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SIMPLE FRACTIONS AND LINEAR DECOMPOSITION OF SOME CONVOLUTIONS OF MEASURES

Jolanta K. Misiewicz

Institute of Mathematics, University of Zielona Góra
Podgórna 50, 65–246 Zielona Góra, Poland

e-mail: J.Misiewicz@im.uz.zgora.pl

and

Roger Cooke

Facultteit Informatietechnologie en System, Technische Universiteit Delft
Mekelweg 4, Postbus 5031, 2600 GA Delft, Holland

e-mail: r.m.cooke@its.tudelft.hl

Abstract

Every characteristic function φ can be written in the following way:

$$\varphi(\xi) = \frac{1}{h(\xi)+1}, \qquad \text{where} \quad h(\xi) = \begin{cases} 1/\varphi(\xi) - 1 & \text{if} \quad \varphi(\xi) \neq 0 \\ \infty & \text{if} \quad \varphi(\xi) = 0. \end{cases}$$

This simple remark implies that every characteristic function can be treated as a simple fraction of the function $h(\xi)$. In the paper, we consider a class $\mathcal{C}(\varphi)$ of all characteristic functions of the form $\varphi_a(\xi) = \frac{a}{h(\xi)+a}$, where $\varphi(\xi)$ is a fixed characteristic function. Using the well known theorem on simple fraction decomposition of rational functions we obtain that convolutions of measures μ_a with $\widehat{\mu_a}(\xi) = \varphi_a(\xi)$ are linear combinations of powers of such measures. This can simplify calculations. It is interesting that this simplification uses signed measures since coefficients of linear combinations can be negative numbers. All the results of this paper except Proposition 1 remain true if we replace probability measures with complex valued measures with finite variation, and replace the characteristic function with Fourier transform.

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1. Introduction

By $\mathcal{P}$ we denote the set of all probability measures on a real line, by $\mathcal{P}(I\!\!E)$ the set of all probability measures on a linear space $I\!\!E$. For every characteristic function $\widehat{\mu}(\xi) = \varphi(\xi)$, $\xi \in I\!\!E$, of a measure μ we define

$$h(\xi) = h_\varphi(\xi) := \begin{cases} \left(\frac{1}{\varphi(\xi)} - 1\right) & \text{for} \quad \varphi(\xi) \neq 0 \\ \infty & \text{for} \quad \varphi(\xi) = 0. \end{cases}$$

Of course $\varphi(\xi) = 1/(h(\xi)+1)$. In the paper, we discuss properties of the set of all characteristc functions $\varphi_a(\xi)$ (and corresponding measures μ_a) having the form $\varphi_a(\xi) = a/(h(\xi)+a)$, with $h(\xi) = h_\varphi(\xi)$, for φ being a fixed characteristic function. According to the methods that we are using here there is no reason to specify precisely if a given characteristic function corresponds to a real random variable or to a random vector taking values in more complicated linear spaces. In order to save generality we will formulate our results for topological linear space $I\!\!E$ with the space of linear functionals $I\!\!E^*$, however in most of the examples we will consider just real random variables and their characteristic functions.

For every probability measure μ on $I\!\!E$ with the characteristic function $\varphi(\xi), \xi \in I\!\!E^*$ we define the following sets:

$$\begin{aligned} \mathbf{T}(\varphi) \ &= \left\{ a : \frac{a}{h(\xi)+a} \text{ is positive definite on } I\!\!E^* \right\}, \\ \mathcal{C}(\varphi) \ &= \left\{ \chi : \chi(\xi) = \frac{a}{h(\xi)+a}, a \in \mathbf{T}(\varphi) \right\}, \\ \mathcal{M}(\varphi) &= \left\{ \mu \in \mathcal{P}(I\!\!E) : \widehat{\mu} \in \mathcal{C}(\varphi) \right\}. \end{aligned}$$

Of course for every characteristic function φ we have that $1 \in \mathbf{T}(\varphi)$, $\varphi \in \mathcal{C}(\varphi)$, and every probability measure μ belongs to the class $\mathcal{M}(\widehat{\mu})$. The set $\mathbf{T} = \mathbf{T}(\varphi)$ is a subset of complex plane $\mathbf{Z}$ or a subset of real line $I\!\!R$ and $0 \notin \mathbf{T}(\varphi)$.

Notice that two characteristic functions χ, ψ belong to the same class $\mathcal{C}(\varphi)$ if and only if the following condition holds:

$$(*) \qquad \exists a, b\ ab \neq 0\ \forall \xi \in I\!E^*\ \chi(\xi)\psi(\xi) \neq 0 \quad \frac{a}{\chi(\xi)} - \frac{b}{\psi(\xi)} = const.$$

About the measures μ and ν we will say that they are *linearly simmilar* if there exists a characteristic function φ such that they both belong to the class $\mathcal{M}(\varphi)$.

