Discussiones Mathematicae
Graph Theory xx (xxxx) 1–11
doi:10.7151/dmgt.2123

ON THE MINIMUM NUMBER OF SPANNING TREES IN CUBIC MULTIGRAPHS

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Abstract

Let G_{2n}, H_{2n} be two non-isomorphic connected cubic multigraphs of order $2n$ with parallel edges permitted but without loops. Let $t(G_{2n}), t(H_{2n})$ denote the number of spanning trees in G_{2n}, H_{2n}, respectively. We prove that for $n \geq 3$ there is the unique G_{2n} such that $t(G_{2n}) < t(H_{2n})$ for any H_{2n}. Furthermore, we prove that such a graph has $t(G_{2n}) = 5^2 2^{n-3}$ spanning trees. Based on our results we give a conjecture for the unique r-regular connected graph H_{2n} of order $2n$ and odd degree r that minimizes the number of spanning trees.

Keywords: cubic multigraph, spanning tree, regular graph, enumeration.

2010 Mathematics Subject Classification: 05C05, 05C38.

1. Introduction

There is an extensive literature devoted to identifying connected graphs G on $V(G)$ vertices and $E(G)$ edges and with either maximum or minimum number of spanning trees $t(G)$ when $|V(G)|$ and $|E(G)|$ are predetermined. Identifying such graphs allows establishing upper and lower bounds for the number of spanning trees in families of connected graphs when $|V(G)|$ and $|E(G)|$ are fixed. Most published papers focused on the maximum number of spanning trees cover just a few restricted families of graphs, e.g., [5, 6, 9]. Determining the graphs with the minimum number of spanning trees was much more successful. In particular, it was determined in [4] that specific threshold graph G minimizes the number of spanning trees over all connected simple graphs with the same number of vertices and edges. However, it was also determined that G was not unique. Based on that, it was subsequently proved in [2] that there is a well-defined class of connected simple graphs that minimize the number of spanning trees among the simple connected graphs on the same number of vertices and edges. Corresponding results for the maximum number of spanning trees in undirected simple graphs have yet to be found.

In addition to identifying the connected simple graphs with minimum number of spanning trees, there were also number of papers recently published devoted to the minimum number of spanning trees in the special families of connected graphs. Kostochka [7] identified the minimum number of spanning trees in a simple cubic graph with fixed number of vertices. In [1] we proved that there is a unique threshold graph that minimizes the number of spanning trees over all 2-connected chordal graphs, and in [3] we identified simple cubic connected graphs that minimize the number of spanning trees over other cubic graphs, on the same number of vertices. Most recently, Ok and Thomassen [8] determined a lower bound on the number of spanning trees in a k-edge-connected graph and identified the extremal k-edge-connected graph.

In this paper we consider all connected cubic graphs of given order $2n$ without loops, and prove/identify that there is the unique graph M_{2n} belonging to this family that minimizes the number of spanning trees. For convenience, throughout the rest of this paper by graph we mean either a multigraph without loops and with at least one pair of parallel edges, or a simple graph. Hence, if G_{2n} is a cubic graph, then either G_{2n} contains induced C_2 or it is simple.

2. Connected Cubic Multigraphs with Minimal Spanning Trees

Let M_3 be a multigraph constructed from a C_2 cycle on two vertices v_1, v_2 by joining a third vertex v_3 with two single edges to vertices v_1 and v_2. Let $M_{2n} = M_{2(3+k)}$, $n \geq 3$, be a connected cubic multigraph on $2n$ vertices that consists of two M_3 subgraphs and k C_2 cycles, all joined with one another by single edges—see Figure 1.

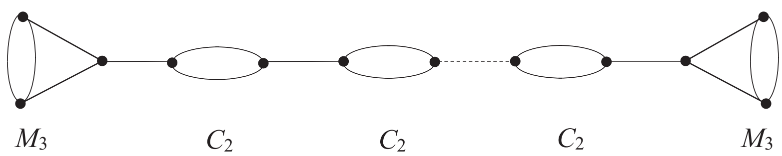

Figure 1. Graph M_{2n}.

