F-TESTS FOR GENERALIZED LINEAR HYPOTHESES IN SUBNORMAL MODELS

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Abstract

When the measurement errors may be assumed to be normal and independent from what is measured a subnormal model may be used. We define a linear and generalized linear hypotheses for these models, and derive F-tests for them. These tests are shown to be UMP for linear hypotheses as well as strictly unbiased and strongly consistent for these hypotheses. It is also shown that the F-tests are invariant for regular transformations, possess structural stability and are almost strongly consistent for generalized linear hypothesis. An application to a mixed model studied by Michalskyi and Zmyślony is shown.

Keywords: F-tests, subnormal models, mixed models, invariance, UMP tests, third type error.

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1. Introduction

In many instances, measurement errors may be assumed to be normal and independent from what is measured. We are thus led to use, in these instances, subnormal models in which the observations vector Y^n is assumed to be the sum of two independent components: Z^n and e^n, symbolically denoted by $Z^n\,(i)\,e^n$. While e^n is a normally distributed error vector no restriction is imposed on the distribution of Z^n thus reducing the risk of third type error. These errors occur when a wrong model is chosen. After considering subnormal models we define linear and generalized linear hypotheses.

Next, we derive F-tests for these hypothesis and study their properties. The well known results on normal vectors we use may be found in Seber (1980). Our results lead to robust F-tests that maintain the good behaviour of these tests in fixed effects models. The search for robust F-tests has also been carried out by other authors, for instance see Agard and Birch (1992), Silvapulle (1992) and Rao (1993), Sutradhar and & Yue (1993). Besides this we tried to enlarge the class of models for which F-tests can be carried out trying to achieve this through a unified approach. Recent works by Michalsky and Zmyślony (1996) and (1999) lead to such a unified treatment under normality assumptions, which we now substitute by sub-normality.

2.   Models

In order to be able to derive F-tests we assume, besides $Z^n(i)e^n$, that $Z^n \epsilon \Omega^m$, with Ω^m a dimension m subspace of R^n, and that $e^n \sim N\left(0^n, \sigma^2 C\right)$ this is e^n normal with null mean vector and regular variance-covariance matrix $\sigma^2 C$, known up to σ^2. The nullity of this mean vector and C being regular may be considered as arising from the removal of systematic bias and linkage between errors. This is clear for the mean vector while, if C was singular it would have one null eigenvalue associated with an eigenvector α^n, so that $Var\left(\alpha^t e\right) = 0$ and $Pr\left(\alpha^t e = 0\right) = 1$. As to C being known, we point out that, when the measurement methods are well established, the corresponding error distributions are well known. We thus have

$$(1) \qquad\qquad\qquad Y^n = Z^n + e^n$$

with $Z^n(i)e^n$, $Z^n \epsilon \Omega^m$ and $e^n \sim N\left(0^n, \sigma^2 C\right)$. Let $U(C)$ be the family of matrices G such that $GCG^t = I_n$. Since C is symmetric there is P orthogonal such that PCP^t is a diagonal matrix $D(r_1, \cdots, r_n)$ with the eigenvalues $r_1, \cdots, r_n$ of C as principal elements. It may be shown, see Mexia (1989), that $r_j > 0$, $j = 1, \ldots, n$, and that

$$(2) \qquad\qquad G_\circ = D\left(r_1^{-\frac{1}{2}}, \ldots, r_n^{-\frac{1}{2}}\right) P \epsilon U(C);$$

$U(C)$ being also the family of matrices $P'G_\circ$ with P' orthogonal. Thus, if $G_1, G_2 \epsilon U(C)$, $G_2 G_1^{-1}$ will be orthogonal. Lastly, if $G \epsilon U(C)$, $GL^{-1} \epsilon U$

$\left(LCL^t\right)$. With $G\epsilon U\left(C\right)$, taking $Y'^n = GY^n, Z'^n = GZ^n, e'^n = Ge^n$ and $\Omega'^m = G\Omega^m$ we will have

(3) $$Y'^n = Z'^n + e'^n$$

where $Z'^n\left(i\right)e'^n$, $Z'^n\epsilon\Omega'^m$ and $e'^n \sim N\left(G0^n, G\left(\sigma^2C\right)G^t\right) = N\left(0^n, \sigma^2I_n\right).$

This new model is subnormal homoscedastic. Given $w^p \subset \Omega^m$ we have $w'^p = Gw^p \subset \Omega'^m$ and with $\overline{w} = w'^\perp \cap \Omega'$, we put

(4) $$U = \frac{1}{\sigma^2}\left\|Z'^n_{\overline{w}}\right\|^2$$

where $Z'^n_{\overline{w}}$ is the orthogonal projection of Z'^n on $\overline{w}$.

