l(G) = \Delta(G)$  and  $\chi''_l(G) = \Delta(G) + 1$  are equivalent to  $\chi'_l(G) = \chi'(G) = \Delta(G)$  and  $\chi''_l(G) = \chi''(G) = \Delta(G) + 1$ . For planar graphs, the best known result is that if  $\Delta(G) \geq 12$ , a planar graph  $G$  is list edge  $\Delta(G)$ -colorable and list total  $(\Delta(G) + 1)$ -colorable [7]. It is proved that  $\chi'_l(G) = \Delta(G)$  and  $\chi''_l(G) = \Delta(G) + 1$  for a planar graph  $G$  with  $\Delta(G) \geq 7$  and no triangle adjacent to a  $C_4$  [6], or with no cycle of length from 4 to  $k$ , where  $k \geq 4$  and  $(\Delta(G), k) \in \{(7, 4), (6, 5), (5, 8)\}$  [10], or with  $\Delta(G) \geq 7$  and no adjacent cycles of length at most 4 [15], or with  $\Delta(G) \geq 8$  and no cycle of length 3 adjacent to a cycle of length 5 [15], or with  $\Delta(G) \geq 8$  and no adjacent 4-cycles [18], or with  $\Delta(G) \geq 8$  and no chordal 5-cycle [19].

In the paper, we discuss the list edge and total colorings of a planar graph  $G$  with  $\Delta(G) \geq 7$  and with structure restrictions. General 5-cycles contains the following cases: (1) 5-cycles; (2) three 3-cycles  $C_1, C_2$  and  $C_3$  with  $E(C_1) \cap E(C_2) \cap E(C_3) \neq \emptyset$ . A wheel  $W_n$  is the join  $C_n \vee K_1$  of an  $n$ -cycle  $C_n = [v_1 v_2 \cdots v_n]$  and a single vertex  $v$ . In  $W_n$ , by inserting  $l$  vertices of degree 2 on each edge  $v_i v_{i+1}$ ,  $i = 1, 2, \dots, n$  with  $v_{n+1} = v_1$ , we call the obtained graph an  $l$ -cycle-subdivision of  $W_n$ . A  $k^-$ -cycle-subdivision of  $W_n$  is an  $l$ -cycle-subdivision of  $W_n$  with  $0 \leq l \leq k$ .

**Theorem 2.** *Let  $G$  be a planar graph without general 5-cycles and  $1^-$ -cycle-subdivisions of  $W_3$ . If  $\Delta(G) \geq 7$ , then  $\chi'_l(G) = \Delta(G)$  and  $\chi''_l(G) = \Delta(G) + 1$ .*

For the restriction of cycles, we should state that the fact that no triangle is adjacent to a 4-cycle not only contains the substructure of no chordal 5-cycle, but also contains the substructure of no chordal 4-cycle. Hence, Theorem 2 is different from that in [6].

In the proof of Theorem 2, we will use the method of contradiction. Let  $G = (V, E)$  be a minimal counterexample to the statement of Theorem 2, in the

sense that the quantity  $|V| + |E|$  is minimum. Then  $G$  is connected. In terms of the edge choosability, a minimal counterexample is called an LEC-minimal if  $G$  is not edge  $\Delta(G)$ -choosable, but it holds for each proper subgraph  $H$  of  $G$ . For the total choosability, we define an LTC-minimal counterexample similarly. In the following, we always assume that  $L$  is a list assignment such that  $G$  is not edge  $L$ -colorable with  $|L(e)| = \Delta(G)$  for any edge  $e \in E(G)$  or is not total  $L$ -colorable with  $|L(x)| = \Delta(G) + 1$  for any element  $x \in V(G) \cup E(G)$ .

In Section 2, we will use two ways to discuss structure properties of LEC-minimal counterexample and LTC-ones, respectively. For LEC-minimal counterexamples in Section 2.1, we will use a theorem of Alon and Tarsi to obtain main results, which focus on the list coloring through special orientations. For LTC-minimal counterexamples in Section 2.2, we will use Combinatorial Nullstellensatz to analyse the structures, which is effective with the help of MATLAB. In Section 3, we will use discharging method to prove Theorem 2.

Some definitions and notations should be introduced. Let  $G$  be a graph with the vertex set  $V(G)$  and edge set  $E(G)$ . If  $v \in V(G)$ , then its neighbor set  $N_G(v)$  (or simply  $N(v)$ ) is the set of the vertices in  $G$  adjacent to  $v$  and the degree  $d_G(v)$  of  $v$  is  $|N_G(v)|$ . We denote the maximum degree and minimum degree of  $G$  by  $\Delta(G)$  and  $\delta(G)$ , respectively. A  $k$ -,  $k^+$ - and  $k^-$ -vertex is a vertex of degree  $k$ , at least  $k$  and at most  $k$ , respectively. A vertex  $u$  is called a  $k$ -neighbor (respectively,  $k^-$ -neighbor,  $k^+$ -neighbor) of a vertex  $v$  if  $uv \in E(G)$  and  $d_G(u) = k$  (respectively,  $d_G(u) \leq k$ ,  $d_G(u) \geq k$ ). An edge  $uv$  is denoted by an  $(a_1, a_2)$ -edge if  $d_G(u) = a_1$  and  $d_G(v) = a_2$ . Similarly, we can define an  $(a_1^+, a_2^+)$ -edge, an  $(a_1^-, a_2^-)$ -edge and an  $(a_1^+, a_2^-)$ -edge. Note that the length of a cycle is the number of its edges, and a cycle of length  $k$  is called a  $k$ -cycle. A  $k$ -cycle  $C$  can be denoted by  $C = [u_1 u_2 \cdots u_k]$ , where  $u_1, u_2, \dots, u_k$  are its consecutive vertices. For convenience, a cycle  $C = [u_1 u_2 \cdots u_n]$  is called an  $(a_1, a_2, \dots, a_n)$ -cycle if the degree of the vertex  $u_i$  is  $a_i$  for  $i = 1, 2, \dots, n$ . For a given plane graph  $G$ ,  $F(G)$  denotes the face set of  $G$ . For  $f \in F(G)$ , we use  $V(f)$  to denote the set of vertices on the boundary of  $f$ . A face of  $G$  is said to be *incident* with all edges and vertices in its boundary. The *degree* of a face  $f$ , denoted by  $d_G(f)$ , is the number of edges incident with it, where a cut edge is counted twice. A  $k$ -,  $k^+$ - and  $k^-$ -face in a plane graph  $G$  is defined analogously to counterparts of a vertex and the notation of a  $k$ -face is the same to that of a  $k$ -cycle. For the terminologies and notations not defined here, we follow [4].

## 2. STRUCTURE PROPERTIES OF MINIMAL COUNTEREXAMPLES

To begin with, we display some known structural properties of a minimal counterexample  $G$ , no matter  $G$  is LEC-minimal or LTC-minimal.

**Lemma 3** [20]. *Let  $uv \in E(G)$  with  $d_G(u) \leq \lfloor \frac{\Delta(G)}{2} \rfloor$ . Then  $d_G(u) + d_G(v) \geq \Delta(G) + 2$ . Moreover,  $\delta(G) \geq 2$ .*

**Lemma 4** [20]. *The subgraph induced by all edges joining 2-vertices to  $\Delta(G)$ -vertices in  $G$  is a forest.*

**Lemma 5** [6]. *If  $\Delta(G) \geq 7$  and there is no triangle adjacent to a  $C_4$  in  $G$ , then  $G$  has no configurations  $(A_1)$  and  $(A_2)$  as follows, see Figure 1.*

(1) *For a vertex  $v_1$  with  $d_G(v_1) = 2$ , let  $u$  and  $w_1$  be its two neighbors, and there is a path  $(v_1, w_1, v_2, \dots, v_p, w_p, v_{p+1})$  ( $p \geq 1$ ) such that  $v_i$  is adjacent to  $u$  for each  $i \in \{1, 2, \dots, p+1\}$ ,  $d_G(v_i) = 3$  for each  $i \in \{2, \dots, p\}$  and  $d_G(v_{p+1}) = 2$ , see  $(A_1)$  in Figure 1.*

(2) *For a vertex  $v_1$  with  $d_G(v_1) = 2$ , let  $u$  and  $w_1$  be its two neighbors, and there is a cycle  $(w_1, v_2, w_2, \dots, w_{p-1}, v_p)$  such that  $v_i$  is adjacent to  $u$ ,  $d_G(v_i) = 3$  for  $i = 2, \dots, p$ , see  $(A_2)$  in Figure 1.*

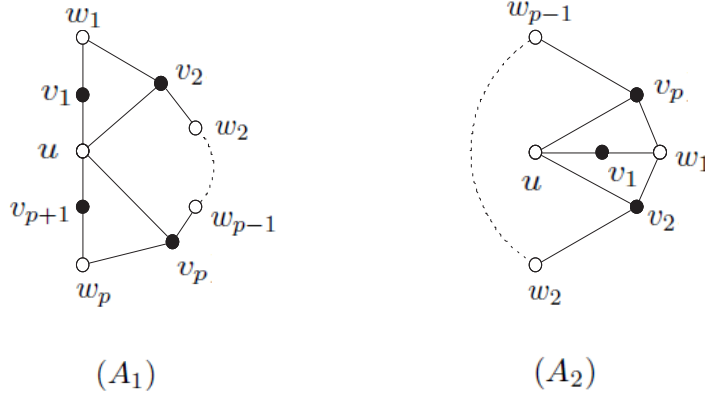


Figure 1

**Lemma 6** [20]. *For any integer  $k$  satisfying  $2 \leq k \leq \lfloor \frac{\Delta(G)}{2} \rfloor$ , let  $X_k = \{x \in V(G) : d_G(x) \leq k\}$  and  $Y_k = \bigcup_{x \in X_k} N_G(x)$ . If  $X_k \neq \emptyset$ , then there exists a bipartite subgraph  $M_k$  of  $G$  with partite sets  $X_k$  and  $Y_k$  such that  $d_{M_k}(x) = 1$  for every  $x \in X_k$  and  $d_{M_k}(y) \leq k - 1$  for every  $y \in Y_k$ .*

In Lemma 6 we call  $y$  the  $k$ -master of  $x$  if  $xy \in M_k$  and  $x \in X_k$ , and we call  $x$  the  $k$ -dependent of  $y$ .

We introduce some coloring notations. In a proper partial edge (respectively, total) coloring  $\phi$ , let  $A_\phi(x)$  denote the set of colors which are still available for coloring the element  $x \in E(G)$  (respectively,  $x \in V(G) \cup E(G)$ ) under the partial coloring  $\phi$ .