For parallel edges e_1, e_2 we assume that two spanning trees containing e_1, e_2 respectively are distinct. In addition, if G is isomorphic to H, then we write

$G \simeq H$, otherwise we write $G \not\simeq H$. The proof of our main result in Theorem 3 is based on graph transformations derived from the following simple lemma.

Lemma 1. *Let $T(G)$ be a spanning tree of connected G that includes an edge e. Let H be a graph obtained from G by contracting e into a vertex. Then contracting e into a vertex in $T(G)$ produces a spanning tree $T(H)$. Furthermore, $t(H)$ equals the number of spanning trees in G that contain e.*

Proof. Clearly, contracting e into a vertex in $T(G)$ does not produce a cycle and results in a connected spanning subgraph of H, which is $T(H)$. Hence, to every unique spanning tree of H there corresponds a unique spanning tree of G that contains edge e. ■

We also need the following lemma.

Lemma 2. *Connected cubic graph G_6 minimizes $t\,(G_6)$ if and only if $G_6 \simeq M_6$.*

Proof. It's easy to verify that there are only six pairwise non-isomorphic connected cubic graphs on six vertices (Figure 2): (1) Möbius ladder H_6, (2) prism $P_6 \simeq C_2 \square C_3$, (3) multigraph $C_2 \times 1$ with one induced C_2 cycle, (4) multigraph $C_2 \times 2$ with two induced C_2 cycles, (5) multigraph $C_2 \times 3$ with three induced C_2 cycles, and (6) M_6.

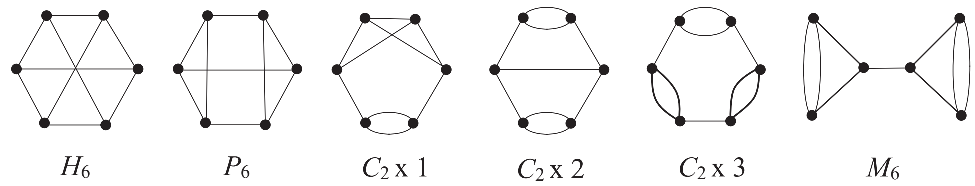

Figure 2. All distinct connected cubic graphs on six vertices.

Furthermore, it's trivial to verify based on a well-known Kirchhoff's matrix-tree theorem that $t(H_6) = 81 > t(P_6) = 75 > t(C_2 \times 1) = 56 > t(C_2 \times 2) = 45 > t(C_2 \times 3) = 36 > t(M_6) = 25$. ■

We can now state the main result as follows.

Theorem 3. *Connected cubic graph G_{2n} minimizes $t(G_{2n})$ for given $n \geq 3$ if and only if $G_{2n} \simeq M_{2n}$.*

Proof. For $n = 3$, according to Lemma 2, G_{2n} minimizes $t(G_{2n})$ if and only if $G_6 \simeq M_6$. Suppose there exists G_{2n} for $n \geq 4$ such that $t(G_{2n}) \leq t(M_{2n})$ and

$G_{2n} \ncong M_{2n}$. Without loss of generality, assume G_{2n} to be with minimum $n \geq 4$ that satisfies $t(G_{2n}) \leq t(M_{2n})$ and $G_{2n} \ncong M_{2n}$.

Suppose G_{2n} contains a simple cycle C_i on i vertices, where C_i is not included in any M_3 component of G_{2n}. If replacing in G_{2n} component X with component Y produces connected cubic graph H_{2m}, then we denote it by $G_{2n}(X \to Y) \to H_{2m}$. If i-th spanning tree in a graph G induces a spanning tree in a subgraph S of G, then such a spanning tree in G we denote by $T_i(G, S)$. Otherwise, i-th spanning tree in G we denote by $\widehat{T}_i(G, S)$. Then we have the following:

Claim 1. *G_{2n} does not contain C_2 that is not included in M_3.*

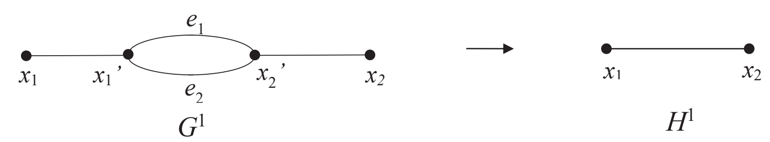

Figure 3. Transformation based on C_2 component.

