Discussiones Mathematicae
General Algebra and Applications 21 (2001) 207–218

MAXIMAL COLUMN RANK PRESERVERS OF FUZZY MATRICES*

Seok-Zun Song and Soo-Roh Park

Department of Mathematics, Cheju National University
Cheju 690-756, South Korea
e-mail: szsong@cheju.cheju.ac.kr

Abstract

This paper concerns two notions of rank of fuzzy matrices: maximal column rank and column rank. We investigate the difference of them. We also characterize the linear operators which preserve the maximal column rank of fuzzy matrices. That is, a linear operator T preserves maximal column rank if and only if it has the form $T(X) = UXV$ with some invertible fuzzy matrices U and V.

Keywords: linear operator on matrices, fuzzy matrix, maximal column rank of a matrix, congruence operator on matrices, chain semiring.

1991 Mathematics Subject Classification: 15A03, 15A04, 15A33, 08A72, 16Y60.

1. Introduction

There are many papers on the study of linear operators that preserve semiring rank and column rank of matrices over several semirings ([2]–[6]). Beasley and Pullman [2] obtained characterizations of linear operators that preserve semiring rank of fuzzy matrices. Song characterized the column rank case in [5]. Hwang, Kim and Song [4] defined maximal column rank of a matrix over a semiring and compared it with column rank. And they obtained characterizations of the linear operators that preserve maximal column rank of binary Boolean matrices. For the case of nonbinary Boolean matrices, the linear operators were characterized in [6].

*This work was supported by grant No. R01-2001- 00004 from the Korean Science & Engineering Foundation

In this paper, we study the extent to which known properties of linear operators preserving the column rank of matrices over 'chain semiring' (see Section 2) carry over to operators preserving maximal column rank. We obtain some characterizations of the linear operators that preserve maximal column rank of matrices over fuzzy semiring and chain semirings which are more general classes than the binary Boolean algebra.

2. Column rank versus maximal column rank of matrices over chain semiring

A *semiring* is a binary system $(\mathbb{S}, +, \times)$ such that $(\mathbb{S}, +)$ is an Abelian monoid (identity 0), $(\mathbb{S}, \times)$ is a monoid (identity 1), $\times$ distributes over $+$, $0 \times s = s \times 0 = 0$ for all s in $\mathbb{S}$ and $1 \neq 0$. Usually $\mathbb{S}$ denotes both the semiring and the set and $\times$ is denoted by juxtaposition.

Let $\mathbb{M}_{m,n}(\mathbb{S})$ denote the set of $m \times n$ matrices with entries in a semiring $\mathbb{S}$. The zero matrix $0_{m,n}$ and the identity matrix I_n are defined as if $\mathbb{S}$ were a field. Addition, multiplication by scalars, and the product of matrices are also defined as if $\mathbb{S}$ were a field. Thus $\mathbb{M}_{m,n}(\mathbb{S})$ is a semiring under matrix addition and multiplication. If $\mathbb{V}$ is a nonempty subset of $\mathbb{S}^k \equiv \mathbb{M}_{k,1}(\mathbb{S})$ that is closed under addition and multiplication by scalars, then $\mathbb{V}$ is called a *vector space* over $\mathbb{S}$. The notions of subspace and of generating set are the same as if $\mathbb{S}$ were a field. A set G of vectors over $\mathbb{S}$ is *linearly dependent* if for some $g \in G$, g is in the subspace generated by $G \backslash \{g\}$. Otherwise, G is *linearly independent*. The *maximal column rank*, $mc(A) = mc_{\mathbb{S}}(A)$, of an $m \times n$ matrix A over $\mathbb{S}$ is the maximal number of the columns of A which are linearly independent over $\mathbb{S}$. As with fields, a basis for a vector space $\mathbb{V}$ is a generating subset of the least cardinality. That cardinality is the *dimension* , $dim(\mathbb{V})$, of $\mathbb{V}$. The *column space* of an $m \times n$ matrix A over $\mathbb{S}$ is the vector space generated by its columns. The *column rank*, $c(A) = c_{\mathbb{S}}(A)$, of an $m \times n$ matrix A over $\mathbb{S}$ is the dimension of the column space. The *semiring rank*, $r(A) = r_{\mathbb{S}}(A)$, of a nonzero matrix A in $\mathbb{M}_{m,n}(\mathbb{S})$ is the least integer k such that $A = BC$ for some $m \times k$ and $k \times n$ matrices B and C over $\mathbb{S}$. The semiring rank, column rank and maximal column rank of the zero matrix are 0.