In the following, we will introduce two important theorems and show how to use them to discuss structure properties of minimal counterexamples.

## 2.1. Structure properties of LEC-minimal counterexamples

### 2.1.1. One lemma deduced by a theorem of Alon and Tarsi

Our approach is similar to the one used by Borodin, Kostochka and Woodall in [7] except that we rely on the following theorem proved by Alon and Tarsi [1]. This intricate theorem reveals the connection between the list coloring of a graph  $G$  and its orientations.

A *digraph* (directed graph)  $D$  is an ordered pair  $(V(D), A(D))$  consisting of the vertex set  $V(D)$  and arc set  $A(D)$ . For any arc  $a = \langle u, v \rangle$ , we say that  $u$  is the *tail* of  $a$  and  $v$  its *head*. The *indegree*  $d_D^-(v)$  of a vertex  $v$  in  $D$  is the number of arcs with head  $v$ , and the *outdegree*  $d_D^+(v)$  of  $v$  is the number of arcs with tail  $v$ . A directed cycle is denoted by a cyclic sequence  $u_1 u_2 \cdots u_k u_1$  in which each vertex dominates its successor. A subdigraph  $H$  of a directed graph  $D$  is called *Eulerian* if the indegree  $d_H^-(v)$  of every vertex  $v$  of  $H$  is equal to its outdegree  $d_H^+(v)$ . Note that we do not assume that  $H$  is connected.  $H$  is *even* if it has an even number of edges, otherwise it is *odd*. Let  $EE(D)$  and  $EO(D)$  denote the numbers of even and odd Eulerian subgraphs of  $D$ , respectively. (For convenience we assume that the empty subgraph is an even Eulerian subgraph.)

**Theorem 7** [1]. *Let  $D$  be a digraph. For each vertex  $v \in V(D)$ , let  $f(v)$  be a set of  $d_D^+(v) + 1$  distinct integers, where  $d_D^+(v)$  is the outdegree of  $v$ . If  $EE(D) \neq EO(D)$ , then there is a proper coloring  $c : V(D) \rightarrow \mathbb{Z}$  such that  $c(v)$  is in  $f(v)$  for every vertex  $v$ . That is, if  $L$  is a list assignment such that  $|L(v)| = d_D^+(v) + 1$  for all vertices  $v$  in  $D$ , then  $D$  is  $L$ -choosable.*

Based on the theorem, we construct orientations for some special graphs, which is useful in the discussion of structural properties.

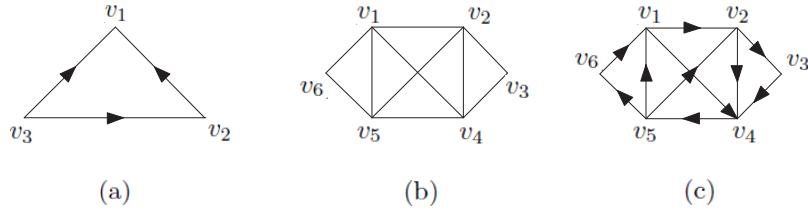


Figure 2. Graphs and their orientations.

**Lemma 8.** *For a graph  $G$ , let  $L$  be a list assignment of  $V(G)$ . If  $G$  is*

- (1) *a 3-cycle  $C = [v_1 v_2 v_3]$  in Figure 2(a) with  $|L(v_1)| \geq 1$ ,  $|L(v_2)| \geq 2$  and  $|L(v_3)| \geq 3$ , or*
- (2) *the graph in Figure 2(b) with  $|L(v_1)| \geq 3$ ,  $|L(v_2)| \geq 3$ ,  $|L(v_3)| \geq 2$ ,  $|L(v_4)| \geq 2$ ,  $|L(v_5)| \geq 4$  and  $|L(v_6)| \geq 2$ , or*

- (3) the graph in Figure 3(a) with  $|L(v_1)| \geq 3$ ,  $|L(v_2)| \geq 2$ ,  $|L(v_3)| \geq 2$ ,  $|L(v_4)| \geq 3$  and  $|L(v_5)| \geq 3$ , then  $G$  is  $L$ -choosable.

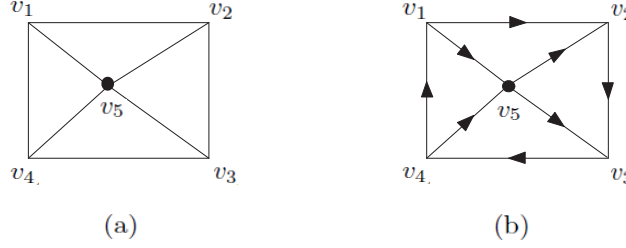


Figure 3. A graph and its orientation.

**Proof.** In the following, we will give an orientation of  $G$ ; in each case, the resulting directed graph is denoted by  $D$ .

(1) We give an orientation of the 3-cycle  $C = [v_1v_2v_3]$  in Figure 2(a). It is easy to check that there is no odd Eulerian subgraph and only one even Eulerian subgraph  $\emptyset$ . Since  $L(v) = d_D^+(v) + 1$  for every vertex  $v$  in  $D$ , by Theorem 7, there exists a proper coloring  $\phi$  of  $G$  such that  $\phi(v) \in L(v)$  for each vertex  $v \in V(G)$ .

(2) An orientation of  $G$  is given in Figure 2(c). In  $D$ , we can check that there are five even Eulerian subgraphs:  $\emptyset$ ,  $C_4^1 = [v_1v_4v_5v_6]$ ,  $C_4^2 = [v_1v_2v_4v_5]$ ,  $C_4^3 = [v_2v_3v_4v_5]$ ,  $C_6^1 = [v_1v_2v_3v_4v_5v_6]$  and four odd Eulerian subgraphs:  $C_3^1 = [v_1v_4v_5]$ ,  $C_3^2 = [v_2v_4v_5]$ ,  $C_5^1 = [v_1v_2v_4v_5v_6]$ ,  $C_5^2 = [v_1v_2v_3v_4v_5]$ . Since  $d_D^+(v_1) = d_D^+(v_2) = 2$ ,  $d_D^+(v_3) = d_D^+(v_4) = d_D^+(v_6) = 1$  and  $d_D^+(v_5) = 3$ , by Theorem 7, there exists a proper coloring  $\phi$  of  $G$  such that  $\phi(v) \in L(v)$  for each vertex  $v \in V(G)$ .

(3) An orientation of  $G$  is given in Figure 3(b). In  $D$ , we can check that there are four even Eulerian subgraphs:  $\emptyset$ ,  $C_4^1 = [v_1v_5v_3v_4]$ ,  $C_4^2 = [v_1v_2v_3v_4]$ ,  $C_4^3 = [v_2v_3v_4v_5]$  and two odd Eulerian subgraphs:  $C_3^1 = [v_3v_4v_5]$ ,  $C_5^1 = [v_1v_5v_2v_3v_4]$ . Since  $d_D^+(v_1) = d_D^+(v_4) = d_D^+(v_5) = 2$  and  $d_D^+(v_2) = d_D^+(v_3) = 1$ , by Theorem 7, there exists a proper coloring  $\phi$  of  $G$  such that  $\phi(v) \in L(v)$  for each vertex  $v \in V(G)$ . ■

### 2.1.2. Structure properties of LEC-minimal counterexamples

**Lemma 9.** Let  $G$  be an LEC-minimal counterexample with  $\Delta(G) \geq 7$ . Then  $G$  contains none of the following configurations (see Figure 4).

- (1)  $(B_1)$  a  $(4, 4, 5^-)$ -cycle.
- (2)  $(B_2)$  consisting of two  $(4, 5, 5)$ -cycles with a common 5-vertex.
- (3)  $(B_3)$  consisting of two  $(4, 5, 5)$ -cycles with a common  $(5, 5)$ -edge.

**Proof.** (1) We claim that  $G$  contains no edge  $uv$  with  $d_G(u) + d_G(v) \leq \Delta(G) + 1$ . Otherwise, by the minimality of  $G$ , there exists a proper edge coloring  $\phi$  of

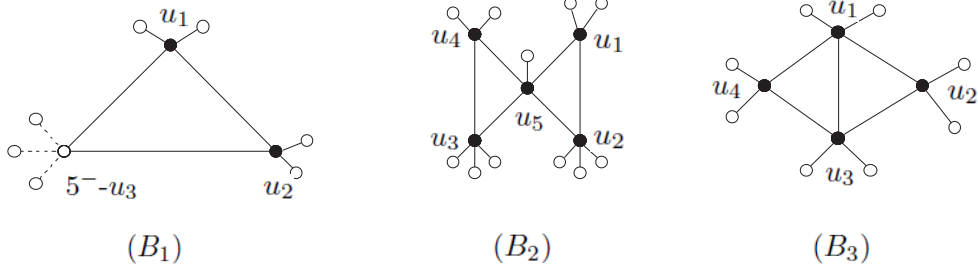


Figure 4. In  $(B_1)$ , the degree of  $u_3$  is at most 5. Note that a vertex  $v$  is black if  $v$  has no other neighbors than the ones already depicted, and a vertex  $v$  is white if it might have more neighbors than the ones shown in the figures.

$G' = G - uv$  for the edge list assignment  $L$ . At the same time,  $d_{G'}(u) + d_{G'}(v) \leq \Delta(G) - 1$ . So there exists a color in  $L(uv) \setminus \{\phi(e) | e \in (E_G(u) \cup E_G(v)) \setminus \{uv\}\}$  to color  $uv$ , where  $E_G(u)$  (respectively,  $E_G(v)$ ) denotes the set of edges incident with  $u$  (respectively  $v$ ) in  $G$ . Hence,  $\phi$  can be extended to a proper edge coloring of  $G$  and  $G$  is edge  $\Delta(G)$ -choosable, a contradiction.

(2) Suppose, to the contrary, that there exists the structure  $(B_2)$  presented in Figure 4. Let  $a = u_1u_2$ ,  $b = u_2u_5$ ,  $c = u_3u_5$ ,  $d = u_3u_4$ ,  $e = u_4u_5$  and  $f = u_5u_1$ ,  $G' = G - \{a, b, c, d, e, f\}$  and let  $H$  be the line graph of the subgraph of  $G$  induced by the edges  $a, b, c, d, e, f$ . Then  $H$  is isomorphic to the graph in Figure 2(b). By the minimality of  $G$ , there exists a proper edge coloring  $\phi$  of  $G'$  for the edge list assignment  $L$ . Since  $\min\{|A_\phi(a)|, |A_\phi(d)|\} \geq \Delta(G) - (2+3) \geq 2$ ,  $\min\{|A_\phi(b)|, |A_\phi(c)|\} \geq \Delta(G) - (1+3) \geq 3$  and  $\min\{|A_\phi(e)|, |A_\phi(f)|\} \geq \Delta(G) - (2+1) \geq 4$ . Then by Lemma 8(2),  $\phi$  can be extended to a proper edge coloring of  $G$  and  $G$  is edge  $\Delta(G)$ -choosable, a contradiction.