Proof. If G_{2n} contains C_2 outside M_3 components, then there is a transformation illustrated in Figure 3. The subgraph G^1 does not have to be an induced subgraph of G (e.g., there might be an edge between x_1 and x_2 in G^1). If $x_1 = x_2$ in Figure 3, then C_2 belongs to M_3—a contradiction. So, transformation in Figure 3 does not produce a loop. Consequently, we can transform G_{2n} as follows $G_{2n}(G^1 \to H^1) \to H_{2n-2}$. Furthermore, for every spanning tree $T_i(H_{2n-2}, H^1)$ there are two unique spanning trees:

1. $T_{i_1}(G_{2n}, G^1)$ with edges $(x_1, x_1'), e_1, (x_2', x_2)$,
2. $T_{i_2}(G_{2n}, G^1)$ with edges $(x_1, x_1'), e_2, (x_2', x_2)$,

and for every spanning tree $\widehat{T}_i(H_{2n-2}, H^1)$ there are five unique spanning trees:

1. $\widehat{T}_{i_1}(G_{2n}, G^1)$ with edges $(x_1, x_1'), e_1$,
2. $\widehat{T}_{i_2}(G_{2n}, G^1)$ with edges $(x_1, x_1'), e_2$,
3. $\widehat{T}_{i_3}(G_{2n}, G^1)$ with edges $e_1, (x_2', x_2)$
4. $\widehat{T}_{i_4}(G_{2n}, G^1)$ with edges $e_2, (x_2', x_2)$,
5. $\widehat{T}_{i_5}(G_{2n}, G^1)$ with edges $(x_1, x_1'), (x_2', x_2)$.

If there is an edge between x_1 and x_2 in G^1, then $3t(H_{2n-2}) < t(G_{2n}) \leq t(M_{2n}) = 2t(M_{2n-2})$ implying $t(H_{2n-2}) < t(M_{2n-2})$—a contradiction. If there is no edge between x_1 and x_2 in G^1, then $2t(H_{2n-2}) \leq t(G_{2n}) \leq t(M_{2n}) = 2t(M_{2n-2})$ and $H_{2n-2} \ncong M_{2n-2}$—a contradiction. These contradictions prove Claim 1. □

Claim 2. *G_{2n} does not contain induced C_3.*

Proof. Suppose a subgraph of G_{2n} exists that includes induced cycle $C_3 = x_1'x_2'x_3'$. Let $(x_1, x_1'), (x_2, x_2'), (x_3, x_3')$ be the edges not in $E(C_3)$. There are two cases to consider based on the vertices x_1, x_2, x_3.

Case 1. x_1, x_2, x_3 are pairwise distinct. In this case there is a transformation $G_{2n}(C_3 \to x_1') \to H_{2n-2}$, which is a contraction of C_3 in G_{2n} into a vertex x_1'. Hence, for every spanning tree $T_i(H_{2n-2}, x_1') = T_i(H_{2n-2})$ there are three unique spanning trees:

1. $T_{i_1}(G_{2n}, C_3)$ with edges $(x_1', x_2'), (x_2', x_3')$,
2. $T_{i_2}(G_{2n}, C_3)$ with edges $(x_2', x_3'), (x_3', x_1')$,
3. $T_{i_3}(G_{2n}, C_3)$ with edges $(x_3', x_1'), (x_1', x_2')$.

Consequently, $3t(H_{2n-2}) \leq t(G_{2n}) \leq t(M_{2n}) = 2t(M_{2n-2})$, implying $t(H_{2n-2}) < t(M_{2n-2})$—a contradiction.

Case 2. x_1, x_2, x_3 are not pairwise distinct. If $x_1 = x_2 = x_3$, then $G_{2n} \simeq K_4$—a contradiction $(2n = 4 < 6)$. So, without loss of generality assume $x_1 \neq x_2 = x_3$. In this case there is a transformation illustrated in Figure 4.