In what follows, we are going to derive $F-$tests for

(5) $$H_0\left(d\right) : \Pr\left(U \leq d\right) = 1.$$

Now $H_0\left(0\right)$ holds if and only if $Z'^n\epsilon w'$ or equivalently, if $Z^n\epsilon w$.

Thus $H_0\left(0\right)$ will be a linear hypothesis of the type considered in normal fixed effects models, see Scheffé (1959), while $H_0\left(d\right)$ will be a generalized linear hypothesis.

3.   Test derivation

Let the line vectors of A and M constitute an orthonormal basis for $\Omega'^{\perp n-m}$ and $\overline{w}^{m-p}$. Then, with $B = \begin{bmatrix} A \\ M \end{bmatrix}$, the line vectors of B will constitute an orthonormal basis for $w'^{\perp n-p}$ and the orthogonal projection matrices on these subspaces will be $Q\left(\Omega'^\perp\right) = A^tA$, $Q\left(\overline{w}\right) = M^tM$ and $Q\left(w'^\perp\right) = B^tB$. Since orthogonal projection matrices are symmetrical and idempotent, for any $v^n\epsilon R^n$ we will have $\left\|v^n_{\Omega'^\perp}\right\|^2 = \left\|Q\left(\Omega'^\perp\right)v^n\right\|^2 = v^tQ\left(\Omega'^\perp\right)^tQ\left(\Omega'^\perp\right)v = v^tQ\left(\Omega'^\perp\right)v = v^tA^tAv = \left\|Av^n\right\|^2$ as well as $\left\|v^n_{\overline{w}}\right\|^2 = \left\|Mv^n\right\|^2$ and $\left\|v^n_{w'^\perp}\right\|^2 = \left\|B\,v^n\right\|^2$.

We write $\left\|V^s\right\|^2 \sim \sigma^2\chi^2_{s,\delta}$, when $\left\|V^s\right\|^2$ is the product by σ^2 of a chi-square with s degrees of freedom and noncentrality parameter δ, while $f_{(1-q,m-p,n-m,\delta)}$ will be the quantile, for probability $1-q$, of $F-$distribution $F\left(z \mid m-p, n-m, \delta\right)$ with $m-p$ and $n-m$ degrees of freedom and non-centrality parameter δ. If $\delta = 0$ we write simply $\chi^2_s, f_{(1-q,m-p,n-m)}$ and $F\left(z \mid m-p, n-m\right)$.

Let us establish:

Proposition 1. *We have* $\|Y_{\overline{w}}'^{n}\|^{2}$ *(i)* $\|Y_{\Omega^{\perp}}'^{n}\|^{2} \sim \sigma^{2}\chi_{n-m}^{2}$. *When* $Z'^{n} = b^{n}\epsilon\Omega'$, *the   conditional distribution of*

$$\Im = \frac{n-m}{m-p} \frac{\|Y_{\overline{w}}'^{n}\|^{2}}{\|Y_{\Omega'\perp}'^{n}\|^{2}}$$

will be $F\left(z \mid m-p, n-m, u\right)$ *with* $u = \frac{1}{\sigma^{2}}\|b_{\overline{w}}^{n}\|^{2}$ *and* $U = u$. *If* $F^{\circ}\left(u\right)$ *is the   distribution of* U, *the unconditional   distribution of* $\Im$ *will be* $F\left(z \mid m-p, n-m\right) = \int_{0}^{+\infty} F\left(z \mid m-p, n-m, u\right) dF^{\circ}\left(u\right)$ *the distribution degenerating into* $F\left(z \mid m-p, n-m\right)$ *whenever* $H_{0}\left(0\right)$ *holds.*