Proposition 1. *For every characteristic function φ the following conditions hold:*

1) $p\mathbf{T}(\varphi) \subset \mathbf{T}(\varphi)$ *for every* $p \in (0, 1]$;

2) $\mathcal{C}(\varphi)$ *is closed under geometric transformation, i.e. for every* $p \in (0, 1]$

$$\chi \in \mathcal{C}(\varphi) \quad \Longrightarrow \quad T_p(\chi)(\xi) \stackrel{def}{=} \frac{p\chi(\xi)}{1-(1-p)\chi(\xi)} \in \mathcal{C}(\varphi);$$

3) *if the distribution $\mathcal{L}(X)$ of the random variable X belongs to $\mathcal{M}(\varphi)$, then also $\mathcal{L}(Y) \in \mathcal{M}(\varphi)$, where*

$$Y = \sum_{k=1}^{\Theta} X_k,$$

$X_1, X_2, \ldots$ are independent copies of X and Θ independent of $X_1, X_2, \ldots$ has geometric distribution with parameter $p \in (0, 1]$.

Proof. We start with proving (3). Calculating the characteristic function of Y we obtain:

$$\psi(\xi) \stackrel{def}{=} \mathbf{E}e^{i<\xi,Y>} = \sum_{k=1}^{\infty} e^{i(<\xi,X_1>+\ldots+<\xi,X_k>)}p(1-p)^{k-1}$$

$$= \sum_{k=1}^{\infty} \left(\widehat{\mu}(\xi)\right)^k p(1-p)^{k-1} = \frac{p\widehat{\mu}(\xi)}{1-(1-p)\widehat{\mu}(\xi)}.$$

Since $\widehat{\mu} \in \mathcal{C}(\varphi)$, then there exists $a \neq 0$ such that $\widehat{\mu}(\xi) = \frac{a}{h(\xi)+a}$, thus, finally we obtain

$$\psi(\xi) = \frac{ap}{h(\xi) + ap} \in \mathcal{C}(\varphi).$$

To prove (2) it is enough to put $\chi(\xi) = \widehat{\mu}(\xi)$ and apply the property (3). To see (1) notice that $a \in \mathbf{T}(\varphi)$ implies that the function $\chi(\xi) = \frac{a}{h(\xi)+a}$ is a characteristic function, thus $\chi(\xi) = \widehat{\mu}(\xi)$ for some probability distribution μ. Finally, from the proof of (3) it follows that if $a \in \mathbf{T}(\varphi)$, then the function $\frac{ap}{h(\xi)+ap}$ belongs to $\mathcal{C}(\varphi)$, so $ap \in \mathbf{T}(\varphi)$. ∎

Theorem 1. *For every choice of $k, n_1, ..., n_k \in \mathbb{N}$ and every choice of different numbers $a_1, ..., a_k \in \mathbf{T}(\varphi)$ there exists a set of numbers $b_{i,j}$, $i = 1, ..., k$, $j = 1, ..., n_i$ such that*

$$\mu_{a_1}^{*n_1} * \ldots * \mu_{a_k}^{*n_k} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} b_{i,j} \mu_{a_i}^{*j}.$$

Proof. Denote by ν the measure $\mu_{a_1}^{*n_1} * \ldots * \mu_{a_k}^{*n_k}$. The characteristic function of ν has the following form

$$\widehat{\nu}(\xi) = \left(\frac{a_1}{h(\xi) + a_1}\right)^{n_1}, \ldots, \left(\frac{a_k}{h(\xi) + a_k}\right)^{n_k}.$$

We can see that $\widehat{\nu}(\xi)$ is a rational function of the argument $h(\xi)$. Using now the theorem on simple fractions decomposition of rational functions (see e.g. [3], [4]) we obtain that

$$\widehat{\nu}(\xi) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} b_{i,j} \left(\frac{a_i}{h(\xi) + a_i}\right)^{j},$$

for suitable coefficients $b_{i,j}$. This ends the proof. ∎

2. Examples of decomposition theorems

Proposition 2. *Let $\mu \neq \nu$ similar two probability measures on $\mathbb{E}$. Measures μ and ν are linearly similar iff there exist constants A and B such that $A + B = 1$ and*

$$\mu * \nu = A\mu + B\nu.$$

Proof. If $\widehat{\mu}(\xi)\widehat{\nu}(\xi) = A\widehat{\mu}(\xi) + B\widehat{\nu}(\xi)$, then the characteristic functions $\widehat{\mu}, \widehat{\nu}$ have the property $(*)$, thus they belong to the same class $\mathcal{C}(\varphi)$ for some characteristic function φ.