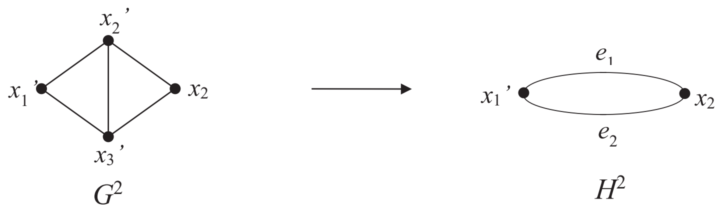

Figure 4. Transformation based on C_3 when $x_1 \neq x_2 = x_3$.

So, there is a transformation $G_{2n}(G^2 \to H^2) \to H_{2n-2}$. Clearly, $t(G^2) = 8$ and $t(H^2) = 2$. This means that there are four times more $T_i(G_{2n}, G^2)$ spanning trees than $T_i(H_{2n-2}, H^2)$ spanning trees. In addition, for every spanning tree $\widehat{T}_i(H_{2n-2}, H^2)$ there are eight unique spanning trees:

1. $\widehat{T}_{i_1}(G_{2n}, G^2)$ with edges $(x_1', x_2'), (x_1', x_3')$,
2. $\widehat{T}_{i_2}(G_{2n}, G^2)$ with edges $(x_1', x_2'), (x_2', x_3')$,
3. $\widehat{T}_{i_3}(G_{2n}, G^2)$ with edges $(x_2', x_3'), (x_1', x_3')$,
4. $\widehat{T}_{i_4}(G_{2n}, G^2)$ with edges $(x_2, x_2'), (x_2, x_3')$,
5. $\widehat{T}_{i_5}(G_{2n}, G^2)$ with edges $(x_2, x_2'), (x_2', x_3')$,
6. $\widehat{T}_{i_6}(G_{2n}, G^2)$ with edges $(x_2', x_3'), (x_2, x_3')$,
7. $\widehat{T}_{i_7}(G_{2n}, G^2)$ with edges $(x_1', x_2'), (x_2, x_3')$,
8. $\widehat{T}_{i_8}(G_{2n}, G^2)$ with edges $(x_2, x_2'), (x_1', x_3')$.

Hence, $2t(H_{2n-2}) < t(G_{2n}) \leq t(M_{2n}) = 2t(M_{2n-2})$, implying $t(H_{2n-2}) < t(M_{2n-2})$, a contradiction.

Consequently, contradictions of Cases 1–2 prove Claim 2. □

Claim 3. *G_{2n} does not contain induced C_4.*

Proof. Suppose that G_{2n} contains induced square C_4—Figure 5. In Figure 5 we allow $x_1 = x_3$ and $x_2 = x_4$ but due to Claim 2 we do not allow other $x_i = x_j$ for $i \neq j$.

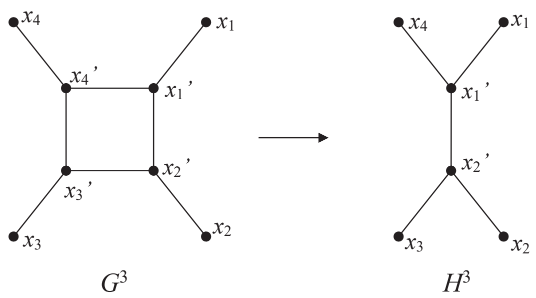

Figure 5. Transformation based on C_4.

So, there is a transformation $G_{2n}(G^3 \rightarrow H^3) \rightarrow H_{2n-2}$. Let X be a subgraph of G_{2n} induced by $\{x'_1, x'_2, x'_3, x'_4\}$, and let Y be a subgraph of H_{2n-2} induced by $\{x'_1, x'_2\}$ corresponding to Figure 5. Clearly, $t(X) = 4$ and $t(Y) = 1$. This means that there are four times more $T_i(G_{2n}, X)$ spanning trees than $T_i(H_{2n-2}, Y)$ spanning trees. In addition, for every spanning tree $\widehat{T}_i(H_{2n-2}, Y)$ there is a path $P_H = x'_1 \cdots x'_2$ on at least three vertices. Hence, for every spanning tree $\widehat{T}_i(H_{2n-2}, Y)$ there are at least three unique spanning trees of G_{2n} based on the following four cases.