Proof. Since the line vectors of A constitute an orthonormal basis for $\Omega'^{\perp}$ and $Z'^{m}\epsilon\Omega'$, $AY'^{n} = Ae'^{n}$. Besides this,

$$\left[\left(Ae'^{n}\right)^{t} \ \left(Me'^{n}\right)^{t}\right]^{t} = Be'^{n} \sim N\left(B0^{n}, B\left(\sigma^{2}I_{n}\right) B^{t}\right) = N\left(0^{n}, \sigma^{2}I_{n-p}\right)$$

so that $Ae'^{n} \sim N\left(0^{n-m}, \sigma^{2}I_{n-m}\right)$ *(i)* $Me'^{n} \sim N\left(0^{m-p}, \sigma^{2}I_{m-p}\right)$ and that

$$\left\|Y_{\Omega'\perp}'^{n}\right\|^{2} = \left\|AY'^{n}\right\|^{2} = \left\|Ae'^{n}\right\|^{2} \sim \sigma^{2}\chi_{n-m}^{2}.$$

Thus $e_{\Omega'\perp}'^{n} = A^{t}Ae'^{n}$ *(i)* $e_{\overline{w}}'^{n} = M^{t} Me'^{n}$ and $Y_{\Omega'\perp}'^{n} = e_{\Omega'\perp}'^{n}$ *(i)* $Y_{\overline{w}}'^{n} = Z_{\overline{w}}'^{n} + e_{\overline{w}}'^{n}$, since Z' *(i)* e'^{n}. Going over to the second part of the thesis, since Z'^{n} *(i)* e'^{n}, whenever $Z'^{n} = b^{n}\epsilon\Omega'$, $Y'^{n} \sim N\left(b^{n}, \sigma^{2}I_{n}\right)$ *(i)* and so,

$$MY'^{n} \sim N\left(Mb^{n}, M\left(\sigma^{2}I_{n}\right)M^{t}\right) = N\left(Mb^{n}, \sigma^{2}I_{m-p}\right)$$

so that

$$\left\|Y_{\overline{w}}'^{n}\right\|^{2} = \left\|MY'^{n}\right\|^{2} \sim \chi_{m-p,u}^{2} \ \text{ with } \ u = \frac{1}{\sigma^{2}}\|Mb^{n}\|^{2} = \frac{1}{\sigma^{2}}\|b_{\overline{w}}^{n}\|^{2}$$

and $U = u$ Since both chi-squares are independent, $\Im$ has conditional distribution $F\left(z \mid m-p, n-m, u\right)$. Thus, deconditioning in order to U, we get

$$F\left(z \mid m-p, n-m, F^{\circ}\right) = \int_{0}^{+\infty} F\left(z \mid m-p, n-m, u\right) dF^{\circ}\left(u\right)$$

and, when $H_0\left(0\right)$ holds, this distribution reduces to $F\left(z \mid m-p, n-m\right)$, since $\Pr\left(U=0\right)=1$.

Corollary 2. *When $H_0\left(d\right)$ holds*

$$F\left(z \mid m-p, n-m, d\right) \leq F\left(z \mid m-p, n-m, F^{\circ}\right).$$

Proof. Since, see Mexia (1989), $F\left(z \mid m-p, n-m, u\right)$ decreases with u the thesis follows from the expression of $F\left(z \mid m-p, n-m, F^{\circ}\right)$.

When we use the statistic $\Im$ and the critical region $]c, +\infty)$ we have test $\left(\Im, c\right)$. Since, when $\Pr\left(U=d\right)=1, H_0\left(d\right)$ holds, the significance level for this hypothesis will be $1 - F\left(c \mid m-p, n-m, d\right)$. Thus if we want a q-significance level test we must take $c = f_{(1-q, m-p, n-m, d)}$. An interesting point is that $\left(\Im, c\right)$ may be used in testing $H_0\left(d\right)$ for any d, its power depending only on c and F°. According to proposition 1 we have

$$(6) \qquad \beta_c\left(F^{\circ}\right) = 1 - F\left(c \mid m-p, n-m, F^{\circ}\right) = \int_{0}^{+\infty} \beta_c\left(u\right) dF^{\circ}\left(u\right)$$