Assume now that $\mu, \nu \in \mathcal{M}(\varphi)$ for some characteristic function φ, and let $h(\xi) = \frac{1}{\varphi(\xi)} - 1$. Then there exist constants a and b, $ab \neq 0$ such that

$$\widehat{\mu}(\xi)\widehat{\nu}(\xi) = \frac{a}{h(\xi)+a}\,\frac{b}{h(\xi)+b}.$$

Treating this product as a product of simple fractions of the variable $h(\xi)$ we easily obtain that

$$\widehat{\mu}(\xi)\widehat{\nu}(\xi) = \frac{b}{b-a}\,\frac{a}{h(\xi)+a} + \frac{a}{a-b}\,\frac{b}{h(\xi)+b} = \frac{b}{b-a}\widehat{\mu}(\xi) + \frac{a}{a-b}\widehat{\nu}(\xi),$$

which ends the proof. ∎

Proposition 3. *Let $\mu_1, ..., \mu_n \in \mathcal{M}(\varphi)$ be different measures with $\widehat{\mu_i}(t) = a_i/(h(t)+a_i)$ where $h(\xi) = \frac{1}{\varphi(\xi)} - 1$. Then:*

$$\mu_1 * \cdots * \mu_n = \sum_{i=1}^{n} A_i\mu_i, \qquad \textit{for} \ \ A_i = \prod_{j\neq i} \frac{a_j}{a_j - a_i}.$$

Proof. The proof is trivial and will be omitted. ∎

Proposition 4. *Let μ_a, μ_b be measures with the characteristic functions φ_a, φ_b, $a, b \in \mathbf{T}(\varphi)$, $a \neq b$. Then for every $m, n \in I\!N$ there exist constants $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ such that*

$$\varphi_a^n(t)\varphi_b^m(t) = \sum_{j=1}^{n} a_j\varphi_a^j(t) + \sum_{k=1}^{m} b_k\varphi_b^k(t),$$

and consequently,

$$\mu_a^{*n}\mu_b^{*m} = \sum_{j=1}^{n} a_j\mu_a^{*j} + \sum_{k=1}^{m} b_k\mu_b^{*k}.$$

Moreover,

$$a_j = a_j(n,m) = \left(-\frac{b}{a}\right)^m \left(\frac{a}{a-b}\right)^{n+m-j} \binom{n+m-1-j}{n-j},$$

$$b_k = b_k(n,m) = \left(-\frac{a}{b}\right)^{m-k} \left(\frac{a}{b-a}\right)^{n+m-k} \binom{n+m-1-k}{m-k}.$$

Proof. The proof of the main part of the theorem follows easily from Lemma 2 and mathematical induction. In order to calculate a_j and b_k we shall first use the mathematical induction to prove that

$$a_j(n,1) = -\frac{b}{a}\left(\frac{a}{a-b}\right)^{n+1-j}, \qquad b_k(n,1) = \left(\frac{a}{a-b}\right)^n$$

which gives us the desired formula for $m = 1$. Applying mathematical induction again, with respect to m, except the classical calculations we shall remember also that the following formula holds:

$$\sum_{k=0}^{N} \binom{M+k}{k} = \binom{N+M+1}{N}.$$

∎

3. Examples of classes $\mathcal{M}(\varphi)$

Example 1. Consider the family of exponential distributions, i.e. probability distributions on $I\!R$ with densities given by the formula:

$$\gamma_{1,a}(x) = \begin{cases} ae^{-ax} & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad < 0 \end{cases} \qquad a > 0.$$

Notice that the characteristic function of such distribution has the form:

$$\widehat{\gamma}_{1,a}(t) = \frac{a}{-it+a},$$

thus the set of exponential distributions is equal to $\mathcal{M}(\varphi)$ with $h(t) = -it$ and $\mathbf{T}(\varphi) = I\!R \setminus \{0\}$, where for negative $a \in \mathbf{T}(\varphi)$ the corresponding distribution is concentrated on the negative half-line. It follows also from Proposition 1 that the convolution mixture of exponential distribution $\gamma_{1,a}$ with respect to the geometric distribution with parameter p is exponential $\gamma_{1,ap}$, so it is the same as the distribution of the random variable $p^{-1}X$, where X has $\gamma_{1,a}$ distribution. From the proof of Proposition 2 it follows easily that the distribution of symmetrization of the distribution $\gamma_{1,a}$ has the form:

$$\gamma_{1,a} * \gamma_{1,-a} = \frac{1}{2}\gamma_{1,a} + \frac{1}{2}\gamma_{1,-a}.$$