Case 1. $\widehat{T}_i(H_{2n-2}, Y)$ contains $P_H = x'_1 x_1 \cdots x_2 x'_2$. For every $\widehat{T}_i(H_{2n-2}, Y)$ there correspond three unique spanning trees $\widehat{T}_{i_j}(G_{2n}, X)$ that contain all edges of $\widehat{T}_i(H_{2n-2}, Y)$ and the following:

1. $\widehat{T}_{i_1}(G_{2n}, X)$ contains additional edges $(x'_1, x'_4), (x'_2, x'_3)$,
2. $\widehat{T}_{i_2}(G_{2n}, X)$ contains additional edges $(x'_2, x'_3), (x'_3, x'_4)$,
3. $\widehat{T}_{i_3}(G_{2n}, X)$ contains additional edges $(x'_1, x'_4), (x'_3, x'_4)$.

Case 2. $\widehat{T}_i(H_{2n-2}, Y)$ contains $P_H = x'_1 x_1 \cdots x_3 x'_2$. For every $\widehat{T}_i(H_{2n-2}, Y)$ there correspond four unique spanning trees $\widehat{T}_{i_j}(G_{2n}, X)$ that contain all edges of $\widehat{T}_i(H_{2n-2}, Y)$ and the following:

1. $\widehat{T}_{i_1}(G_{2n}, X)$ contains additional edges $(x'_1, x'_4), (x'_2, x'_3)$,
2. $\widehat{T}_{i_2}(G_{2n}, X)$ contains additional edges $(x'_2, x'_3), (x'_3, x'_4)$,
3. $\widehat{T}_{i_3}(G_{2n}, X)$ contains additional edges $(x'_1, x'_2), (x'_1, x'_4)$,
4. $\widehat{T}_{i_4}(G_{2n}, X)$ contains additional edges $(x'_1, x'_2), (x'_3, x'_4)$.

Case 3. $\widehat{T}_i(H_{2n-2}, Y)$ contains $P_H = x'_1 x_4 \cdots x_3 x'_2$. For every $\widehat{T}_i(H_{2n-2}, Y)$ there correspond three unique spanning trees $\widehat{T}_{i_j}(G_{2n}, X)$ that contain all edges of $\widehat{T}_i(H_{2n-2}, Y)$ and the following:

1. $\widehat{T}_{i_1}(G_{2n}, X)$ contains additional edges $(x'_1, x'_4), (x'_2, x'_3)$,
2. $\widehat{T}_{i_2}(G_{2n}, X)$ contains additional edges $(x'_1, x'_2), (x'_2, x'_3)$,
3. $\widehat{T}_{i_3}(G_{2n}, X)$ contains additional edges $(x'_1, x'_2), (x'_1, x'_4)$.

Case 4. $\widehat{T}_i(H_{2n-2}, Y)$ contains $P_H = x'_1 x_4 \cdots x_2 x'_2$. For every $\widehat{T}_i(H_{2n-2}, Y)$ there correspond four unique spanning trees $\widehat{T}_{i_j}(G_{2n}, X)$ that contain all edges of $\widehat{T}_i(H_{2n-2}, Y)$ and the following:

1. $\widehat{T}_{i_1}(G_{2n}, X)$ contains additional edges $(x'_1, x'_4), (x'_2, x'_3)$,
2. $\widehat{T}_{i_2}(G_{2n}, X)$ contains additional edges $(x'_1, x'_2), (x'_2, x'_3)$,
3. $\widehat{T}_{i_3}(G_{2n}, X)$ contains additional edges $(x'_1, x'_4), (x'_3, x'_4)$,
4. $\widehat{T}_{i_4}(G_{2n}, X)$ contains additional edges $(x'_1, x'_2), (x'_3, x'_4)$.