(3) Suppose, to the contrary, that there exists the structure  $(B_3)$  presented in Figure 4. Let  $a = u_1u_2$ ,  $b = u_2u_3$ ,  $c = u_3u_4$ ,  $d = u_4u_1$ ,  $e = u_1u_3$ ,  $G' = G - \{a, b, c, d, e\}$  and  $H$  be the line graph of the subgraph of  $G$  induced by the edges  $a, b, c, d, e$ . Then  $H$  is isomorphic to the graph in Figure 3(a). By the minimality of  $G$ , there exists a proper edge coloring  $\phi$  of  $G'$  for the edge list assignment  $L$ . Since  $\min\{|A_\phi(a)|, |A_\phi(b)|, |A_\phi(c)|, |A_\phi(d)|, |A_\phi(e)|\} \geq \Delta(G) - (2+2) \geq 3$ . Then by Lemma 8(3),  $\phi$  can be extended to a proper edge coloring of  $G$  and  $G$  is edge  $\Delta(G)$ -choosable, a contradiction. ■

## 2.2. Structure properties of LTC-minimal counterexamples

In this section, we will construct special polynomials according to the definition of total colorings, then we can deduce whether the corresponding graph is total  $(\Delta(G) + 1)$ -choosable by Combinatorial Nullstellensatz. The process above aims at solving the problem of forbidden configurations of LTC-minimal counterexam-



ples. Firstly, we introduce Combinatorial Nullstellensatz.

### 2.2.1. Combinatorial Nullstellensatz

**Lemma 10** [2] (The Combinatorial Nullstellensatz). *Let  $F$  be an arbitrary field, and let  $P = P(x_1, \dots, x_n)$  be a polynomial in  $F[x_1, \dots, x_n]$ . Suppose the degree  $\deg(P)$  of  $P$  equals  $\sum_{i=1}^n k_i$ , where each  $k_i$  is a non-negative integer, and suppose the coefficient of  $\prod_{i=1}^n x_i^{k_i}$  in  $P$  is non-zero. Then if  $S_1, \dots, S_n$  are subsets of  $F$  with  $|S_i| > k_i$ , there are  $s_1 \in S_1, \dots, s_n \in S_n$  so that  $P(s_1, \dots, s_n) \neq 0$ .*

From the assumptions of Combinatorial Nullstellensatz, we can see that some special coefficients should be calculated. Since the constructed polynomials are always complicated, we use partial derivatives and MATLAB to determine the coefficients of certain monomials. In fact, if  $P(x_1, \dots, x_m)$  is a polynomial with  $\deg(P) = n$ ,  $k_1, \dots, k_m$  are nonnegative integers with  $\sum_{i=1}^m k_i = n$  and  $c_P \left( x_1^{k_1} x_2^{k_2} \dots x_m^{k_m} \right)$  is the coefficient of the monomial  $\prod_{i=1}^m x_i^{k_i}$  in  $P$ , then  $\frac{\partial^n P}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_m^{k_m}} = c_P \left( x_1^{k_1} x_2^{k_2} \dots x_m^{k_m} \right) \prod_{i=1}^m k_i!$ .

### 2.2.2. Structure properties of LTC-minimal counterexamples

**Lemma 11.** *Let  $G$  be an LTC-minimal counterexample. Then  $G$  contains no configurations  $(B_1)$ ,  $(B_2)$  and  $(B_3)$  (see Figure 4).*

**Proof.** (1) To the contrary, suppose that there exists the structure  $(B_1)$  in  $G$ . Let  $G' = G \setminus \{u_1u_2, u_2u_3, u_3u_1\}$ . Then by the minimality of  $G$ , there exists a total coloring of  $G'$  for the list assignment  $L$ . In the total coloring above, erase the colors of  $u_1, u_2, u_3$  and denote this partial total coloring by  $\phi$ . Now, we use colors  $x_1, x_2, \dots, x_6$  to color  $u_1, u_2, u_3, u_1u_2, u_2u_3, u_3u_1$ , respectively. Then we have

$$P(x_1, x_2, \dots, x_6) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_1 - x_6)(x_2 - x_3)(x_2 - x_4) \\ (x_2 - x_5)(x_3 - x_5)(x_3 - x_6)(x_4 - x_5)(x_4 - x_6)(x_5 - x_6).$$

In  $P$ , the coefficient of monomial  $x_1^3 x_2^3 x_3 x_4 x_5^2 x_6^2$  is  $c_P \left( x_1^3 x_2^3 x_3 x_4 x_5^2 x_6^2 \right) = -2 \neq 0$  by MATLAB. Furthermore,  $\min \{ |A_\phi(u_1)|, |A_\phi(u_2)|, |A_\phi(u_1u_2)| \} \geq \Delta(G) + 1 - 4 \geq 8 - 4 = 4$ ,  $|A_\phi(u_3)| \geq \Delta(G) + 1 - 6 \geq 8 - 6 = 2$ ,  $\min \{ |A_\phi(u_2u_3)|, |A_\phi(u_1u_3)| \} \geq \Delta(G) + 1 - (2+3) \geq 8 - 5 = 3$ . By Lemma 10, we can find  $s_i \in A_\phi(u_i)$  for  $i = 1, 2, 3$ ,  $s_4 \in A_\phi(u_1u_2)$ ,  $s_5 \in A_\phi(u_2u_3)$  and  $s_6 \in A_\phi(u_3u_1)$  such that  $P(s_1, s_2, \dots, s_6) \neq 0$ . So  $\phi$  can be extended to a total coloring of  $G$  and  $G$  is total  $L$ -choosable, a contradiction.

(2) To the contrary, suppose that there exists the structure  $(B_2)$  in  $G$ . Let  $G' = G \setminus \{u_1u_5, u_1u_2, u_2u_5, u_3u_5, u_3u_4, u_4u_5\}$ . Then by the minimality of  $G$ , there exists a total coloring of  $G'$  for the list assignment  $L$ . In the total coloring above,

erase the colors of  $u_1, u_2, u_3, u_4, u_5$  and denote this partial total coloring by  $\phi$ . Now, we use colors  $x_1, x_2, \dots, x_{11}$  to color  $u_1, u_2, u_3, u_4, u_5, u_1u_5, u_1u_2, u_2u_5, u_3u_5, u_3u_4, u_4u_5$ , respectively. Then we have

$$\begin{aligned} P(x_1, x_2, \dots, x_{11}) = & (x_1 - x_2)(x_1 - x_5)(x_1 - x_6)(x_1 - x_7)(x_2 - x_5) \\ & (x_2 - x_7)(x_2 - x_8)(x_3 - x_5)(x_3 - x_4)(x_3 - x_9)(x_3 - x_{10}) \\ & (x_4 - x_5)(x_4 - x_{10})(x_4 - x_{11})(x_5 - x_6)(x_5 - x_8)(x_5 - x_9) \\ & (x_5 - x_{11})(x_6 - x_7)(x_6 - x_8)(x_6 - x_9)(x_6 - x_{11})(x_7 - x_8) \\ & (x_8 - x_9)(x_8 - x_{11})(x_9 - x_{10})(x_9 - x_{11})(x_{10} - x_{11}). \end{aligned}$$

In  $P$ , we get that  $c_P(x_1^3x_4^3x_5^5x_6^4x_7^2x_8^2x_9^3x_{10}^2x_{11}^4) = -2 \neq 0$  by MATLAB. Let  $(w_1, w_2, \dots, w_{11}) = (u_1, u_2, u_3, u_4, u_5, u_1u_5, u_1u_2, u_2u_5, u_3u_5, u_3u_4, u_4u_5)$ ,  $a_i \leq |A_\phi(w_i)|$  and  $w_i$  is corresponding to  $x_i$ ,  $i = 1, 2, \dots, 11$ . Then we show  $a_1, a_2, \dots, a_{11}$  in Table 1.

$w_i$	$w_1$	$w_2$	$w_3$	$w_4$	$w_5$	$w_6$	$w_7$	$w_8$	$w_9$	$w_{10}$	$w_{11}$
$a_i$	4	2	2	4	6	5	3	4	4	3	5

Table 1

By Lemma 10, we can find  $s_i \in A_\phi(w_i)$  for  $i = 1, 2, \dots, 11$  such that  $P(s_1, s_2, \dots, s_{11}) \neq 0$ . So  $\phi$  can be extended to a total coloring of  $G$  and  $G$  is total  $L$ -choosable, a contradiction.

(3) To the contrary, suppose that there exists the structure  $(B_3)$  in  $G$ . Let  $G' = G \setminus \{u_1u_2, u_2u_3, u_3u_4, u_4u_1, u_1u_3\}$ . Then by minimality of  $G$ , there exists a total coloring of  $G'$  for the list assignment  $L$ . In the total coloring above, erase the colors of  $u_1, u_2, u_3, u_4$  and denote this partial total coloring by  $\phi$ . Now, we use colors  $x_1, x_2, \dots, x_9$  to color  $u_1, u_2, u_3, u_4, u_1u_2, u_2u_3, u_3u_4, u_4u_1, u_1u_3$ , respectively. Then we have

$$\begin{aligned} P(x_1, x_2, \dots, x_9) = & (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_1 - x_5)(x_1 - x_9) \\ & (x_1 - x_8)(x_2 - x_3)(x_2 - x_5)(x_2 - x_6)(x_3 - x_4)(x_3 - x_6) \\ & (x_3 - x_9)(x_3 - x_7)(x_4 - x_7)(x_4 - x_8)(x_5 - x_9)(x_5 - x_6) \\ & (x_5 - x_8)(x_6 - x_7)(x_6 - x_9)(x_7 - x_8)(x_7 - x_9)(x_8 - x_9). \end{aligned}$$

In  $P$ , we get that  $c_P(x_1^3x_2^3x_3^3x_4^3x_5^3x_6^3x_7^2x_8^2) = -2 \neq 0$  by MATLAB. Let  $(w_1, w_2, \dots, w_9) = (u_1, u_2, u_3, u_4, u_1u_2, u_2u_3, u_3u_4, u_4u_1, u_1u_3)$ ,  $a_i \leq |A_\phi(w_i)|$  and  $w_i$  is corresponding to  $x_i$ ,  $i = 1, 2, \dots, 9$ . Then we show  $a_1, a_2, \dots, a_9$  in Table 2.