Example 2. Consider a symmetric α-stable random vector X taking values in $I\!E$ with characteristic function $\varphi(\xi) = \exp\{-\|\Re(\xi)\|_\alpha^\alpha\}$, where $\alpha \in (0,2]$, and $\Re$ is the linear operator from $I\!E^*$ into some L_α-space. Let Θ_a be a random variable with distribution $\gamma_{1,a}$, $a > 0$, Θ independent of X. The characteristic function of the random vector $Y = X\Theta_a^{1/\alpha}$ is given by

$$\begin{aligned}\psi(\xi) &= \mathbf{E}e^{i<\xi,Y>} = \int_0^\infty \left(\mathbf{E}e^{i<\xi s^{1/\alpha},X>}\right) ae^{-as}ds \\ &= a\int_0^\infty \exp\left\{-(\|\Re(\xi)\|^\alpha + a)s\right\} ds = \frac{a}{\|\Re(\xi)\|^\alpha + a}.\end{aligned}$$

This means that for every $\alpha \in (0,2]$ we have

$$\mathbf{T}(\psi_\alpha) = (0,\infty),$$

$$\mathcal{M}(\psi_\alpha) = \left\{\mathcal{L}\left(X\Theta_a^{1/\alpha}\right) : a > 0\right\}.$$

Example 3. For a fixed $b > 0$ consider the family of probability distributions $Sh_b = \{\mu_{a,b} : |a| < b\}$, where $\mu_{a,b}$ has the density function of the form:

$$f_{a,b} = \frac{1 + \cos(\pi\frac{a}{b})}{\pi\sin(\pi\frac{a}{b})}\frac{\sinh(ax)}{\sinh(bx)}.$$

According to formula 28 in the table 17.34 of Gradshtejn and Ryzhik [2] the characteristic function of the measure $\mu_{a,b}$ is given by:

$$\varphi_{a,b}(t) = \frac{1+\cos(\pi\frac{a}{b})}{\cosh(\pi\frac{t}{b})+\cos(\pi\frac{a}{b})}.$$

Notice that the class Sh_b is closed under convolution mixtures with respect to the geometric distribution. To see this consider Θ with geometric distribution with parameter $p \in (0,1)$ and let $Y = \sum_{k=1}^{\Theta} X_k$, where X_k are independent random variables with density $f_{a,b}$. The characteristic function $\psi(t)$ of the random variable Y is of the form:

$$\psi(t) = \frac{p\varphi_{a,b}(t)}{1-q\varphi_{a,b}(t)} = \frac{1+p\cos(\pi\frac{a}{b})-q}{\cosh(\pi\frac{t}{b})+p\cos(\pi\frac{a}{b})-q}.$$

It is easy to see that

$$-1 = -p-q < p\cos\left(\pi\frac{a}{b}\right) - q < p-q < 1,$$

thus there exists $0 < c < b$ such that $p\cos(\pi\frac{a}{b}) - q = \cos(\pi\frac{c}{b})$, and consequently $\psi(t) = \varphi_{c,b}(t)$. This shows that the class Sh_b is equal $\mathcal{M}(\varphi_{a,b})$, where a appearing here can be arbitrarily chosen from $(0,b)$. Finally, it is easy to see that

$$\mathbf{T}(\varphi_{a,b}) = \left\{ \frac{1+\cos\left(\pi\frac{c}{b}\right)}{1+\cos\left(\pi\frac{a}{b}\right)} = \frac{\cos^2\left(\frac{\pi c}{2b}\right)}{\cos^2\left(\frac{\pi a}{2b}\right)} : |c| < b, |a| < b \right\} = (0,\infty).$$

Example 4. Consider the family of symmetric exponential distributions $\Lambda = \{\lambda_a : a > 0\}$, where λ_a has the following density and the corresponding characteristic function:

$$g_a = \frac{a}{2}e^{-a|x|}, \qquad \widehat{\lambda_a}(t) = \frac{a^2}{t^2+a^2}.$$

It is easy to see that $\Lambda = \mathcal{M}(\widehat{\lambda_1})$, and $\mathbf{T}(\widehat{\lambda_1}) = (0, \infty)$. In order to use Theorem 2 we shall know the density of the measure $\lambda_a^{*(n+1)}$, which according to formula 3.737.1 of [2] is given by:

$$g_a^{*(n+1)}(x) = \frac{ae^{-a|x|}}{2^{2n+2}n!} \sum_{k=0}^{n} \frac{(2n-k)!}{k!n!} (2a|x|)^k.$$

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