None of the added edges, or combination of these edges, in Cases 1–4 could result in a cycle in $\widehat{T}_{i_j}(G_{2n}, X)$ because it would imply a cycle in $\widehat{T}_i(H_{2n-2}, Y)$ from which it was constructed. So, by Cases 1–4, there are at least three times more $\widehat{T}_i(G_{2n}, X)$ spanning trees than $T_i(H_{2n-2}, Y)$ spanning trees. Hence, $3t(H_{2n-2}) \leq t(G_{2n}) \leq t(M_{2n}) = 2t(M_{2n-2})$, implying $t(H_{2n-2}) < t(M_{2n-2})$—a contradiction, which proves Claim 3. □

Claim 4. *G_{2n} does not contain induced C_k for $k \geq 5$.*

Proof. Suppose that G_{2n} contains induced cycle C_k for $k \geq 5$—Figure 6. In Figure 6 we allow $x_1 = x_4$ but all other vertices x_i, x_j are pairwise distinct for $i \leq 4$ and $j \leq 4$. Otherwise, either Claim 2 or Claim 3 would be violated. In particular $x_2 \neq x_3$. Let G^4 be a subgraph of G_n. So, there is a transformation $G_{2n}(G^4 \to H^4) \to H_{2n-2}$. Let X be a subgraph of G^4 induced by $\{x_2, x_3, x'_1, x'_2, x'_3, x'_4\}$, and let Y be a subgraph of H^4 induced by $\{x_2, x_3, x'_1, x'_4\}$ indicated in Figure 6 with thick solid lines each. There are important properties of the subgraphs X, Y, G^4, H^4 in Figure 6 as follows: (1) edges of Y do not belong to $E(G_n)$, (2) other edges of H^4 than the ones in Y belong to $E(G_n)$, (3) edges of X do not belong to $E(H_{n-2})$, and (4) edges of X belong to $E(G_n)$. We explore these properties in the following four cases.

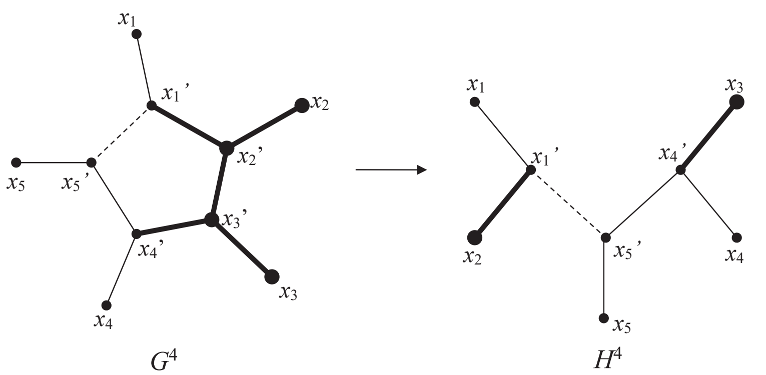

Figure 6. Transformation based on C_k for $k \geq 5$.

Case 1. $\widehat{T}_i(H_{2n-2}, Y)$ contains neither (x'_1, x_2) nor (x_3, x'_4).

1. $\widehat{T}_{i_1}(G_{2n}, X)$ contains edges $(x'_1, x'_2), (x'_2, x'_3) \in E(X)$,
2. $\widehat{T}_{i_2}(G_{2n}, X)$ contains edges $(x_2, x'_2), (x'_2, x'_3) \in E(X)$,
3. $\widehat{T}_{i_3}(G_{2n}, X)$ contains edges $(x'_2, x'_3), (x_3, x'_3) \in E(X)$,
4. $\widehat{T}_{i_4}(G_{2n}, X)$ contains edges $(x'_2, x'_3), (x'_3, x'_4) \in E(X)$.

In addition, each $\widehat{T}_{i_j}(G_{2n}, X)$ contains edges $E(\widehat{T}_i(H_{2n-2}, Y))$, for $4 \geq j \geq 1$, which together represent all edges in $\widehat{T}_{i_1}(G_{2n}, X)$.

Case 2. $\widehat{T}_i(H_{2n-2}, Y)$ contains (x'_1, x_2) but not (x_3, x'_4).

1. $\widehat{T}_{i_1}(G_{2n}, X)$ contains edges $(x'_1, x'_2), (x_2, x'_2), (x'_2, x'_3) \in E(X)$,
2. $\widehat{T}_{i_2}(G_{2n}, X)$ contains edges $(x'_1, x'_2), (x_2, x'_2), (x_3, x'_3) \in E(X)$,
3. $\widehat{T}_{i_3}(G_{2n}, X)$ contains edges $(x'_1, x'_2), (x_2, x'_2), (x'_3, x'_4) \in E(X)$.