By Lemma 10, we can find  $s_i \in A_\phi(w_i)$  for  $i = 1, 2, \dots, 9$  such that  $P(s_1, s_2, \dots, s_9) \neq 0$ . So  $\phi$  can be extended to a total coloring of  $G$  and  $G$  is total  $L$ -choosable, a contradiction.  $\blacksquare$

$w_1$	$w_1$	$w_2$	$w_3$	$w_4$	$w_5$	$w_6$	$w_7$	$w_8$	$w_9$
$a_i$	4	4	4	4	4	4	4	4	4

Table 2

## 3. PROOF OF THEOREM 2

Let  $G = (V, E)$  be a minimal-counterexample to the statement of Theorem 2 which is stated in Section 1. Let  $f_k(y)$  (respectively,  $f_{k^+}(y)$ ,  $f_{k^-}(y)$ ) denote the number of  $k$ -faces (respectively,  $k^+$ -faces,  $k^-$ -faces) incident with  $y$  for any element  $y \in V(G) \cup E(G)$ . For any face  $f \in F(G)$ ,  $n_k(f)$ ,  $n_{k^-}(f)$  and  $n_{k^+}(f)$  denote the number of the  $k$ -vertices,  $k^-$ -vertices and  $k^+$ -vertices incident with the face  $f$ , respectively. Since  $G$  contains no general 5-cycles, we have the following observation.

**O1.** For any vertex  $v \in V(G)$ ,  $f_3(v) \leq \left\lfloor \frac{2d(v)}{3} \right\rfloor$ .

By Lemma 3, we can get the following result immediately.

**O2.** For any face  $f \in F(G)$ , we have

- (1)  $n_{3^-}(f) \leq \left\lfloor \frac{d(f)}{2} \right\rfloor$ .
- (2) If  $n_{3^-}(f) \geq 1$  and  $2n_{3^-}(f) + 1 \leq d(f) - 1$ , then  $n_4(f) \leq d(f) - (2n_{3^-}(f) + 1)$ .

By Lemmas 3 and 6, it is easy to obtain the following.

**O3.** For any  $\Delta(G)$ -vertex  $v$  in  $V(G)$ ,  $v$  has at most one 2-dependent, and for any 2-vertex  $u$ ,  $u$  has a 2-master.

For any  $\Delta(G)$ -vertex  $v$  in  $V(G)$ , except its 2-dependent, other 2-neighbors of  $v$  are called 2-non-dependents of  $v$ . Since  $G$  contains no chordal 5-cycle, we have the following results.

**O4.** (1) There is no 3-face adjacent to a 4-face.

- (2) There are no consecutive adjacent three faces  $f_1, f_2$  and  $f_3$  with  $d_G(f_1) = d_G(f_2) = 3$  and  $d_G(f_3) \leq 4$ ,  $f_i$  is incident with  $vv_i$  and  $vv_{i+1}$  for  $i = 1, 2, 3$ .

In [19], Theorem 2 was proved for  $\Delta(G) \geq 8$ . Hence, to prove Theorem 2, we just consider the case  $\Delta(G) = 7$ . In the following, we will use discharging method to show the proof. By Euler's formula, we have that  $\sum_{y \in V(G) \cup F(G)} (d(y) - 4) = -8$ . Let  $c(y) = d(y) - 4$  for each  $y \in V(G) \cup F(G)$ . Then  $\sum_{y \in V(G) \cup F(G)} c(y) = -8 < 0$ . We will give discharging rules to redistribute the charges and check the

final charge  $c'(y) \geq 0$  for each  $y \in V(G) \cup F(G)$ . Then a contradiction arises and the proof is completed.

We use  $\tau(y_1 \rightarrow y_2)$  to denote the charge moved from  $y_1$  to  $y_2$ , for  $y_1, y_2 \in V(G) \cup F(G)$ . Suppose that  $f = [v_1v_2v_3]$  is a 3-face with  $d_G(v_1) \leq d_G(v_2) \leq d_G(v_3)$ . Let  $(d_G(v_1), d_G(v_2), d_G(v_3)) \rightarrow (c_1, c_2, c_3)$  denote that the vertex  $v_i$  gives  $f$  the charge  $c_i$  for  $i = 1, 2, 3$ .

**R1.** Let  $f \in F(G)$  and  $v$  be its incident vertex. Then  $\tau(f \rightarrow v)$  equals

- R1.1.**  $\frac{1}{2}$  if  $d(f) = 6$  and  $d(v) \leq 3$ .
- R1.2.**  $\frac{1}{4}$  if  $d(f) = 6$  and  $d(v) = 4$ .
- R1.3.**  $\frac{1}{4}$  if  $d(f) = 6$ ,  $d(v) \geq 5$  and  $f$  is incident with a  $(6^+, 6^+)$ -edge.
- R1.4.**  $\frac{1}{6}$  if  $d(f) = 6$  and  $d(v) \geq 5$  and  $f$  is not incident with a  $(6^+, 6^+)$ -edge.
- R1.5.**  $\frac{2}{3}$  if  $d(f) \geq 7$  and  $d(v) = 2$ .
- R1.6.**  $\frac{1}{2}$  if  $d(f) \geq 7$  and  $d(v) = 3$ .
- R1.7.**  $\frac{1}{4}$  if  $d(f) \geq 7$  and  $d(v) \geq 4$ .

**R2.** Let  $f = [v_1v_2v_3]$  be a 3-face with  $d_G(v_1) \leq d_G(v_2) \leq d_G(v_3)$ .

- R2.1.**  $(3^-, 6^+, 6^+) \rightarrow (0, \frac{1}{2}, \frac{1}{2})$ .
- R2.2.**  $(4, 4, 6^+) \rightarrow (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ .
- R2.3.**  $(4, 5, 5) \rightarrow (0, \frac{1}{2}, \frac{1}{2})$ .
- R2.4.**  $(4, 5, 6^+) \rightarrow (\frac{1}{4}, \frac{1}{3}, \frac{1}{2})$ .
- R2.5.**  $(4, 6^+, 6^+) \rightarrow (0, \frac{1}{2}, \frac{1}{2})$ .
- R2.6.**  $(5^+, 5^+, 5^+) \rightarrow (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ .

**R3.** Let  $v$  be a 3-vertex with  $f_{4^-}(v) = 2$  in  $G$  and  $uv \in E(G)$ . Then  $\tau(u \rightarrow v)$  is

- R3.1.**  $\frac{1}{4}$  if  $f_{4^-}(uv) = 2$ .
- R3.2.**  $\frac{1}{8}$  if  $f_{4^-}(uv) = 1$ .

**R4.** Let  $v$  be a 2-vertex in  $G$  and  $uv \in E(G)$ . If  $v$  is a 2-dependent of  $u$ , then  $\tau(u \rightarrow v)$  equals

- R4.1.**  $\frac{5}{3}$  if  $f_3(v) = f_4(v) = 1$ .
- R4.2.**  $\frac{5}{3}$  if  $f_4(v) = 2$ .
- R4.3.**  $\frac{7}{6}$  if  $f_3(v) = 1$  and  $f_{7^+}(v) = 1$ .
- R4.4.**  $\frac{4}{3}$  if  $f_4(v) = 1$  and  $f_{6^+}(v) = 1$ .
- R4.5.** 1 if  $f_{6^+}(v) = 2$ .

If  $v$  is a 2-non-dependent of  $u$ , then  $\tau(u \rightarrow v)$  equals

- R4.6.**  $\frac{1}{3}$  if  $f_3(v) = f_4(v) = 1$ .  
**R4.7.**  $\frac{1}{3}$  if  $f_4(v) = 2$ .  
**R4.8.**  $\frac{1}{6}$  if  $f_3(v) = 1$  and  $f_{7^+}(v) = 1$ .  
**R4.9.**  $\frac{1}{6}$  if  $f_4(v) = 1$  and  $f_{6^+}(v) = 1$ .

According to the discharging rules for 2-vertices, we divide 2-vertices into five types. Let  $v$  be a 2-dependent of  $u$  in  $G$ .  $v$  is said to be a *A-2-dependent* of  $u$  if  $f_3(v) = f_4(v) = 1$ , *B-2-dependent* of  $u$  if  $f_4(v) = 2$ , *C-2-dependent* of  $u$  if  $f_3(v) = 1$ ,  $f_{7^+}(v) = 1$ , *D-2-dependent* of  $u$  if  $f_4(v) = 1$  and  $f_{6^+}(v) = 1$  and *E-2-dependent* of  $u$  if  $f_{6^+}(v) = 2$ . Similarly, we can define *A-*, *B-*, *C-*, *D-*, *E-2-non-dependents* of  $u$ . At the same time, for a 3-vertex  $v$  in  $G$  and  $uv \in E(G)$ ,  $v$  is called a *3-TB-vertex* of  $u$  if  $f_3(uv) = 2$  and *3-QB-vertex* of  $u$  if  $f_4(uv) = 2$ .

Firstly, we discuss the final charge of each face  $f \in F(G)$ . Let  $f = [v_1v_2v_3]$  be a 3-face with  $d_G(v_1) \leq d_G(v_2) \leq d_G(v_3)$ . If  $d_G(v_1) \leq 3$ , then by Lemma 3,  $d_G(v_1), d_G(v_2) \geq 6$ , and it follows that  $c'(f) \geq 3 - 4 + 2 \times \frac{1}{2} = 0$  by R2.1. If  $d_G(v_1) = 4$ , then by  $(B_1)$ ,  $f$  is a  $(4, 4, 6^+)$ -face or a  $(4, 5, 5)$ -face or a  $(4, 5, 6^+)$ -face or a  $(4, 6^+, 6^+)$ -face, and it follows that  $c'(f) \geq 3 - 4 + \min\{2 \times \frac{1}{4} + \frac{1}{2}, 2 \times \frac{1}{2}, \frac{1}{4} + \frac{1}{3} + \frac{1}{2}\} = 0$  by R2.2–R2.5. If  $d_G(v_1) \geq 5$ , then  $c'(f) \geq 3 - 4 + 3 \times \frac{1}{3} = 0$  by R2.6 and Lemma 3. Let  $f$  be a 4-face in  $G$ . Then  $c'(f) = c(f) = 0$ . There is no 5-face in  $G$ .