It follows directly from the definitions that for all $m \times n$ matrices A over $\mathbb{S}$:

$$(2.1) \quad 0 \leq r_{\mathbb{S}}(A) \leq c_{\mathbb{S}}(A) \leq mc_{\mathbb{S}}(A) \leq n.$$

The maximal column rank of a matrix may actually exceed its column rank over some semirings. For example, see the matrix A in Example 2.1.

Let $\mathbb{S}$ be any set of two or more elements. If $\mathbb{S}$ is totally odered by $<$, that is, $\mathbb{S}$ is a chain under $<$ (i.e. $x < y$ or $y < x$ for all distinct x, y in $\mathbb{S}$), then define $x + y$ as $max(x, y)$ and xy as $min(x, y)$ for all x, y in $\mathbb{S}$. If $\mathbb{S}$ has a universal lower bound and a universal upper bound, then $\mathbb{S}$ becomes a semiring: a *chain semiring.*

Let H be any nonempty family of sets nested by inclusion, $0 = \bigcap_{x\in H} x$, and $1 = \bigcup_{x\in H} x$. Then $\mathbb{S} = H \cup \{0, 1\}$ is a chain semiring. Let a, b be real numbers with $a < b$. Define $\mathbb{S} = \{r : a \leq \ r \leq \ b\}$. Then $\mathbb{S}$ is a chain semiring with $a = 0$ and $b = 1$. It is isomorphic to the chain semiring in the previous example with $H = \{[a, r] : a \leq \ r \leq \ b\}$. If in particular we choose the real numbers 0 and 1 as a and b in H, then $\mathbb{F} = \{r : 0 \leq \ r \leq 1, \ \text{r is real}\}$ is a *fuzzy semiring* and each $m \times n$ matrix over $\mathbb{F}$ is a *fuzzy matrix.*

Hereafter, otherwise specified, $\mathbb{K}$ will denote a chain semiring which is not the binary Boolean algebra $\mathbb{B}$, and all matrices will denote the $m \times n$ matrices over $\mathbb{K}$.

Since 1 is the only invertible member of the multiplicative monoid of $\mathbb{K}$, the permutation matrices (obtained by permuting the columns of I_n) are the only invertible members of $\mathbb{M}_{n,n}(\mathbb{K})$.

It is already known that:

(2.2) The semiring rank of a nonzero matrix A is the minimum number of semiring rank 1 matrices which sum to A ([2]).

(2.3) The column rank of a matrix over a chain semiring is unchanged by pre- or post-multiplication by an invertible matrix. Furthermore, the column rank of a 2×2 matrix is unchanged by transposition ([5]).

If we take $\mathbb{H}$ in the above to be a singleton, say $\{a\}$, and denote the empty subset by 0 and $\{a\}$ by 1, the resulting chain semiring is merely the binary Boolean algebra, and denoted by $\mathbb{B}$.

Let $\alpha(\mathbb{S}, m, n)$ be the largest integer k such that for all $m \times n$ matrices A over $\mathbb{S}$, $c(A) = mc(A)$ if $c(A) \leq k$.

In [1] Beasley and Pullman compared semiring rank with column rank over several semirings. Similarly, we compare column rank with maximal column rank over chain semiring and investigate the value α.

