Discussiones Mathematicae
Probability and Statistics 21 (2001) 11–20

TESTING ON THE FIRST-ORDER AUTOREGRESSIVE MODEL WITH CONTAMINATED EXPONENTIAL WHITE NOISE FINITE SAMPLE CASE

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Abstract

The testing problem on the first-order autoregressive parameter in finite sample case is considered. The innovations are distributed according to the exponential distribution. The aim of this paper is to study how much the size of this test changes when, at some time k, an innovation outlier contaminant occurs. We show that the test is rather sensitive to these changes.

Keywords: autoregressive model, exponential distribution, outlier, test.

2000 Mathematics Subject Classification: 62F11, 62M10.

1. Introduction

Consider the following autoregressive model

$$Y_t = \rho Y_{t-1} + \varepsilon_t \qquad t = \ldots, -1, 0, 1, \ldots \tag{1}$$

where the ε_t's are iid and distributed according to exponential distribution, i.e., the density of ε_t is

$$f_{\varepsilon_t}(x) = e^{-y}, \quad y > 0.$$

This model is useful for modelling a wide range of phenomena which do not allow negative values (see, for example, Gaver and Levis, 1980).

Many authors have studied this model. Bell and Smith (1986) studied the estimating and testing problem on the parameter ρ. Turkman (1990) proposed a bayesian estimator of ρ for the same model.

Now, suppose that we observe the model

$$X_t = \rho X_{t-1} + \varepsilon_t + \delta\, \delta_{t,k} \qquad 0 < k \leq n \qquad n \quad \text{fixed} \tag{2}$$

where

$$\delta_{t,k} = \begin{cases} 1 & \text{if} \quad t = k \\ 0 & \text{if} \quad t \neq k \end{cases}$$

instead of (1). δ is a known magnitude of contamination of model (1) which occurs at $t = k$.

This process, called innovation oultier (IO) model, has been proposed for the first time by Fox (1972).
Assume that all what we observe is the segment of observations

$$x_0, x_1, x_2, \ldots, x_n \qquad n \quad \text{fixed} \tag{3}$$

and $X_0 = Y_0$ is distributed according to an exponential distribution of parameter $1 - \rho$. When $\delta = 0$, the process (X_t) is mean stationary. This is not true for $\delta \neq 0$. We want to test

$$H_0 : \rho = \rho_0 \qquad \text{against} \qquad H_1 : \rho > \rho_0. \tag{4}$$

Note that, in our problem, the value of the parameter ρ is known. It takes the value ρ_0 under H_0 or another one $\rho_1 > \rho_0$ under H_1. Then, in the original model ($\delta = 0$), Bell and Smith (1986, p. 2274) proposed the following statistic

$$T(\rho) = 2 \sum_{t=1}^{n} (Y_t - \rho Y_{t-1}) \tag{5}$$

and showed that, under H_0, $T(\rho)$ is distributed as a chi-square (χ_2) with $2n$ degrees of freedom. Also, in their paper, the authors used the Kolmogorov-Smirnov statistic to obtain confidence intervals for ρ. However, assessing goodness of this statistic is rather difficult and then improving the criteria is needed.

In this paper, we propose to study the effect of the given contaminant δ on the size of test (4) when the observations are provided by model

(2) instead of model (1). Only the finite sample case is considered. A similar problem has been studied in ANOVA and Student t test in iid case (see, for example, Berkoun *et al.,* 1996).

2. THE SIZE OF THE TEST

Given a sample $X_1, X_2, \ldots, X_n$ from model (2), the hypothesis H_0 is tested at a significance level α. If the statistic $T(\rho)$ of Bell and Smith (1986) is used, we can find an appropriate critical value c such that the hypothesis is rejected if $t(\rho) > c$, i.e. $P_{H_0}(T(\rho) > c) = \alpha$.