Let  $f$  be a 6-face in  $G$ . Suppose that  $f$  is not incident with any  $(6^+, 6^+)$ -edge. If  $n_{3^-}(f) \geq 1$  and  $2n_{3^-}(f) + 1 \leq 5$ , then by  $O2(2)$ , R1.1, R1.2 and R1.4,  $c'(f) \geq d(f) - 4 - [\frac{1}{2}n_{3^-}(f) + \frac{1}{4}n_4(f) + \frac{1}{6}(d(f) - n_{3^-}(f) - n_4(f))] = d(f) - 4 - (\frac{1}{3}n_{3^-}(f) + \frac{1}{12}n_4(f) + \frac{1}{6}d(f)) \geq d(f) - 4 - (\frac{1}{3}n_{3^-}(f) + \frac{d(f)}{12} - \frac{1}{6}n_{3^-}(f) - \frac{1}{12} + \frac{1}{6}d(f)) = \frac{3}{4}d(f) - \frac{1}{6}n_{3^-}(f) - \frac{47}{12} \geq \frac{1}{4} > 0$ , where  $n_{3^-}(f) \leq 2$ . If  $n_{3^-}(f) \geq 1$  and  $2n_{3^-}(f) = 6$ , then by Lemma 3,  $n_4(f) = 0$ . Hence, by R1.1 and R1.4,  $c'(f) \geq 6 - 4 - 3 \times \frac{1}{2} - 3 \times \frac{1}{6} = 0$ . If  $n_{3^-}(f) = 0$ , then by R1.2 and R1.4,  $c'(f) \geq 6 - 4 - (\frac{1}{4}n_4(f) + \frac{1}{6}n_{5^+}(f)) \geq \frac{1}{2}$ . Suppose that  $f$  is incident with a  $(6^+, 6^+)$ -edge. By Lemma 3 and R1.1 and R1.3,  $n_{3^-}(f) \leq 2$  and it follows that  $c'(f) \geq 6 - 4 - 6 \times \frac{1}{4} = \frac{1}{2}$  if  $n_{3^-}(f) = 0$ ,  $c'(f) \geq 6 - 4 - \frac{1}{2} - 5 \times \frac{1}{4} = \frac{1}{4}$  if  $n_{3^-}(f) = 1$ , and  $c'(f) \geq 6 - 4 - 2 \times \frac{1}{2} - 4 \times \frac{1}{4} = 0$  if  $n_{3^-}(f) = 2$ .

Let  $f$  be a  $7^+$ -face in  $G$  and  $d(f) = k$ . By Lemma 3 and R1.5–R1.7,  $c'(f) \geq k - 4 - [\frac{2}{3}n_2(f) + \frac{1}{2}n_3(f) + \frac{1}{4}(k - n_2(f) - n_3(f))] = \frac{3}{4}k - 4 - \frac{5}{12}(n_2(f) + n_3(f)) + \frac{1}{6}n_3(f) \geq \frac{3}{4}k - 4 - \frac{5}{12}(n_2(f) + n_3(f))$ . By  $O2(1)$ ,  $n_2(f) + n_3(f) \leq \lfloor \frac{k}{2} \rfloor$ . Hence, by R1.5–R1.7, we have  $c'(f) \geq \frac{3}{4}(2t + 1) - 4 - \frac{5}{12}t \geq 0$  if  $k = 2t + 1$  and  $t \geq 3$ , and  $c'(f) \geq \frac{3}{4}(2t) - 4 - \frac{5}{12}t \geq \frac{1}{3}$  if  $k = 2t$  and  $t \geq 4$ .

Now, we will check that the final charge  $c'(v) \geq 0$  for each vertex  $v \in V(G)$ . Let  $f^i$  be the face incident with  $vv_i, vv_{i+1}$ ,  $i = 1, 2, \dots, d(v)$  with  $v_{d(v)+1} = v_1$ . Let  $v$  be a 2-vertex in  $G$ . Then by Lemma 6,  $v$  has a 2-master. If  $f_{4^-}(v) = 2$ , then  $c'(v) \geq 2 - 4 + \frac{5}{3} + \frac{1}{3} = 0$  by R4.1, R4.2, R4.6 and R4.7. If  $f_3(v) = 1$  and  $f_{5^+}(v) = 1$ , without loss of generality, assume that  $f_1$  is a 3-face, then since

$G$  contains no 5-cycle,  $f_2$  is a  $7^+$ -face. Hence,  $c'(v) \geq 2 - 4 + \frac{7}{6} + \frac{1}{6} + \frac{2}{3} = 0$  by R1.5, R4.3 and R4.8. If  $f_4(v) = 1$ , then  $f_{6^+}(v) = 1$  and it follows that  $c'(v) \geq 2 - 4 + \frac{4}{3} + \frac{1}{6} + \frac{1}{2} = 0$  by R1.1, R4.4 and R4.9. If  $f_{6^+}(v) = 2$ , then  $c'(v) \geq 2 - 4 + 1 + 2 \times \frac{1}{2} = 0$  by R1.1 and R4.5.

Let  $v$  be a 3-vertex in  $G$ . Since  $G$  contains no  $1^-$ -cycle-subdivisions of  $W_3$ , then  $f_{4^-}(v) \leq 2$ . If  $f_{4^-}(v) = 2$ , then  $c'(v) \geq 3 - 4 + \frac{1}{4} + 2 \times \frac{1}{8} + \frac{1}{2} = 0$  by R1.1, R1.6, R2.1 and R3. If  $f_{4^-}(v) \leq 1$ , then  $c'(v) \geq 3 - 4 + 2 \times \frac{1}{2} = 0$  by R1.1, R1.6 and R2.1. Let  $v$  be a 4-vertex in  $G$ , then by O1,  $f_3(v) \leq 2$  and it follows that  $f_{6^+}(v) \geq 2$ . Hence,  $c'(v) \geq 4 - 4 - 2 \times \frac{1}{4} + 2 \times \frac{1}{4} = 0$  by R1.2, R1.3, R1.7 and R2.2–R2.5.

Let  $v$  be a 5-vertex in  $G$ . By O1,  $f_3(v) \leq 3$  and it follows that  $f_{6^+}(v) \geq 2$ . If  $f_3(v) \leq 2$ , then  $c'(v) \geq 5 - 4 - 2 \times \frac{1}{2} = 0$  by R2.3–R2.6. If  $f_3(v) = 3$ , assume that  $v$  is incident with two adjacent 3-faces with the common edge  $uv$ . If  $d(u) = 5$ , since  $G$  contains no  $(B_2)$  and  $(B_3)$ , we have  $c'(v) \geq 5 - 4 - \frac{1}{2} - 2 \times \frac{1}{3} + \frac{1}{6} = 0$  by R1.4, R2.3 and R2.4. If  $d(u) = 4$ , since  $G$  contains no  $(B_2)$ , we have  $c'(v) \geq 5 - 4 - 2 \times \frac{1}{2} - \frac{1}{3} + 2 \times \frac{1}{6} = 0$  by R1.4, R2.3 and R2.4. If  $d(u) \geq 6$ , then  $v$  is incident with at most one  $(4, 5, 5)$ -face, and it follows that  $c'(v) \geq 5 - 4 - 2 \times \frac{1}{3} - \frac{1}{2} + \frac{1}{6} = 0$  by R1.4, R2.3 and R2.4. Let  $v$  be a 6-vertex in  $G$ . By O1,  $f_3(v) \leq 4$  and it follows that  $f_{6^+}(v) \geq 2$ . If  $f_3(v) \leq 3$ , then  $c'(v) \geq 6 - 4 - 3 \times \frac{1}{2} - \max\{\frac{1}{4} + \frac{1}{8}, 3 \times \frac{1}{8}\} = \frac{1}{8}$  by R2 and R3. If  $f_3(v) = 4$ , then by O4,  $f_{6^+}(v) = 2$ . Assume that  $f^1, f^2, f^4, f^5$  are four 3-faces, then by O4 (2),  $f_{4^-}(v_1) = f_{4^-}(v_3) = f_{4^-}(v_4) = f_{4^-}(v_6) = 1$ , and it follows that  $c'(v) \geq 6 - 4 - 4 \times \frac{1}{2} - 2 \times \frac{1}{4} + 2 \times \frac{1}{4} = 0$  by Lemma 3, R1.3, R1.4, R2 and R3.

Let  $v$  be a 7-vertex in  $G$ . By O1,  $f_3(v) \leq 4$ . Let  $n_{2A}, n_{2B}, n_{2C}, n_{2D}, n_{2E}, n_{3TB}$  and  $n_{3QB}$  denote the number of  $A$ -,  $B$ -,  $C$ -,  $D$ -,  $E$ -2-non-dependents, 3-TB-vertices and 3-QB-vertices of  $v$  in  $G$ , respectively. We will divide this problem into several cases in terms of the number of different kinds of 2-dependents and 2-non-dependents.

(1)  $v$  has no 2-dependents in  $G$ . Then  $f_3(v) \leq 4$  by O1.

(1.1) If  $f_3(v) = 4$ , then  $v$  has no  $A$ -,  $B$ -,  $D$ -2-non-dependents and at most two  $C$ -2-non-dependents by O4. If  $v$  has two  $C$ -2-non-dependents, then  $n_{3TB} = 0$  by Lemma 4 and it follows that  $c'(v) \geq 7 - 4 - 2 \times \frac{1}{6} - 4 \times \frac{1}{2} = \frac{2}{3}$  by R2.1 and R4.8. If  $v$  has one  $C$ -2-non-dependent, then  $n_{3TB} \leq 1$  and it follows that  $c'(v) \geq 7 - 4 - \frac{1}{4} - \frac{1}{6} - 4 \times \frac{1}{2} = \frac{7}{12}$  for  $n_{3TB} = 1$  by R2.1, R3.1 and R4.8;  $c'(v) \geq 7 - 4 - \frac{1}{6} - 4 \times \frac{1}{2} = \frac{5}{6}$  for  $n_{3TB} = 0$  by R2.1 and R4.8. If  $n_{2C} = 0$ , then  $c'(v) \geq 7 - 4 - 2 \times \frac{1}{4} - 4 \times \frac{1}{2} = \frac{1}{2}$  by R2.1 and R3.1.