Theorem 2.1 ([1])**.** *Let $\mu(\mathbb{S}, m, n)$ be the largest integer k such that for all $m \times n$ matrices A over $\mathbb{S}$, $r(A) = c(A)$ if $r(A) \leq k$. Then we have*

(i) *for any chain semiring $\mathbb{K}$, we have*

$$\mu(\mathbb{K}, m, n) = \begin{cases} 2 & \text{if } m \geq 2 \text{ and } n = 2, \\ 1 & \text{otherwise.} \end{cases}$$

(ii) *for the binary Boolean algebra $\mathbb{B}$,*

$$\mu(\mathbb{B}, m, n) = \begin{cases} 1 & \text{whenever } \min(m, n) = 1, \\ 3 & \text{for all } m \geq 3 \text{ and } n = 3, \\ 2 & \text{otherwise.} \end{cases}$$

We give the following example for Theorem 2.2.

Example 2.1. Let p be a nonzero nonunit element of $\mathbb{K}$. Consider,

$$A = \begin{pmatrix} 1 & 0 & p & 1 \\ 0 & 1 & 0 & p \end{pmatrix}.$$

Then $mc(A) = 3$, since the last three columns of A are linearly independent. But $c(A) = 2$, since the first two columns generate the column space of A.

Lemma 2.1 ([4])**.** *Over any semiring $\mathbb{S}$, if $mc(A) > c(A)$ for some $p \times q$ matrix A, then for all $m \geq p$ and $n \geq q$, $\alpha(\mathbb{S}, m, n) < c(A)$.*

Lemma 2.2. *If the columns of $A \in \mathbb{M}_{m,n}(\mathbb{K})$ are linearly independent, then $c(A) = n$ and $mc(A) = n$.*

Proof. Let $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$. Then the column space of A is

$$\mathbb{V} = \{k_1\mathbf{a}_1 + \ldots + k_n\mathbf{a}_n : k_i \in \mathbb{K}\}.$$

Let G be any subset of $\mathbb{V}$ generating $\mathbb{V}$. If $\mathbf{a}_1 \notin G$, then $\mathbf{a}_1 = k_2\mathbf{a}_2 + \ldots + k_n\mathbf{a}_n$ for some $k_i \in \mathbb{K}$. Then the columns of A are not linearly independent, which is a contradiction. Hence $\mathbf{a}_1$ is in G. Similarly, all the columns $\mathbf{a}_i$ of A are in G. Thus $c(A) = n$. ∎

Corollary 2.1 ([1]). *If the columns of $A \in \mathbb{M}_{m,n}(\mathbb{B})$ are linearly independent, then $c(A) = n$ and $mc(A) = n$.*

Theorem 2.2. *Let $\mathbb{K}$ be a chain semiring. Then we have*

$$\alpha(\mathbb{K}, m, n) = \begin{cases} 3 & \text{if } m = 2 \text{ and } n = 3, \\ 2 & \text{if } m = n = 2, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Consider the matrix A in Example 2.1. Then by Lemma 2.1, we may conclude that

$$\alpha(\mathbb{K}, m, n) \leq 1 \quad \text{if m} \geq 2 \text{ and n} \geq 4.$$

Clearly we have that $c(B) = 1$ if and only if $mc(B) = 1$, for any matrix $B \in \mathbb{M}_{m,n}(\mathbb{K})$. Thus ,

$$\alpha(\mathbb{K}, m, n) = 1 \quad \text{if m } \geq 2 \text{ and n} \geq 4.$$

Suppose $m = 2$ and $n = 3$. If $c(B) = 3$, then $mc(B) \geq 3$ by (2.1). Thus $mc(B) = 3$. Conversely, if $mc(B) = 3$, then all the columns of B are linearly independent. Thus $c(B) = 3$ by Lemma 2.2. If $c(B) = 2$, then $mc(B) \geq 2$. But $mc(B) \neq 3$ by above case. Thus $mc(B) = 2$. Conversely, if $mc(B) = 2$, then $c(B) \leq 2$. But $c(B) \neq 1$. Hence $c(B) = 2$. Therefore