with $\beta_c\left(u\right) = 1 - F\left(c \mid m-p, n-m, u\right)$. Since

$$\left\|Y_{\overline{w}}^{\prime n}\right\|^2 = \left\|Y_{\overline{w}^{\prime\perp}}^{\prime n}\right\|^2 - \left\|Y_{\Omega^{\prime\perp}}^{\prime n}\right\|^2, \text{ with } S_{w^{\prime}} = \left\|Y_{\overline{w}^{\prime\perp}}^{\prime n}\right\|^2 \text{ and } S_{\Omega^{\prime}} = \left\|Y_{\Omega^{\prime\perp}}^{\prime n}\right\|^2,$$

we get

$$(7) \qquad \Im = \frac{n-m}{m-p} \frac{\left\|Y_{\overline{w}^{\prime\perp}}^{\prime n}\right\|^2 - \left\|Y_{\Omega^{\prime\perp}}^{\prime n}\right\|^2}{\left\|Y_{\Omega^{\prime\perp}}^{\prime n}\right\|^2} = \frac{n-m}{m-p} \frac{S_{w^{\prime}} - S_{\Omega^{\prime}}}{S_{\Omega^{\prime}}}$$

and so we obtained, see Scheffé (1959), the canonical form of the F test statistic for fixed effect models. Thus, when it may be assumed that $C = I_n$, the usual algorithms for computing the test statistic may be used.

We now establish:

Proposition 3. $\Im$ *and* U *do not depend on which matrix* $G\epsilon U\left(C\right)$ *is used and are unchanged by regular linear transformations.*

Proof. If $G_1, G_2\epsilon U\left(C\right)$ we saw that $P = G_2 G_1^{-1}$ is orthogonal. Taking $Y_i'^n = G_i Y^n, Z_i'^n = G_i Z^n, \Omega_i'^n = G_i \Omega^m$ and $w_i'^p = G_i w^p, i = 1, 2,$ we get $Y_2'^n = PY_1'^n, Z_2'^n = PZ_1'^n \Omega_2'^n = P\Omega_1'^m$ and $w_2'^p = Pw_1'^p$ as well as, see Mexia (1989),

$$\left\|\left(Y_2'^n\right)_{\Omega_2'^\perp}\right\|^2 = \left\|\left(Y_1'^n\right)_{\Omega_1'^\perp}\right\|^2 \quad \text{and} \quad \left\|\left(Y_2'^n\right)_{w_2'^\perp}\right\|^2 = \left\|\left(Y_1'^n\right)_{w_1'^\perp}\right\|^2.$$

Thus, with $\overline{w_i} = w_i'^\perp \cap \Omega_i', i = 1, 2,$, we get

$$\left\|\left(Y_2'^n\right)_{\overline{w}_2}\right\|^2 = \left\|\left(Y_1'^n\right)_{\overline{w}_1}\right\|^2 , \left\|\left(Y_2'^n\right)_{\Omega_2'^\perp}\right\|^2 = \left\|\left(Y_1'^n\right)_{\Omega_1'^\perp}\right\|^2$$

$$\text{and} \quad \left\|\left(Z_2'^n\right)_{\overline{w}_2}\right\|^2 = \left\|\left(Z_1'^n\right)_{\overline{w}_1}\right\|^2$$

and the first part of the thesis is established. Let L be the matrix of a regular linear transformation. Taking $Y^{+n} = LY^n$, $Z^{+n} = LZ^n$ and $e^{+n} = Le^n$, we get $Y^{+n} = Z^{+n} + e^{+n}$, with $Z^{+n}\left(i\right)$ $e^{+n} \sim N\left(L0^n, L\left(\sigma^2 C\right) L^t\right) = N\left(0^n, \sigma^2 LCL^t\right)$. Now, if $G^+\epsilon U\left(LCL^t\right)$ and, $G\epsilon U\left(C\right)$, we will have $GL^{-1}G^+\epsilon U\left(LCL^t\right)$ and, according to the first part of the thesis, $\Im$ and U are the same whichever of these two matrices is used. We now have only to point out that using GL^{-1} we get $Y'^{+n} = GL^{-1}Y'^{+n} = Y'^n$ as well as $Z'^{+n} = GL^{-1}Z'^{+n} = Z'^n$.