(1.2) If  $f_3(v) = 3$ , then  $v$  has at most one  $A$ -2-non-dependent by O4 and no  $B$ -2-non-dependent. If  $v$  has one  $A$ -2-non-dependent, then  $c'(v) \geq 7 - 4 - \frac{1}{3} - \frac{1}{4} - 3 \times \frac{1}{2} = \frac{17}{12}$  by R2.1, R3.1 and R4.6. If  $v$  has no  $A$ -2-non-dependent, then  $v$  has at most one  $D$ -2-non-dependent by O4. If  $n_{2A} = 0$  and  $n_{2D} = 1$ , then  $n_{2C} \leq 2$  by O4. Hence, by R2.1, R3.1, R3.2, R4.8 and R4.9, we have

$c'(v) \geq 7-4-3 \times \frac{1}{6} - \frac{1}{8} - 3 \times \frac{1}{2} = \frac{7}{8}$  if  $n_{2C} = 2$ ;  $c'(v) \geq 7-4-2 \times \frac{1}{6} - \frac{1}{4} - \frac{1}{8} - 3 \times \frac{1}{2} = \frac{19}{24}$  if  $n_{2C} = 1$ ;  $c'(v) \geq 7-4-\frac{1}{6}-\frac{1}{4}-\frac{1}{8}-\frac{1}{8}-3 \times \frac{1}{2} = \frac{5}{12}$  if  $n_{2C} = 0$ . If  $n_{2A} = 0$  and  $n_{2D} = 0$ , then  $n_{2C} \leq 3$  by *O4*. Hence, by R2.1, R3.1, R3.2 and R4.8, we have  $c'(v) \geq 7-4-3 \times \frac{1}{6} - 3 \times \frac{1}{2} = 1$  if  $n_{2C} = 3$ ;  $c'(v) \geq 7-4-2 \times \frac{1}{6} - 2 \times \frac{1}{8} - 3 \times \frac{1}{2} = \frac{11}{12}$  if  $n_{2C} = 2$ ;  $c'(v) \geq 7-4-\frac{1}{6}-\frac{1}{4}-2 \times \frac{1}{8} - 3 \times \frac{1}{2} = \frac{19}{24}$  if  $n_{2C} = 1$ ;  $c'(v) \geq 7-4-\frac{1}{4}-3 \times \frac{1}{8} - 3 \times \frac{1}{2} = \frac{7}{8}$  if  $n_{2C} = 0$ .

(1.3) If  $f_3(v) \leq 2$ , then  $c'(v) \geq 7-4-\max\{5 \times \frac{1}{3} + 2 \times \frac{1}{2}, 6 \times \frac{1}{3} + \frac{1}{2}, 7 \times \frac{1}{3}\} = \frac{1}{3}$  by R2.1 and R4.

(2)  $v$  has one  $E$ -2-dependent in  $G$ . Then  $f_3(v) \leq 4$  by *O1*.

(2.1) If  $f_3(v) = 4$ , then  $v$  has no  $A$ -,  $B$ - and  $D$ -2-non-dependent and  $n_{2C} \leq 2$  by Lemma 4 and  $f_{6^+}(v) = 3$  by *O4*. Then by R2.1, R1.3–R1.7, R3.1, R4.5 and R4.8, we have  $c'(v) \geq 7-4-1-2 \times \frac{1}{6} - 4 \times \frac{1}{2} + 3 \times \frac{1}{6} = \frac{1}{6}$  if  $n_{2C} = 2$ ;  $c'(v) \geq 7-4-1-\frac{1}{6}-\frac{1}{4}-4 \times \frac{1}{2} + 3 \times \frac{1}{6} = \frac{1}{12}$  if  $n_{2C} = 1$ ;  $c'(v) \geq 7-4-1-2 \times \frac{1}{4} - 4 \times \frac{1}{2} + 3 \times \frac{1}{6} = 0$  if  $n_{2C} = 0$ .

(2.2) If  $f_3(v) = 3$ , then  $n_{2A}(v) \leq 1$  and  $n_{2B}(v) = n_{2D}(v) = 0$  by Lemma 4 and *O4*. If  $n_{2A} = 1$ , then  $n_{2C} \leq 1$  by Lemma 4. Hence, by R2.1, R1.3–R1.7, R3.1, R4.5, R4.6 and R4.8, we have  $c'(v) \geq 7-4-1-\frac{1}{3}-\frac{1}{6}-3 \times \frac{1}{2} = 0$  if  $n_{2C} = 1$ ;  $c'(v) \geq 7-4-1-\frac{1}{3}-\frac{1}{4}-3 \times \frac{1}{2} + \frac{1}{6} = \frac{1}{12}$  if  $n_{2C} = 0$ . If  $n_{2A} = 0$ , then  $n_{2C} \leq 3$  by Lemma 4. Hence,  $c'(v) \geq 7-4-1-3 \times \frac{1}{6} - 3 \times \frac{1}{2} = 0$  by R2.1, R4.5 and R4.8.

(2.3) If  $f_3(v) = 2$ , then  $n_{2A} \leq 2$ . If  $n_{2A} = 2$ , then  $n_{2B} = n_{2C} = n_{2D} = 0$  by Lemma 4 and it follows that  $c'(v) \geq 7-4-1-2 \times \frac{1}{3} - 2 \times \frac{1}{2} = \frac{1}{3}$  by R2.1, R4.5 and R4.6. If  $n_{2A} = 1$ , then  $n_{2B} = n_{2D} = 0$  and  $n_{2C} \leq 1$  by Lemma 4. Hence,  $c'(v) \geq 7-4-1-\frac{1}{3}-\frac{1}{6}-2 \times \frac{1}{2} = \frac{1}{2}$  if  $n_{2C} = 1$  by R2.1, R4.5, R4.6 and R4.8;  $c'(v) \geq 7-4-1-\frac{1}{3}-\frac{1}{8}-2 \times \frac{1}{2} = \frac{7}{8}$  if  $n_{2C} = 0$  by R2.1, R3.2, R4.5 and R4.6. If  $n_{2A}(v) = 0$ , then  $n_{2B}(v) \leq 1$  by *O4*. If  $n_{2B}(v) = 1$ , then  $c'(v) \geq 7-4-1-\frac{1}{3}-\frac{1}{4}-2 \times \frac{1}{8} - 2 \times \frac{1}{2} = \frac{1}{6}$  by R2.1, R3.1, R3.2, R4.5 and R4.7. If  $n_{2B}(v) = 0$ , then  $n_{3QB} \leq 1$  by *O4*. By Lemma 5, Lemma 4, R2.1, R3.1, R3.2, R4.5, R4.8 and R4.9, we have  $c'(v) \geq 7-4-1-\frac{1}{6}-\frac{1}{4}-\frac{1}{4}-\frac{1}{8}-2 \times \frac{1}{2} = \frac{5}{24}$  for  $n_{3QB} = 1$  and now  $n_{2D} \leq 1$ ;  $c'(v) \geq 7-4-1-3 \times \frac{1}{6} - \frac{1}{8} - 2 \times \frac{1}{2} = \frac{3}{8}$  for  $n_{3QB} = 0$  and equality holds if  $n_{2D} = 1$  and  $n_{2C} = 2$ .

(2.4) If  $f_3(v) \leq 1$ , then  $n_{3QB} \leq 4$  by Lemma 5 and Lemma 4. Since  $G$  contains no  $(A_1)$  and  $(A_2)$ , we discuss this problem in two cases. If  $f_3(v) = 1$ , then  $c'(v) \geq 7-4-1-\frac{1}{3}-\frac{1}{6}-\frac{1}{4}-2 \times \frac{1}{8} - \frac{1}{2} = \frac{1}{2}$  by R2.1, R3.1, R3.2, R4.5 and R4.7–R4.9. If  $f_3(v) = 0$ , then  $c'(v) \geq 7-4-1-\frac{1}{3}-3 \times \frac{1}{4} - 2 \times \frac{1}{8} = \frac{2}{3}$  by R3.1, R3.2, R4.5 and R4.7.

(3)  $v$  has one  $D$ -2-dependent in  $G$ . Then  $n_{2A} \leq 1$  by *O4*.

(3.1) Suppose  $n_{2A} = 1$ . Then by *O4*,  $1 \leq f_3(v) \leq 2$ . If  $f_3(v) = 2$ , then  $c'(v) \geq 7-4-\frac{4}{3}-\frac{1}{3}-2 \times \frac{1}{8} - 2 \times \frac{1}{2} = \frac{1}{12}$  by R2.1, R3.2, R4.4 and R4.6. If  $f_3(v) = 1$ , then  $n_{3QB} \leq 2$  and it follows that  $c'(v) \geq 7-4-\frac{4}{3}-\frac{1}{3}-2 \times \frac{1}{4} - \frac{1}{8} - \frac{1}{2} = \frac{5}{24}$  by R2.1, R3.1, R3.2, R4.4 and R4.6.

(3.2) Suppose  $n_{2A} = 0$ . Then by *O4*,  $f_3(v) \leq 3$ . In the following, we will use R1.3–R1.7, R2.1, R3.1, R3.2, R4.4 and R4.8. If  $f_3(v) = 3$ , then  $n_{2C} \leq 2$  by Lemma 4 and *O4*. Hence,  $c'(v) \geq 7 - 4 - \frac{4}{3} - 2 \times \frac{1}{6} - \frac{1}{8} - 3 \times \frac{1}{2} + 3 \times \frac{1}{6} = \frac{5}{24}$  for  $n_{2C} = 2$ ;  $c'(v) \geq 7 - 4 - \frac{4}{3} - \frac{1}{6} - \frac{1}{4} - \frac{1}{8} - 3 \times \frac{1}{2} + 3 \times \frac{1}{6} = \frac{1}{8}$  for  $n_{2C} = 1$ ;  $c'(v) \geq 7 - 4 - \frac{4}{3} - \frac{1}{4} - 2 \times \frac{1}{8} - 3 \times \frac{1}{2} + 3 \times \frac{1}{6} = \frac{1}{6}$  for  $n_{2C} = 0$ . If  $f_3(v) = 2$ , then  $n_{2C} \leq 2$  by Lemma 4 and *O4*. Hence,  $c'(v) \geq 7 - 4 - \frac{4}{3} - 2 \times \frac{1}{6} - \frac{1}{4} - \frac{1}{8} - 2 \times \frac{1}{2} + \frac{1}{6} = \frac{1}{8}$  for  $n_{2C} = 2$ ;  $c'(v) \geq 7 - 4 - \frac{4}{3} - \frac{1}{6} - 2 \times \frac{1}{4} - \frac{1}{8} - 2 \times \frac{1}{2} + \frac{1}{6} = \frac{1}{24}$  for  $n_{2C} = 1$ ;  $c'(v) \geq 7 - 4 - \frac{4}{3} - 3 \times \frac{1}{4} - \frac{1}{8} - 2 \times \frac{1}{2} + 2 \times \frac{1}{6} = \frac{1}{8}$  for  $n_{2C} = 0$ . If  $f_3(v) = 1$ , then  $n_{2C} \leq 1$  and  $c'(v) \geq 7 - 4 - \frac{4}{3} - \frac{1}{6} - 3 \times \frac{1}{4} - \frac{1}{8} - \frac{1}{2} = \frac{1}{8}$  for  $n_{2C} = 1$ ;  $c'(v) \geq 7 - 4 - \frac{4}{3} - 3 \times \frac{1}{4} - 2 \times \frac{1}{8} - \frac{1}{2} = \frac{1}{6}$  for  $n_{2C} = 0$ . If  $f_3(v) = 0$ , then by Lemma 5,  $c'(v) \geq 7 - 4 - \frac{4}{3} - 5 \times \frac{1}{4} - \frac{1}{8} = \frac{7}{24}$ .