$$\alpha(\mathbb{K}, 2, 3) = 3.$$

Suppose $m = 2$ and $n = 2$. Then we have that $c(B) = 2$ if and only if $mc(B) = 2$. Hence

$$\alpha(\mathbb{K}, 2, 2) = 2.$$

It is trivial that $\alpha(\mathbb{K}, 1, n) = 1$ for all $n \geq 1$ and $\alpha(\mathbb{K}, m, 1) = 1$ for all $m \geq 1$. ■

Lemma 2.3. *The maximal column rank of a matrix is unchanged by pre- or post-multiplication by an invertible matrix. Furthermore, the maximal column rank of a 2×2 matrix is unchanged by transposition.*

Proof. The results follow from Theorem 2.2 using (2.3). ■

Let $\mathbf{j}_k$ denote the column vector of length k all of whose entries are 1, and J_{mn} the $m \times n$ matrix all of whose entries are 1. When the orders are understood, we may drop the subscript on $\mathbf{j}_k$ and J_{mn}. Let E_{ij} be the $m \times n$ matrix all of whose entries are 0 except the (i, j)th, which is 1.

We define the norm of an arbitrary $X \in \mathbb{M}_{m,n}(\mathbb{K})$ by $\|X\| = \mathbf{j}^t X \mathbf{j}$, the sum of all entries in X. That is, $\|X\|$ is the maximum entry in X. Note the mapping $X \to \|X\|$ preserves matrix addition and scalar multiplication.

Lemma 2.4 ([5]). *Suppose*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $c(A) = 2$ if and only if $ad \neq bc$.

Lemma 2.5. *Suppose*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $mc(A) = 2$ if and only if $ad \neq bc$.

Proof. We have that $mc(A) = 2$ if and only if $c(A) = 2$, by Theorem 2.2. Thus the result follows from Lemma 2.4. ∎

Lemma 2.6. *If H is a submatrix of A, then $mc(H) \leq mc(A)$.*

Proof. It is clear from the definition of maximal column rank. ∎

3. LINEAR OPERATORS THAT PRESERVE MAXIMAL COLUMN RANK OVER $\mathbb{M}_{m,n}(\mathbb{K})$

A function T mapping $\mathbb{M}_{m,n}(\mathbb{S})$ into $\mathbb{M}_{m,n}(\mathbb{S})$ is called an *operator* on $\mathbb{M}_{m,n}(\mathbb{S})$. The operator T

(i) is *linear* if $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all $\alpha, \beta \in \mathbb{S}$ and all $A, B \in \mathbb{M}_{m,n}(\mathbb{S})$,

(ii) *preserves semiring rank h* if, for any $A \in \mathbb{M}_{m,n}(\mathbb{S})$ with $r(A) = h$, $r(T(A)) = r(A)$,

(iii) *preserves column rank k* if, for any $A \in \mathbb{M}_{m,n}(\mathbb{S})$ with $c(A) = k$, $c(T(A)) = c(A)$,

(iv) *preserves maximal column rank* l if, for any $A \in \mathbb{M}_{m,n}(\mathbb{S})$ with $mc(A) = l$, $mc(T(A)) = mc(A)$,

(v) is a *congruence operator* if there exist invertible matrices U and V in $\mathbb{M}_{m,m}(\mathbb{S})$ and $\mathbb{M}_{n,n}(\mathbb{S})$, respectively, such that $T(A) = UAV$ for all $A \in \mathbb{M}_{m,n}(\mathbb{S})$

(vi) is a *transposition operator* if $m = n$ and $T(A) = A^t$ for all $A \in \mathbb{M}_{m,n}(\mathbb{S})$.