In addition, each $\widehat{T}_{i_j}(G_{2n}, X)$ contains edges $E(\widehat{T}_i(H_{2n-2}, Y)) \backslash \{(x'_1, x_2)\}$, for $3 \geq j \geq 1$, which together represent all edges in $\widehat{T}_{i_1}(G_{2n}, X)$.

Case 3. $\widehat{T}_i(H_{2n-2}, Y)$ contains (x_3, x'_4) but not (x'_1, x_2).

1. $\widehat{T}_{i_1}(G_{2n}, X)$ contains edges $(x_3, x'_3), (x'_3, x'_4), (x'_2, x'_3) \in E(X)$,
2. $\widehat{T}_{i_2}(G_{2n}, X)$ contains edges $(x_3, x'_3), (x'_3, x'_4), (x_2, x'_2) \in E(X)$,
3. $\widehat{T}_{i_3}(G_{2n}, X)$ contains edges $(x'_2, x'_3), (x'_3, x'_4), (x'_1, x'_2) \in E(X)$.

In addition, each $\widehat{T}_{i_j}(G_{2n}, X)$ contains edges $E(\widehat{T}_i(H_{2n-2}, Y)) \backslash \{(x_3, x'_4)\}$, for $3 \geq j \geq 1$, which together represent all edges in $\widehat{T}_{i_1}(G_{2n}, X)$.

Case 4. $\widehat{T}_i(H_{2n-2}, Y)$ contains (x'_1, x_2) and (x_3, x'_4). In this case there is a $\widehat{T}_{i_1}(G_{2n}, X)$ that contains edges $(x'_1, x'_2), (x_2, x'_2), (x_3, x'_3), (x'_3, x'_4) \in E(X)$. Furthermore, removing either (x_3, x'_3) or (x'_3, x'_4) from $\widehat{T}_{i_1}(G_{2n}, X)$ induces forest Z_n of two trees. Clearly, vertices x_3, x'_4 must belong to two different trees in Z_n. This implies that either x_3, x'_1 or x'_4, x'_1 belong to two separate trees in either case. If x_3, x'_1 belong to two separate trees, then we obtain second $\widehat{T}_{i_2}(G_{2n}, X)$ with edges $(x'_1, x'_2), (x_2, x'_2), (x'_2, x'_3), (x_3, x'_3) \in E(X)$. Otherwise, x'_4, x'_1 belong to two separate trees and we obtain another second $\widehat{T}_{i_2}(G_{2n}, X)$ with edges $(x'_1, x'_2), (x_2, x'_2), (x'_2, x'_3), (x'_3, x'_4) \in E(X)$ instead.

On the other hand, removing either (x'_1, x'_2) or (x_2, x'_2) from $\widehat{T}_{i_1}(G_{2n}, X)$ induces different forest Z_n of two trees. Clearly, vertices x'_1, x_2 must belong to two different trees in Z_n. This implies that either x'_1, x'_4 or x_2, x'_4 belong to two separate trees in either case. If x'_1, x'_4 belong to two separate trees, then we obtain third $\widehat{T}_{i_3}(G_{2n}, X)$ with edges $(x'_1, x'_2), (x'_2, x'_3), (x_3, x'_3), (x'_3, x'_4) \in E(X)$. Otherwise, x_2, x'_4 belong to two separate trees and we obtain another third $\widehat{T}_{i_3}(G_{2n}, X)$ with edges $(x_2, x'_2), (x'_2, x'_3), (x_3, x'_3), (x'_3, x'_4) \in E(X)$ instead. In addition, each $\widehat{T}_{i_j}(G_{2n}, X)$ contains edges $E(\widehat{T}_i(H_{2n-2}, Y)) \backslash \{(x'_1, x_2), (x_3, x'_4)\}$, for $3 \geq j \geq 1$, which together with edges of previous three $\widehat{T}_{i_1}(G_{2n}, X)$ trees represent all edges in $\widehat{T}_{i_1}(G_{2n}, X)$.