(4)  $v$  has one  $C$ -2-dependent in  $G$ . Then  $n_{2A} \leq 1$  by *O4*.

(4.1) Suppose that  $n_{2A} = 1$ . Then by *O4*,  $2 \leq f_3(v) \leq 3$ . If  $f_3(v) = 3$ , then  $n_{2C} \leq 1$  and  $n_{2B} = n_{2D} = 0$  by *O4*. Hence,  $c'(v) \geq 7 - 4 - \frac{7}{6} - \frac{1}{3} - \frac{1}{6} - 3 \times \frac{1}{2} + \frac{1}{6} = 0$  for  $n_{2C} = 1$  by R1.3–R1.7, R2.1, R4.3, R4.6 and R4.8;  $c'(v) \geq 7 - 4 - \frac{7}{6} - \frac{1}{3} - \frac{1}{8} - 3 \times \frac{1}{2} + \frac{1}{6} = \frac{1}{24}$  for  $n_{2C} = 0$  by R1.3–R1.7, R2.1, R3.2, R4.3 and R4.6. If  $f_3(v) = 2$ , then  $c'(v) \geq 7 - 4 - \frac{7}{6} - \frac{1}{3} - 2 \times \frac{1}{8} - 2 \times \frac{1}{2} = \frac{1}{4}$  by R2.1, R3.1, R3.2, R4.3 and R4.6.

(4.2) Suppose that  $n_{2A} = 0$ . Then by *O4*,  $1 \leq f_3(v) \leq 4$ .

(4.2.1) If  $f_3(v) = 4$ , then  $n_{2B} = n_{2D} = 0$ ,  $n_{2C} \leq 2$  and  $n_{3TB} \leq 1$  by *O4*. If  $f_3(v) = 4$ ,  $n_{2C} = 2$ , then  $n_{3TB} = 0$  by Lemma 3 and it follows that  $c'(v) \geq 7 - 4 - \frac{7}{6} - 2 \times \frac{1}{6} - 4 \times \frac{1}{2} + 3 \times \frac{1}{6} = 0$  by R1.3–R1.7, R2.1, R4.3 and R4.8. If  $f_3(v) = 4$ ,  $n_{2C} = 1$  and  $n_{3TB} = 1$ , then  $v$  is incident with at least one  $7^+$ -face by the division of 2-neighbors of  $v$ . Hence,  $c'(v) \geq 7 - 4 - \frac{7}{6} - \frac{1}{6} - \frac{1}{4} - 4 \times \frac{1}{2} + 2 \times \frac{1}{6} + \frac{1}{4} = 0$  by R1.3–R1.7, R2.1, R3.1, R4.3 and R4.8. If  $f_3(v) = 4$ ,  $n_{2C} = 1$  and  $n_{3TB} = 0$ , then similarly we can check that  $c'(v) \geq 0$ . If  $f_3(v) = 4$ ,  $n_{2C} = 0$ , then  $c'(v) \geq 7 - 4 - \frac{7}{6} - \frac{1}{4} - \frac{1}{8} - 4 \times \frac{1}{2} + 2 \times \frac{1}{6} + \frac{1}{4} = \frac{1}{24}$  by Lemma 3, R1.3–R1.7, R2.1, R3.1, R3.2 and R4.3.

(4.2.2) If  $f_3(v) = 3$ , then  $n_{2C} \leq 2$  and  $n_{2B} = 0$  by *O4*. If  $n_{2C} = 2$ , then  $n_{2D} = 0$  by *O4* and it follows that  $c'(v) \geq 7 - 4 - \frac{7}{6} - 2 \times \frac{1}{6} - 3 \times \frac{1}{2} = 0$  by R2.1, R4.3 and R4.8. If  $n_{2C} = 1$ , then  $n_{2D} \leq 1$  by *O4* and it follows that  $c'(v) \geq 7 - 4 - \frac{7}{6} - 2 \times \frac{1}{6} - \frac{1}{8} - 3 \times \frac{1}{2} + \frac{1}{6} = \frac{1}{24}$  for  $n_{2D} = 1$  by R1.3–R1.7, R2.1, R4.3, R4.8 and R4.9;  $c'(v) \geq 7 - 4 - \frac{7}{6} - \frac{1}{6} - 2 \times \frac{1}{8} - 3 \times \frac{1}{2} + \frac{1}{6} = \frac{1}{12}$  for  $n_{2D} = 0$  by R1.3–R1.7, R2.1, R3.2, R4.3 and R4.8. If  $n_{2C} = 0$ , then  $n_{2D} \leq 1$  by *O4* and it follows that  $c'(v) \geq 7 - 4 - \frac{7}{6} - \frac{1}{6} - \frac{1}{4} - \frac{1}{8} - 3 \times \frac{1}{2} + 2 \times \frac{1}{6} = 0$  for  $n_{2D} = 1$  by R1.3–R1.7, R2.1, R3.1, R3.2, R4.3 and R4.9;  $c'(v) \geq 7 - 4 - \frac{7}{6} - \frac{1}{4} - 2 \times \frac{1}{8} - 3 \times \frac{1}{2} + \frac{1}{6} = 0$  for  $n_{2D} = 0$  by R1.3–R1.7, R2.1, R3.1, R3.2 and R4.3.

(4.2.3) If  $f_3(v) = 2$ , then  $n_{2C} \leq 1$  by Lemma 4. If  $f_3(v) = 2$  and  $n_{2C} = 1$ , then  $n_{2B} \leq 1$  by *O4*. If  $n_{2B} = 1$ , then  $c'(v) \geq 7 - 4 - \frac{7}{6} - \frac{1}{6} - \frac{1}{3} - 2 \times \frac{1}{8} - 2 \times \frac{1}{2} = \frac{1}{12}$  by R2.1, R3.2, R4.3 and R4.7–R4.8. If  $n_{2B} = 0$ , then  $c'(v) \geq 7 - 4 - \frac{7}{6} - \frac{1}{6} - \frac{1}{4} - 2 \times \frac{1}{8} - 2 \times \frac{1}{2} = \frac{1}{6}$  by R2.1, R3.1, R3.2, R4.3 and R4.8. If  $f_3(v) = 2$  and  $n_{2C} = 0$ , then  $n_{2B} \leq 1$



by  $O4$ . At the same time, by Lemma 5, R2.1, R3.1, R3.2, R4.3 and R4.7, we have  $c'(v) \geq 7 - 4 - \frac{7}{6} - \frac{1}{3} - \frac{1}{4} - 2 \times \frac{1}{8} - 2 \times \frac{1}{2} = 0$ .

(4.2.4) If  $f_3(v) = 1$ , then  $n_{2A} = n_{2C} = 0$  and  $n_{2B} \leq 2$  by Lemma 5 and Lemma 4. Hence,  $c'(v) \geq 7 - 4 - \frac{7}{6} - \frac{1}{3} - 2 \times \frac{1}{4} - 2 \times \frac{1}{8} - \frac{1}{2} = \frac{1}{4}$  by R2.1, R3.1, R3.2, R4.3 and R4.7.

(5)  $v$  has one  $B$ -2-dependent in  $G$ . Then  $n_{2A} \leq 1$  by  $O4$ .

(5.1) If  $n_{2A} = 1$ , then by  $O4$ ,  $f_3(v) = 1$ ,  $n_{2B} = 0$  and  $n_{3QB} \leq 1$ . Hence,  $c'(v) \geq 7 - 4 - \frac{5}{3} - \frac{1}{3} - \frac{1}{4} - 2 \times \frac{1}{8} - \frac{1}{2} = 0$  by Lemma 5, R2.1, R3.1, R3.2, R4.2 and R4.6.

(5.2) If  $n_{2A} = 0$ , then by  $O4$ ,  $f_3(v) \leq 2$ .

(5.2.1) If  $f_3(v) = 2$ , then  $n_{2D} \leq 1$  and  $n_{2B} = 0$  by Lemma 4.

(5.2.1.1) If  $f_3(v) = 2$  and  $n_{2D} = 1$ , then  $n_{3TB} \leq 1$  and  $n_{3QB} = 0$  by Lemma 5. If  $n_{3TB} = 1$ , then  $v$  is incident with two  $6^+$ -faces each of which is incident with a  $(6^+, 6^+)$ -edge, and it follows that  $c'(v) \geq 7 - 4 - \frac{5}{3} - \frac{1}{6} - \frac{1}{4} - \frac{1}{8} - 2 \times \frac{1}{2} + 2 \times \frac{1}{4} = \frac{7}{24}$  by R1.3–R1.7, R2.1, R3.1, R3.2, R4.2 and R4.9. If  $n_{3TB} = 0$ , then  $n_{2C} \leq 1$  by  $O4$  and it follows that  $c'(v) \geq 7 - 4 - \frac{5}{3} - 2 \times \frac{1}{6} - \frac{1}{8} - 2 \times \frac{1}{2} + \frac{1}{6} = \frac{1}{24}$  by R1.3–R1.7, R2.1, R3.2, R4.2 and R4.8–R4.9.

(5.2.1.2) If  $f_3(v) = 2$  and  $n_{2D} = 0$ , then  $n_{3TB} \leq 1$  and  $n_{3QB} \leq 1$  by Lemma 5. If  $n_{3TB} = 1$ , then  $n_{2C} = n_{2B} = 0$  by Lemma 3,  $v$  is incident with two  $6^+$ -face each of which is incident with a  $(6^+, 6^+)$ -edge, and it follows that  $c'(v) \geq 7 - 4 - \frac{5}{3} - 2 \times \frac{1}{4} - 2 \times \frac{1}{8} - 2 \times \frac{1}{2} + 2 \times \frac{1}{4} = \frac{1}{12}$  by R1.3, R2.1, R3.1, R3.2 and R4.2. If  $n_{3TB} = 0$ , then  $n_{2C} \leq 2$  and it follows that  $c'(v) \geq 7 - 4 - \frac{5}{3} - \max\{\frac{1}{6} + \frac{1}{4} + 2 \times \frac{1}{8}, 2 \times \frac{1}{6} + 2 \times \frac{1}{8}\} - 2 \times \frac{1}{2} + 2 \times \frac{1}{6} = 0$  by R1.3–R1.7, R2.1, R3.1, R3.2, R4.2 and R4.8.