This proposition shows tests $\left(\Im, c\right)$ to be invariant for regular linear transformations, thus for a wider class of linear transformations than is usually considered for F-tests, for instance see Lehmann (1959). The reason for these enhanced invariance properties is that, instead of the usual assumption of homocedasticity, the variance-covariance matrix of e^n was only required to be regular.

4. Test power

Let $V\left[V\right]$ be the family of tests for $H_0\left(d\right)$ whose conditional power, given $Z'^n = b^n\epsilon\Omega'$, is a [an increasing] function of $u = \frac{1}{\sigma^2}\left\|b_{\overline{w}}^n\right\|^2$. The class V

is similar to the class V_0 of tests in fixed effects models whose power is a

function of a noncentrality parameter δ. We point out that $\Im$ tests are UMP in class V_0. We now extend this property to subnormal models deriving the:

Proposition 4. *The $(\Im, c)$ tests for $H_0(0)$ are strictly unbiased and UMP in class V.*

Proof. Since, see Mexia (1989), $\beta_c(u)$ increases with u, the first part of the thesis follows from $H_0(0)$ holding when $\Pr(U = 0) = 1$. Let now $\overline{\beta}(u)$ be the conditional power for a test in V. If $\overline{\beta}(0) = q$ this test will have significance level q for $H_0(0)$ the same as $(\Im, c^0)$ with $c^0 = f_{(1-q,m-p,n-m)}$. Moreover, if $\overline{\beta}(u) \leq \beta_{c^0}(u)$ did not hold for $u = \delta$ we could use the statistic and critical region of this test for, in fixed effects models, obtaining a test with higher power for δ than the F-test of the same significance level which, as we know, is impossible. To complete the proof we have only to point out that $\overline{\beta}(F_0) = \int_0^{+\infty} \overline{\beta}(u) \, dF^0(u) \leq \int_0^{+\infty} \beta_{c^0}(u) \, dF^0(u) = \beta_{c^0}(F^0)$.

Proposition 5. *If a test in V', with conditional power $\overline{\beta}(u)$, has the same significance level for $H_0(d)$ and $(\Im, c)$, we have $\overline{\beta}(0) \geq \beta_c(u)$.*

Proof. Since $\overline{\beta}(u)$ and $\beta_c(u)$ increase with u and $H_0(d)$ holds when $\Pr(U = d) = 1$ we will have $\overline{\beta}(d) = \beta_c(d)$ since this is the significance level for both tests. Now, if $\overline{\beta}(0) < \beta_c(0)$, there would be $c' > c$ such that $\overline{\beta}(0) = \beta_{c'}(0)$ and so $\overline{\beta}(d) \leq \beta_{c'}(d) < \beta_c(d)$ which is impossible. The thesis follows from this impossibility.

5.   Other properties

Given the observations vector Y^n, statistics $\Im$ and U depend on C, Ω and ω. The limits, in this section, are taken for $N \longrightarrow \infty$. To show that this dependency is continuous we write $W_N \longrightarrow W$, with $W_N = [w_{N,i,j}]$ and $W = [w_{i,j}]$ both $n \times n$ matrices, whenever $w_{N,i,j} \longrightarrow w_{i,j} i = 1, \cdots, n, j = 1, \cdots, n$. Likewise we will have $\nabla_N \longrightarrow \nabla$, with ∇_N and ∇ subspaces, if $Q(\nabla_N) \longrightarrow Q(\nabla)$.

When $C_N \longrightarrow C$, there exists, see Vaquinhas and Mexia (1995), $G_N \epsilon U(C_N)$ and $G \epsilon U(C)$ such that $G_N \longrightarrow G$, thus

$$Y_N'^n = G_N Y^n \longrightarrow Y'^n = G Y^n$$

and $Z_N'^n = G_N Z^n \longrightarrow Z'^n = GZ^n$. And, if $\Omega_N^m \longrightarrow \Omega^m$ and $\omega_N^p \longrightarrow \omega^p$, we will also have

$$\Omega_N'^m = G\Omega_N^m \longrightarrow \Omega'^m = G\Omega^m$$

and

$$\omega_N'^p = G_N \omega_N^p \longrightarrow \omega'^p = G\omega^p$$

as well as $\Omega_N'^\perp \longrightarrow \Omega'^\perp$ and

$$\overline{\omega}_N = \left(\omega_N'^\perp \bigcap \Omega_N' \right) \longrightarrow \overline{\omega} = \left(\omega'^\perp \bigcap \Omega' \right).$$