In this section, we characterize the linear operators that preserve maximal column rank over $\mathbb{M}_{m,n}(\mathbb{K})$.

Lemma 3.1. *Congruence operators on $\mathbb{M}_{m,n}(\mathbb{K})$ are linear, are bijective, and preserve all maximal column ranks.*

Proof. Linearity follows from the linearity of matrix multiplication. The others follow from Lemma 2.3. ■

Hereafter, we shall adopt the convention $m \leq n$, and the set of matrices of maximal column rank 1 over a fixed chain semiring $\mathbb{K}$ is denoted by C_1. Two maximal column rank 1 matrices A, B are said to be *separable* if there is a matrix X with $mc(X) = 1$ such that either $1 = mc(A+X) < mc(B+X)$ or $1 = mc(B+X) < mc(A+X)$. In this case, X is said to *separate* A from B. Using Theorem 2.2 we can apply some results in [5] for column rank 1 matrices to those for maximal column rank 1 matrices. Thus we obtain the following Theorem 3.1 by the analogue proof of that in [5].

Theorem 3.1. *Distinct maximal column rank* 1 *matrices are separable if and only if at least one of them is not a scalar multiple of J.*

The symbol $\leq$ is read entrywise, i.e. $X \leq Y$ if and only if $x_{ij} \leq y_{ij}$ for all (i, j). We recall that the norm of a matrix A, $\|A\|$, is the maximum entry in A.

Lemma 3.2 ([2])**.** *If T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{K})$, $m > 1$, T preserves norm, and $A \leq T(A)$, then $T^q(A) = T^{mn-1}(A)$ for all $q \geq mn$.*

Lemma 3.3. *Let T be a linear operator on $\mathbb{M}_{m,n}(\mathbb{K})$ with $m > 1$. If T preserves norm and maximal column rank* 1 *but is not injective on C_1, then T reduces the maximal column rank of some matrix from $k(\geq 2)$ to* 1.

Proof. Since T is not injective on C_1, $T(A) = T(B)$ for some A, B in C_1 with $A \neq B$. If $A = \alpha J$ and $B = \beta J$, then $\alpha = \beta$ because T preserves norm, contradicting our assumption that $A \neq B$. Therefore by Theorem 3.1, some matrix X of maximal column rank 1 separates A from B. Say, $mc(X+A) = 1$ and $mc(X+B) = k \geq 2$. Since

$$T(X+B) = T(X) + T(B) = T(X) + T(A) = T(X+A),$$

T reduces the maximal column rank of $X+B$ from k to 1. ∎

We say that a linear operator T on $\mathbb{M}_{m,n}(\mathbb{K})$ *strongly preserves maximal column rank 1*, provided that $mc(X) = 1$ if and only if $mc(T(X)) = 1$ for $X \in \mathbb{M}_{m,n}(\mathbb{K})$.

Lemma 3.4. *If T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{K})$ with $m > 1$, and T strongly preserves maximal column rank 1, then T preserves norm.*

Proof. Let $A \in \mathbb{M}_{m,n}(\mathbb{K}), \alpha = \|A\|$ and $\beta = \|T(A)\|$; then $A = \alpha A$ and $\beta = \|T(A)\| = \|T(\alpha A)\| = \alpha\|T(A)\| \leq \alpha$. Suppose $\beta < \alpha$. Then for some (i,j), $a_{ij} = \alpha$. Let Y be the matrix whose entries are all α except for $y_{ij} = 0$. Then $\alpha J = A + Y$. So $mc(A+Y) = 1$. Since $mc(\beta A + Y) \geq 2$ by Lemma 2.5 and Lemma 2.6, but $mc(\beta A + Y) \leq 2$ by construction, we have $mc(\beta A + Y) = 2$. By the linearity of T and the definition of β, we have $T(\beta A) = \beta T(A) = T(A)$. Hence $T(\beta A + Y) = T(\beta A) + T(Y) = T(A) + T(Y) = T(A+Y) = \alpha T(J)$. So T reduces the maximal column rank of $\beta A + Y$ from 2 to 1, contrary to our hypothesis. Thus T preserves norm. ∎