In Cases 1–4 we examined all possible spanning trees of H_{2n-2}. We conclude that for every spanning tree of H_{2n-2} there are at least three unique spanning trees of G_n. Hence, $3t(H_{2n-2}) \leq t(G_{2n}) \leq t(M_{2n}) = 2t(M_{2n-2})$, implying $t(H_{2n-2}) < t(M_{2n-2})$—a contradiction, which proves Claim 4. □

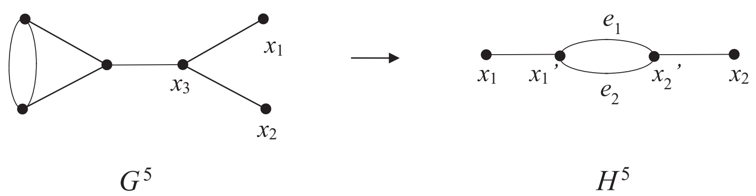

Figure 7. Transformation based on M_3.

Based on Claims 1–4 we conclude that every cycle of G_{2n} must belong to some component M_3 in G_{2n}. So, G_{2n} must consist of at least three M_3 components. This implies that transformation $G_{2n}(G^5 \to H^5) \to H_{2n-2}$ illustrated in Figure 7 is possible.

Edges $(x_1, x_3), (x_2, x_3)$ in G^5 do not belong to any cycle. The number of spanning trees in G^5 is $t(G^5) = 5$, while in H^5 is $t(H^5) = 2$. Consequently, $\frac{5}{2}t(H_{2n-2}) \leq t(G_{2n}) \leq t(M_{2n}) = 2t(M_{2n-2})$, implying $t(H_{2n-2}) < t(M_{2n-2})$, a contradiction. This contradiction proves Theorem 3. ■

We note that M_{2n} is the unique graph as opposed to the simple connected cubic graphs of order $2n$ that minimize the number of spanning trees, which were identified in [3].

3. Extension of M_{2n} to All Connected, Odd-Regular Multigraphs

We define a regular multigraph $M_{2n}^{\frac{d-1}{2}}$ of odd degree d, $d \geq 3$, and on $2n$ verices as follows:

(1) $M_{2n}^1 := M_{2n}$, and it consists of components $M_3^1 := M_3$, $C_2^1 := C_2$,

(2) M_{2n}^{i+1} is constructed from M_{2n}^i as follows:

   (i) add one edge for every pair of vertices in both M_3^i components of M_{2n}^i,

   (ii) for every component C_2^i not included in M_3^i add two parallel edges.

First, consider the number of spanning trees in M_{2n}^i.

Theorem 4. *$t(M_{2n}^k) = k^2(3k+2)^2(k+1)^{n-3}$, for $n \geq 3$ and $k \geq 1$.*

Proof. Based on the definition, M_{2n}^k contains two M_3^k components and $n-3$ C_2^k components. It is easy to see that $t(M_3^k) = (k^2 + k(k+1) + k(k+1)) = k(3k+2)$ and $t(C_2^k) = k+1$. Since these components do not belong to any cycle, the number of spanning trees in M_{2n}^k equals

$$(t(M_3^k))^2 \cdot (t(C_2^k))^{n-3} = k^2(3k+2)^2(k+1)^{n-3}.$$

■

In particular, for the connected cubic graphs we get lower sharp bound for the number of spanning trees as follows.

Corollary 4. *Let G_{2n} be a connected cubic graph of order $2n \geq 6$. Then $t(G_{2n}) \geq 5^2 2^{n-3}$.*

Proof. According to Theorem 3, $t(G_{2n}) \geq t(M_{2n})$, and according to Theorem 4, $t(M_{2n}) = 5^2 2^{n-3}$ for $n \geq 3$. ■

Finally, based on our results we propose the following.

Conjecture 5. *Connected r-regular graph G_{2n} of order $2n$ minimizes $t(G_{2n})$ for r odd and $r, n \geq 3$ if and only if $G_{2n} \simeq M_{2n}^{\frac{r-1}{2}}$.*

If our conjecture is true then $((r-1)/2)^2((3r+1)/2)^2((r+1)/2)^{n-3}$ is lower sharp bound for the number of spanning trees in connected r-regular graphs of order $2n$ and r odd.

Acknowledgments

I like to thank the referees for valuable comments and suggestions.

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Received 18 October 2016
Revised 20 February 2018
Accepted 21 February 2018