Now, when $v_N^n \longrightarrow v^n$ and $\nabla_N \longrightarrow \nabla$, we have

$$\left\| (v_N^n) \nabla_N \right\|^2 \longrightarrow \left\| (v^n)_\nabla \right\|^2.$$

We have thus established:

Proposition 6. *If* $C_N \longrightarrow C, \Omega_N^m \longrightarrow \Omega^m$ *and* $\omega_N^p \longrightarrow \omega^p$, *then*

$$\Im_N = \frac{n-m}{m-p} \frac{\left\| (Y_N'^n)_{\overline{w}_N} \right\|}{\left\| (Y_N'^n)_{\Omega_N'^\perp} \right\|^2} \longrightarrow \Im = \frac{n-m}{m-p} \frac{\left\| (Y_{\overline{w}}'^n) \right\|^2}{\left\| (Y_{\Omega'^\perp}'^n) \right\|^2},$$

$$U = \frac{1}{\sigma^2} \left\| (Z_N'^n)_{\overline{\omega}_N} \right\|^2 \longrightarrow U = \frac{1}{\sigma^2} \left\| Z_{\overline{\omega}}'^n \right\|^2.$$

From this proposition we conclude that $(\Im, c)$ tests have structural stability. The main relevance of this result rests in C being, in most cases, only approximately known. Structural stability ensures robustness of $(\Im, c)$ tests. We now consider what happens when information increases. We start by establishing the:

Lemma 7. *If the iid vectors* $X_j^2 = \begin{bmatrix} X_{1,j} \\ X_{2,j} \end{bmatrix}, j = 1, \cdots,$ *have mean vector*

$\mu^2 = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ *and variance-covariance matrix* $K = \begin{bmatrix} \sigma_1^2 & \sigma_{2,1} \\ \sigma_{2,1} & \sigma_2^2 \end{bmatrix},$ *the distribu-*
tion of

$$Z_N = \sqrt{N} \left[\frac{g}{h} \frac{N^{-1} \sum\limits_{j=1}^{N} X_{1,j}}{N^{-1} \sum\limits_{j=1}^{N} X_{2,j}} - \frac{g}{h} \frac{\mu_1}{\mu_2} \right]$$

converges uniformly towards $N\left(0, \bar{\sigma}^2\right)$ *with*

$$\bar{\sigma}^2 = \frac{g^2}{h^2} \left(\frac{\sigma_1^2}{\mu_1^2} - \frac{2\mu_2}{\mu_1 \mu_1^2} \sigma_{1,2} + \frac{\mu_2^2}{\mu_1^2} \sigma_2^2 \right),$$

whenever $\mu_2 \neq 0$, *and then, with* $0 < r < \frac{1}{2}$,

$$\Pr \left[\left| \frac{g}{h} \frac{\sum_{j=1}^{N} X_{1,j}}{\sum\limits_{j=1}^{N} X_{2,j}} - \frac{g}{h} \frac{\mu_1}{\mu_2} \right| < \frac{1}{N^r} \right] \longrightarrow 1.$$

Proof. According to the central limit theorem, the limit distribution of $\sqrt{N}(\frac{1}{N}\sum_{j=1}^{N} X_j^2 - \mu^2)$ is $N\left(0^2, K\right)$ so that, see Rao (1952), the limit distribution of

$$Z_N = \sqrt{N} \left[\frac{g}{h} \frac{\sum_{j=1}^{N} X_{1,j}}{\sum\limits_{j=1}^{N} X_{2,j}} - \frac{g}{h} \frac{\mu_1}{\mu_2} \right] \quad \text{will be } N\left(0, \sigma^2\right)$$

whenever $\mu_2 \neq 0$. Since $N\left(0, \sigma^2\right)$ is continuous the distribution of Z_N will, due to Polya's theorem, see Fisz (1963), converge uniformly to $N\left(0, \sigma^2\right)$ and so

$$
\Pr\left(| \frac{g}{h} \frac{\sum_{j=1}^{N} X_{1,j}}{\sum\limits_{j=1}^{N} X_{2,j}} - \frac{g}{h} \frac{\mu_1}{\mu_2} | < \frac{1}{N^r} \right) = \Pr\left(N^{\frac{1}{2}-r} < Z_N < N^{\frac{1}{2}-r} \right) \longrightarrow 1
$$

and the proof is complete.