(5.2.2) If  $f_3(v) = 1$ , then  $n_{2C} \leq 1$ . If  $n_{2C} = 1$ , then  $n_{2B} \leq 1$  by Lemma 5 and  $O4$ . If  $n_{2B} = 1$ , then  $c'(v) \geq 7 - 4 - \frac{5}{3} - \frac{1}{6} - \frac{1}{3} - 2 \times \frac{1}{8} - \frac{1}{2} = \frac{1}{12}$  by R2.1, R3.2, R4.2 and R4.7–R4.8. If  $n_{2B} = 0$ , then  $n_{3QB} \leq 2$  and  $f_{6^+}(v) \geq 1$  by  $O4$ . Hence,  $c'(v) \geq 7 - 4 - \frac{5}{3} - \frac{1}{6} - 2 \times \frac{1}{4} - 2 \times \frac{1}{8} - \frac{1}{2} + \frac{1}{6} = \frac{1}{12}$  by R1.3–R1.7, R2.1, R3.1, R3.2, R4.2 and R4.8.

If  $n_{2C} = 0$ , then  $n_{3QB} \leq 2$  and  $f_{6^+}(v) \geq 1$  by  $O4$ . If  $n_{3QB} = 2$ , then by Lemma 5 and Lemma 4,  $n_{2B} = n_{2D} = 0$  and it follows that  $c'(v) \geq 7 - 4 - \frac{5}{3} - 2 \times \frac{1}{4} - 3 \times \frac{1}{8} - \frac{1}{2} + \frac{1}{6} = \frac{1}{8}$  by R1.3–R1.7, R2.1, R3.1, R3.2 and R4.2. If  $n_{3QB} = 1$ , then by Lemma 5 and Lemma 4,  $n_{2B} = 0$  and  $n_{2D} \leq 1$  and it follows that  $c'(v) \geq 7 - 4 - \frac{5}{3} - \frac{1}{6} - \frac{1}{4} - 2 \times \frac{1}{8} - \frac{1}{2} = \frac{1}{6}$  by R2.1, R3.1, R3.2, R4.2 and R4.9. If  $n_{3QB} = 0$ , then by Lemma 5 and Lemma 4,  $n_{2B} \leq 1$ . If  $n_{2B} = 1$ , then by Lemma 5 and Lemma 4,  $n_{2D} = 0$  and it follows that  $c'(v) \geq 7 - 4 - \frac{5}{3} - \frac{1}{3} - 3 \times \frac{1}{8} - \frac{1}{2} = \frac{1}{8}$  by R2.1, R3.2, R4.2 and R4.7. If  $n_{2B} = 0$ , then by Lemma 4,  $n_{2D} \leq 1$  and it follows that  $c'(v) \geq 7 - 4 - \frac{5}{3} - \frac{1}{6} - 4 \times \frac{1}{8} - \frac{1}{2} = \frac{1}{6}$  by R2.1, R3.1, R4.2 and R4.9.

(5.2.3) If  $f_3(v) = 0$ , then  $n_{3QB} \leq 5$  by Lemma 5. Hence,  $c'(v) \geq 7 - 4 - \frac{5}{3} - 5 \times \frac{1}{4} = \frac{1}{12}$  by R3.1 and R4.2.

(6)  $v$  has one  $A$ -2-dependent in  $G$ , then  $n_{2A} \leq 1$  by  $O4$ .

(6.1) If  $n_{2A} = 1$ , then  $f_3(v) = 2$  by *O4*. Hence,  $c'(v) \geq 7 - 4 - \frac{5}{3} - \frac{1}{3} - 2 \times \frac{1}{2} = 0$  by R2.1, R4.1 and R4.6.

(6.2) If  $n_{2A} = 0$ , then  $1 \leq f_3(v) \leq 3$  by *O4*.

(6.2.1) If  $f_3(v) = 3$ , then  $f_{6^+}(v) = 3$ ,  $n_{2B} = n_{2D} = 0$  and  $n_{3TB} \leq 1$  by *O4*. If  $n_{3TB} = 1$ , then  $c'(v) \geq 7 - 4 - \frac{5}{3} - \frac{1}{4} - 3 \times \frac{1}{2} + 3 \times \frac{1}{6} = \frac{1}{12}$  by R1.3–R1.7, R2.1, R3.1 and R4.1. If  $n_{3TB} = 0$ , then  $n_{2C} \leq 2$  by *O4* and it follows that  $c'(v) \geq 7 - 4 - \frac{5}{3} - 2 \times \frac{1}{6} - 3 \times \frac{1}{2} + 3 \times \frac{1}{6} = 0$  by R1.3–R1.7, R2.1, R4.1 and R4.8.

(6.2.2) If  $f_3(v) = 2$ , then  $f_{6^+}(v) \geq 3$ ,  $n_{2C} \leq 1$  and  $n_{2B} = 0$  by *O4*. If  $n_{2C} = 1$ , then  $n_{2D} \leq 1$  by *O4*. Hence, by R1.3–R1.7, R2.1, R3.2, R4.1 and R4.8–R4.9, we have  $c'(v) \geq 7 - 4 - \frac{5}{3} - 2 \times \frac{1}{6} - \frac{1}{8} - 2 \times \frac{1}{2} + \frac{1}{6} = \frac{1}{24}$  for  $n_{2D} = 1$ ;  $c'(v) \geq 7 - 4 - \frac{5}{3} - \frac{1}{6} - 2 \times \frac{1}{8} - 2 \times \frac{1}{2} + \frac{1}{6} = \frac{1}{12}$  for  $n_{2D} = 0$ . If  $n_{2C} = 0$ , then  $n_{2D} \leq 1$  by *O4*. Hence, by R1.3–R1.7, R2.1, R3.2, R4.1 and R4.9, we have  $c'(v) \geq 7 - 4 - \frac{5}{3} - \frac{1}{6} - 2 \times \frac{1}{8} - 2 \times \frac{1}{2} + \frac{1}{6} = \frac{1}{12}$  for  $n_{2D} = 1$ ;  $c'(v) \geq 7 - 4 - \frac{5}{3} - 3 \times \frac{1}{8} - 2 \times \frac{1}{2} + \frac{1}{6} = \frac{1}{8}$  for  $n_{2D} = 0$ .

(6.2.3) If  $f_3(v) = 1$ , then by Lemma 4 and *O4*,  $f_{6^+}(v) \geq 2$ ,  $n_{2C} = 0$  and  $n_{2B} \leq 1$ . If  $n_{2B} = 1$ , then  $c'(v) \geq 7 - 4 - \frac{5}{3} - \frac{1}{3} - \frac{1}{4} - 2 \times \frac{1}{8} - \frac{1}{2} = 0$  by R2.1, R3.1, R3.2, R4.1 and R4.7. If  $n_{2B} = 0$ , then  $n_{2D} \leq 2$ . Hence,  $c'(v) \geq 7 - 4 - \frac{5}{3} - 2 \times \frac{1}{6} - \frac{1}{4} - \frac{1}{2} = \frac{1}{4}$  for  $n_{2D} = 2$  by Lemma 5, R2.1, R3.1, R4.1 and R4.9;  $c'(v) \geq 7 - 4 - \frac{5}{3} - \frac{1}{6} - 2 \times \frac{1}{4} - \frac{1}{8} - \frac{1}{2} = \frac{1}{24}$  for  $n_{2D} = 1$  by R2.1, R3.1, R3.2, R4.1 and R4.9;  $c'(v) \geq 7 - 4 - \frac{5}{3} - 2 \times \frac{1}{4} - 2 \times \frac{1}{8} - \frac{1}{2} = \frac{1}{12}$  for  $n_{2D} = 0$  by R2.1, R3.1, R3.2 and R4.1.

Now, the proof of Theorem 2 is completed.

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## APPENDIX

```

%Input
syms x1 x2 x3 x4 x5 x6 x7 x8 x9 x10 x11 x12
%Lemma 11 (1)
Q1=(x1-x2)*(x1-x3)*(x1-x4)*(x1-x6)*(x2-x3)*(x2-x4)*(x2-x5)*(x3-x5)*(x3-x6)*(x4-x5)
*(x4-x6)*(x5-x6);
c1=diff(diff(diff(diff(diff(diff(Q1,x1,3),x2,3),x3,1),x4,1),x5,2),x6,2) /factorial(3)/factorial(3)
/factorial(1)/factorial(1)/factorial(2)/factorial(2)
%Lemma 11 (2)
Q2=(x1-x2)*(x1-x5)*(x1-x6)*(x1-x7)*(x2-x5)*(x2-x7)*(x2-x8)*(x3-x5)*(x3-x4)*(x3-x9)
*(x3-x10)*(x4-x5)*(x4-x10)*(x4-x11)*(x5-x6)*(x5-x8)*(x5-x9)*(x5-x11)*(x6-x7)*(x6-x8)
*(x6-x9)*(x6-x11)*(x7-x8)*(x8-x9)*(x8-x11)*(x9-x10)*(x9-x11)*(x10-x11);
c2=diff(diff(diff(diff(diff(diff(diff(diff(diff(Q1,x1,3),x2,0),x3,0), x4,3),x5,5),x6,4),x7,2),
x8,2),x9,3),x10,2),x11,4)/factorial(3)/factorial(0)/factorial(0) /factorial(3)/factorial(5)
/factorial(4)/factorial(2)/factorial(2)/factorial(3)/factorial(2)/factorial(4)
%Lemma 11 (3)
Q3=(x1-x2)*(x1-x3)*(x1-x4)*(x1-x5)*(x1-x9)*(x1-x8)*(x2-x3)*(x2-x5)*(x2-x6)*(x3-x4)
*(x3-x6)*(x3-x9)*(x3-x7)*(x4-x7)*(x4-x8)*(x5-x9)*(x5-x6)*(x5-x8)*(x6-x7)*(x6-x9)
*(x7-x8)*(x7-x9)*(x8-x9);
c3=diff(diff(diff(diff(diff(diff(diff(diff(Q1,x1,3),x2,3),x3,3),x4,3),x5,3), x6,3),x7,3),
x8,2),x9,0)/factorial(3)/factorial(3)/factorial(3)/factorial(3)/factorial(3)
/factorial(3)/factorial(3)/factorial(2)

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