Lemma 3.5. *Suppose T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{K})$ with $m > 1$. If T strongly preserves maximal column rank 1, then T permutes Γ, where $\Gamma = \{E_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$*

Proof. By Lemma 3.4, T preserves norm. Therefore by Lemma 3.3, T is injective on C_1. Suppose $T(E_{pq})$ is not in Γ for some (p,q). Now $T(E_{pq}) = \sum \tau_{ij} E_{ij}$, for some τ_{ij}. But $\|T(E_{pq})\| = 1$, so $\tau_{uv} = 1$ for some (u,v). Without loss of generality, we may assume that $(u,v) = (p,q)$, because if P, Q are permutation matrices, then the linear operator $X \rightarrow PT(X)Q$ preserves the maximal column rank that T preserves by Lemma 2.3 and the linear operator permutes Γ if and only if T does. Let $E = E_{pq}$. Then $E \leq T(E)$, so $E \neq T(E) \leq T^2(E) \leq \cdots \leq T^k(E) = T^{k+h}(E)$, where k is the least integer for which equality holds and $h \geq 0$ is arbitrary. By Lemma 3.2, we are assured that k exists and is less than mn. Let $B = T^{k-1}(E)$. Then $B \neq T(B)$ but $T(B) = T(T(B))$, despite the fact that B, $T(B)$ are both in C_1 and T is injective on C_1. This contradiction implies that T maps Γ into Γ. By injectivity, T permutes Γ. ■

Let $\mathbb{B}$ be the two element subsemiring $\{0, 1\}$ of $\mathbb{K}$, and α be a fixed member of $\mathbb{K}$, other than 1. For each x in $\mathbb{K}$ define $x^\alpha = 0$ if $x \leq \alpha$, and $x^\alpha = 1$ otherwise. Then the mapping $x \to x^\alpha$ is a homomorphism of $\mathbb{K}$ onto $\mathbb{B}$. Its entrywise extension to a mapping $A \to A^\alpha$ of $\mathbb{M}_{m,n}(\mathbb{K})$ onto $\mathbb{M}_{m,n}(\mathbb{B})$ preserves matrix sum and product and multiplication by scalars. We call A^α the *α-pattern* of A.

Example 3.1. For a nonzero nonunit $p \in \mathbb{K}$, consider

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & p & 1 \\ p & p & p \end{pmatrix}.$$

Then $mc(A) = 3$, because all the three columns of A are linearly independent. But $mc(A^t) = 2$ since the second column of A^t generates the third column of it. Consider $B = A \oplus 0_{m-3,m-3}$ for all $m \geq 3$. If T is a transposition operator over $\mathbb{M}_{m,m}(\mathbb{K})$, then $T(B) = B^t$ has maximal column rank 2 while $mc(B) = 3$. Thus a transposition operator does not preserve maximal column rank 3.

For our purpose, we write some known results as follows.

Lemma 3.6 (i) ([2]). *Suppose T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{K})$ with $m \geq 1$. Then T is bijective and preserves semiring rank 1 if and only if it is in the group of operators generated by congruence and transposition operators.*

(ii) ([4]) *Suppose T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$ with $m \geq 4$. Then T preserves maximal column ranks 1, 2 and 3 if and only if it is a congruence operator. Moreover the transposition operator on $\mathbb{M}_{m,m}(\mathbb{B})$ does not preserve maximal column rank 3 for $m \geq 4$.*

We say that an $m \times n$ matrix X is a *column matrix* if $X = \mathbf{x}(\mathbf{e}_i)^t$ for some $\mathbf{x} \in \mathbb{S}^m$ and $\mathbf{e}_i \in \mathbb{S}^n$, where $\mathbf{e}_i$ is the vector with 1 in the ith position and 0 elsewhere.