We now establish:

Proposition 8. *If the $Y_1'^n$ are iid and the variance $\bar{\sigma}^2$ of the $X_{1,j} = \frac{1}{\sigma^2}\left\|(Y_j'^n)\overline{w}\right\|^2$ is defined, with*

$$
\Im_N = \frac{n-m}{m-p} \frac{\sum\limits_{j=1}^{N} \|(Y'^n)\overline{w}\|^2}{\sum\limits_{j=1}^{N} \|(Y'^n)_{\Omega'\perp}\|^2} \quad \text{and} \quad c_N = \frac{m-p+d}{n-m} + \frac{1}{N^r}
$$

we have a succession $\{(\Im_N, c_N)\}$ of tests for $H_0(d)$ whose probabilities of errors of first type and second type, for alternatives such that the mean value of the variable $X_{1,j}$ exceeds $m - p + d$, tend to zero.

Proof. Taking $X_{2,j} = \frac{1}{\sigma^2}\|(Y'^n)_{\Omega'\perp}\|^2$ we have $X_{1,j}(i)X_{2,j} \sim \sigma^2\chi_{n-m}^2$ $j = 1, \cdots$, so that the conditions of Lemma 7 hold if and only if $\bar{\sigma}^2$ is defined in which case we have $\sigma_{1,2} = \sigma_{2,1} = 0, \sigma_2^2 = 2(n-m)$ and $\mu_2 = n - m$. Moreover, when $U = u, X_{1,j} \sim \chi_{m-p,u}^2$ with mean value $m - p + u$ so that, when $H_0(d)$ holds, $\mu_1 < m - p + d$. The rest of the proof is straightforward since

$$
\frac{\sum\limits_{j=1}^{N} \left\|(Y_j'^n)\overline{w}\right\|^2}{\sum\limits_{j=1}^{N} \left\|(Y_j'^n)_{\Omega'\perp}\right\|^2} = \frac{\sum\limits_{j=1}^{N} X_{1,j}}{\sum\limits_{j=1}^{N} X_{2,j}}
$$

When we are testing $H_0(0)$ and σ_1^2 is defined we have, for all alternatives, $\mu_1 > m - p$ and, with $c_N = \frac{m-p}{n-m} + \frac{1}{N^r}$, the probabilities of first and second type errors for the tests $\{(\Im_N, c_N)\}$ will tend to zero. Following Tiago de Oliveira (1980) and (1982) we conclude that the F- tests are strongly consistent for $H_0(0)$ and almost strongly consistent for $H_0(d)$.

6.   Mixed models

We will write the sum of subspaces as $\bigcup_{i=1}^r \triangle_i$ and $\underline{R}(B)$ will be the range space of matrix B. In mixed models Z^n will have mean vector $\mu^n = X\beta^k$ and $Var(Z^n) = \sum_{i=1}^a \theta_i V_i$, and $e^n \sim N(0^n, \sigma^2 I_n)$. Thus, with $\nabla_\circ = \underline{R}(X)$ and $\nabla_i = \underline{R}(V_i), i = 1, \cdots, a$, we have $\mu^n \epsilon \nabla_\circ$ and $\underline{R}(Var(Z^n)) = \sum_{i=1}^a \nabla_i$, so that

$$(8) \qquad\qquad Z^n \epsilon \Omega^m = \bigcup_{i=0}^a \nabla_i$$

Given a proper subspace $\triangle$ of $\nabla_\circ$ such that $\overline{\omega}_\circ = \triangle^\perp \cap \nabla_\circ$ is orthogonal to $\underline{R}(Var(Z^n))$ we will have $Y_{\overline{\omega}_\circ}^n = \mu_{\overline{\omega}_\circ}^n + e_{\overline{\omega}_\circ}^n$ and $H_{\circ,\circ,}\mu^n \epsilon \triangle$ can be rewritten as a hypothesis on an estimable vector.

To test $H_{\circ,\circ,}$ against $H_{\circ,1} : \mu^n \notin \triangle$ we can use the test statistic

$$(9) \qquad\qquad \Im_\circ = \frac{n-m}{m-p_\circ} \frac{\left\| Y_{\overline{\omega}_\circ}^n \right\|^2}{\left\| Y_{\Omega^\perp}^n \right\|^2}$$

where $m - p_\circ = \dim(\overline{\omega}_\circ)$. Statistic $\Im_\circ$ will have distribution $F(z \mid m - p_\circ, n - m, \delta)$, with $\delta = \frac{1}{\sigma^2} \left\| \mu_{\overline{\omega}_\circ}^n \right\|^2$. Since $\Im_\circ$ depends on Z^n through μ^n this test will behave as in a fixed effects model, being UMP in the family of tests whose power is a function of δ. Besides this, we can rewrite the hypothesis as $H_{\circ,\circ,} : \theta_\circ = 0$ and $H_{\circ,1,} : \theta_\circ > 0$, with $\theta_\circ = \left\| \mu_{\overline{\omega}}^n \right\|^2$, and

$$\widetilde{\theta}_\circ = \|Y_{\overline{\omega}}^n\|^2 - \frac{m-p_\circ}{n-m} \|Y_{\Omega^\perp}^n\|^2 = Y^{nt} \left[Q(\overline{\omega}) - \frac{m-p_\circ}{n-m} Q(\Omega^\perp) \right] Y^n$$

will be a quadratic unbiased estimator. The matrices for the positive and negative parts of this estimator will be $K_\circ^+ = Q(\overline{w}_\circ)$ and $K_\circ^- = \frac{m-p_\circ}{n-m}Q(\Omega^\perp)$, so that

$$(10) \qquad \Im_\circ = \frac{Y^{nt}K_\circ^+Y^n}{Y^{nt}K_\circ^-Y^n}.$$

Going over to the hypothesis on variance components let the

$$(11) \qquad \overline{\omega}_j = \left(\underset{i\neq j}{\cup}\nabla_i\right)^\perp \cap \Omega; j = 1, \cdots, a$$

have dimension $m - p_j, j = 1, \cdots, a$. When $m - p_j > 0, Z_{\overline{\omega}_j}^n$ will have a null mean vector and $Var(Z_{\overline{\omega}_j}^n) = \theta_j W_j$ with $W_j = Q(\overline{\omega}_j)V_jQ(\overline{\omega}_j)^t$. Thus $U_j = \frac{1}{\sigma^2}\|Z_{\overline{\omega}_j}^n\|^2$ is not identically null if and only if $\theta_j > 0$ and we can use

$$(12) \qquad \Im_j = \frac{n-m}{m-p_j}\frac{\left\|Y_{\overline{\omega}_j}^n\right\|^2}{\left\|Y_{\Omega^\perp}^n\right\|^2}$$

to test $H_{j,\circ}, : \theta_j = 0$ against $H_{j,1}, : \theta_j > 0$. Besides this, with $Q(\overline{\omega}_j) = M_j^t \, M_j$, and $c_j = Trace(W_j)$, $\left\|Y_{\overline{\omega}}^n\right\|^2 = \left\|M_jY^n\right\|^2$ will have mean value $\theta_j c_j + (m - p_j)\sigma^2$. Thus we have the quadratic unbiased estimator

$$(13) \qquad \widehat{\theta}_j = Y^{nt}\left(\frac{1}{c_j}Q\left(\overline{\omega}_j\right) - \frac{m-p_j}{c_j\left(n-m\right)}Q\left(\Omega^\perp\right)\right)Y^n$$

and, with $K_j^+ = \frac{1}{c_j}Q\left(\overline{\omega}_j\right)$ and $K_j^- = \frac{m-p_j}{c_j\left(n-m\right)}Q\left(\Omega^\perp\right)$, we get

$$(14) \qquad \Im_j = \frac{Y^{nt}K_j^+Y^n}{Y^{nt}K_j^-Y^n}.$$

As final remarks we point out that the results on the test power given above hold for the tests presented in this section, and that the use of the positive and negative parts of quadratic estimators in deriving the test statistics was introduced by Michalski and Zmyślony (1996) and (1999).

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