Theorem 3.2. *Suppose T is a linear operator on the $m \times n$ matrices over a chain semiring $\mathbb{K}$, where $m \geq 2$ and $n \geq 3$. If T strongly preserves maximal column rank 1, and it preserves maximal column rank 3, then T is a congruence operator.*

Proof. Let $\bar{\mathbb{M}} = \mathbb{M}_{m,n}(\mathbb{B})$. Lemma 3.5 and linearity imply that T maps $\bar{\mathbb{M}}$ into itself. Let $\bar{T}$ denote the restriction of T to $\bar{\mathbb{M}}$. From the definition of maximal column rank, the maximal column rank $mc_{\mathbb{B}}(X)$ of a member X of $\bar{\mathbb{M}}$ is at least $mc_{\mathbb{K}}(X)$, its maximal column rank as a member of $\mathbb{M}_{m,n}(\mathbb{K})$, because $\mathbb{B} \subseteq \mathbb{K}$. On the other hand, the mapping that takes a matrix A in $\mathbb{M}_{m,n}(\mathbb{K})$ to its 0-pattern A^0 in $\bar{\mathbb{M}}$ preserves matrix sums and multiplication by scalars. Hence $mc_{\mathbb{B}}(X) = mc_{\mathbb{K}}(X)$ for all X in $\bar{\mathbb{M}}$. Therefore $\bar{T}$ strongly preserves maximal column rank 1, and it preserves maximal column rank 3.

Case 1 ($m \geq 4$). Since $\bar{T}$ also permutes Γ by Lemma 3.5 and it strongly preserves maximal column rank 1, $\bar{T}$ must map a column matrix to either a column matrix or transpose of a column matrix if $m = n \geq 4$. For the latter case, $\bar{T}$ is a composition of a transposition operator and pre-multiplication by a permutation matrix. Since transposition operator cannot preserve maximal column rank 3 by Lemma 3.6 (ii), $\bar{T}$ must map a column matrix to a column matrix. Thus the linearity of $\bar{T}$ implies that $mc_{\mathbb{B}}(\bar{T}(X)) \leq mc_{\mathbb{B}}(X)$ for all X in $\bar{\mathbb{M}}$. In particular, $\bar{T}$ preserves maximal column rank 2. Hence $\bar{T}$ is a congruence operator on $\bar{\mathbb{M}}$ by Lemma 3.6 (ii). Then $\bar{T}(X) = UXV$ for some invertible matrices U of order m and V of order n. Notice that the matrices U, V are also invertible in $\mathbb{M}_{m,n}(\mathbb{K})$; in fact, they are just permutation matrices. Let $A \in \mathbb{M}_{m,n}(\mathbb{K})$. Then $T(A) = \sum a_{ij}T(E_{ij}) = \sum a_{ij}\bar{T}(E_{ij})$, because each E_{ij} is in $\bar{\mathbb{M}}$. Since $\bar{T}(E_{ij}) = UE_{ij}V$ for all i, j, by definition of congruence operator, the result follows directly from the linearity of matrix multiplication.

Case 2 ($n = 3$ and $2 \leq m \leq 3$). Theorem 2.2 guarantees that $\bar{T}$ strongly preserves column rank 1. Note that $mc_{\mathbb{B}}(X) = 3$ if and only if $c_{\mathbb{B}}(X) = 3$ by Corollary 2.1 and (2.1). Hence it preserves column rank 3, because if $c_{\mathbb{B}}(X) = 3$, then $3 = mc_{\mathbb{B}}(X) = mc_{\mathbb{B}}(\bar{T}(X)) = c_{\mathbb{B}}(\bar{T}(X))$. Also, $\bar{T}$ strongly preserves semiring rank 1 and it preserves semiring rank 3, by Theorem 2.1 (ii). If $r_{\mathbb{B}}(X) = 2$ for $X \in \bar{\mathbb{M}}$, then X can be factored as a sum of two matrices X_1 and X_2 in $\bar{\mathbb{M}}$ whose semiring ranks are 1, by (2.2). Thus $\bar{T}(X) = \bar{T}(X_1) + \bar{T}(X_2)$ has semiring rank two or less. Since $\bar{T}$ strongly preserves semiring rank 1, $r_{\mathbb{B}}(\bar{T}(X)) = 2$. That is, $\bar{T}$ preserves semiring rank 2. Therefore $\bar{T}$ is in the group of operators generated by congruence (and if $m = n = 3$, also the transposition) operators by Lemma 3.6 (i). Let

$A \in \mathbb{M}_{m,3}(\mathbb{K})$. Then $T(A) = \sum a_{ij}T(E_{ij}) = \sum a_{ij}\bar{T}(E_{ij})$, since each E_{ij} is in $\bar{\mathbb{M}}$. By similar argument as in case 1, there are permutation matrices U of order m and V of order n such that in the case $n = 3$ and $m = 2$, $T(A) = UAV$, while in the case $m = n = 3$, $T(A)$ is either UAV or UA^tV. However, since transposition operator does not preserve maximal column rank 3 by Example 3.1, we see that in fact, T must be a congruence operator. ∎

Theorem 3.3. *Suppose T is a linear operator on the $m \times n$ matrices over a chain semiring with $m \geq 2$ and $n \geq 3$. Then the following statements are equivalent:*

(i) *T preserves all maximal column ranks.*

(ii) *T strongly preserves maximal column rank 1 and it preserves maximal column rank* 3.

(iii) *T is a congruence operator.*

(iv) *T is bijective and preserves maximal column ranks* 1 *and* 3.

Proof. It is obvious that (i) implies (ii). Theorem 3.2 establishes that (ii) implies (iii). According to Lemma 3.1, (iii) implies (i) and (iv). If T satisfies (iv), then T is in the group of operators generated by congruence and transposition operators by Lemma 3.6 (i) and Theorem 2.2. Since the transposition operator does not preserve maximal column rank 3, T must be a congruence operator. Therefore, (iv) implies (iii). ∎

How necessary is it that $m \geq 2$ and $n \geq 3$? If $m \leq 2$ and $n \leq 3$, then a linear operator that preserves all maximal column ranks is the same as a linear operator that preserves all column ranks by Theorem 2.2. The characterizations of the column rank preservers were obtained in [5]. Thus we have characterizations of the linear operators that preserve the maximal column rank of matrices over a chain semiring and in particular, of fuzzy matrices.

Acknowledgement

The authors would like to thank the referee for his helpful comment on Example 2.1.

References

[1] L.B. Beasley and N.J. Pullman, *Semiring rank versus column rank*, Linear Algebra Appl. **101** (1988), 33–48.

[2] L.B. Beasley and N.J. Pullman, *Fuzzy rank-preserving operators*, Linear Algebra Appl. **73** (1986), 197–211.

[3] L.B. Beasley and N.J. Pullman, *Boolean rank-preserving operators and Boolean rank-*1 *spaces*, Linear Algebra Appl. **59** (1984), 55–77.

[4] S.G. Hwang, S.J. Kim and S.Z. Song, *Linear operators that preserve maximal column rank of Boolean matrices*, Linear and Multilinear Algebra **36** (1994), 305–313.

[5] S.Z. Song, *Linear operators that preserve column rank of fuzzy matrices*, Fuzzy Sets and Systems, **62** (1994), 311–317.

[6] S.Z. Song, S.D. Yang, S.M. Hong, Y.B. Jun and S.J. Kim, *Linear operators preserving maximal column ranks of nonbinary Boolean matrices*, Discussiones Math. - Gen. Algebra Appl., **20** (2000), 255–265.

Received 31 March 2001
Revised 27 August 2001
Revised 11 December 2001