Our aim is to find how much the size of the test changes if we observe the segment $x_0, x_1, \ldots, x_n$ instead of $y_0, y_1, \ldots, y_n$. When we observe segment (3) from model (2), the statistic $T(\rho)$ is rewritten as follows:

$$T^*(\rho) = 2\sum_{t=1}^{n}(X_t - \rho X_{t-1}). \tag{6}$$

The relation between (X_t) and (Y_t) is as follows

$$\begin{aligned} X_t &= Y_t \quad \forall t < k, \\ X_{k+j} &= Y_k + \rho^j \delta \quad \forall j = 0, 1, \ldots, n-k. \end{aligned}$$

Assume that $0 < k < n$. Then, when we observe (2) for a given δ, the statistic $T(\rho)$ becomes

$$T^*(\rho) = T(\rho) + 2\delta. \tag{7}$$

Note that the statistic $T^*(\rho)$ is independent of the position k. Hence, the test is not influenced by the position of the contaminant. Also, we know that (Saporta, 1990, p. 474)

$$P(\chi^2_{2n} > x) = e^{-x/2}\sum_{i=0}^{n-1}\frac{(x/2)^i}{i!}.$$

The size $s(\delta)$ of the test is equal to

$$(8)\quad P_{H_0}(T^*(\rho)>c)=P_{H_0}\,(T(\rho)>c-2\delta)=Exp\left\{\frac{c-2\delta}{2}\right\}\sum_{i=0}^{n-1}\frac{1}{i!}\left(\frac{c-2\delta}{2}\right)^i.$$

Since, $T(\rho)$ is a χ^2_{2n} when H_0 is true ($\rho=\rho_0$), the above formula has sense if only if

$$c-2\delta>0\Longrightarrow\delta<\frac{c}{2}.$$

Hence, in the following, we assume that $\delta\in]-\infty,c/2[$. Figure 1 presents the density function of the statistic $T^*(\rho)$ for some values of δ.

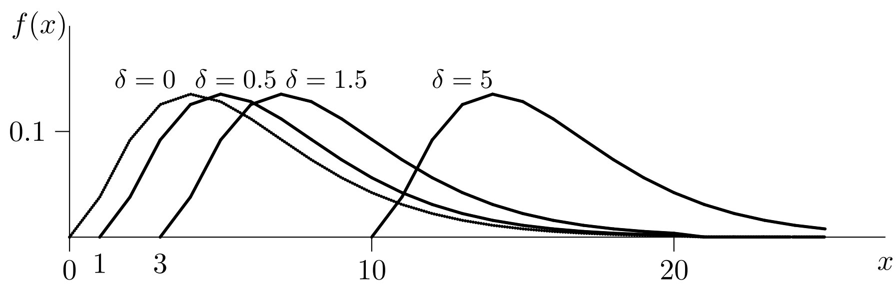

Figure 1. Density of $T^*(\rho)$ for $\delta=0,0.5,1.5,5$ and for $n=3$.

The study of the function $s(\delta)$ allows us to say that:

(i) The function $s(\delta)$ is non-decreasing.

(ii) The size vanishes when δ grows to $-\infty$.

(iii) The maximal size is obtained when $\delta=c/2$, i.e. $\alpha_{max}=s(c/2)=1$.

3. Numerical study

In what follows, we propose to make a numerical study of the effect of a contaminant δ on the size of the test. To get an idea of how much this contaminant increases the size of the test, some numerical results are given in Table 1 for $\delta=0.5,1.5,5$.

n	$\delta = 0.5$		
	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.1$
3	0.1406	0.0717	0.0148
10	0.1241	0.0634	0.0132
15	0.1206	0.0614	0.0127
20	0.1177	0.0600	0.0124

n	$\delta = 1.5$		
	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.1$
3	0.2656	0.1430	0.0318
10	0.1862	0.1000	0.0225
15	0.1705	0.0907	0.0203
20	0.1603	0.0852	0.0189

n	$\delta = 5$		
	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.1$
3	0.9957	0.8582	0.3387
10	0.5604	0.3733	0.1202
15	0.4539	0.2901	0.0887
20	0.3925	0.2453	0.0738

Table 1. Size of the test for $\delta = 0.5, 1.5, 5$ and $\alpha = 0.1, 0.05, 0.01$.

The variation of the size $s(\delta)$ when δ varies is given by the following figure

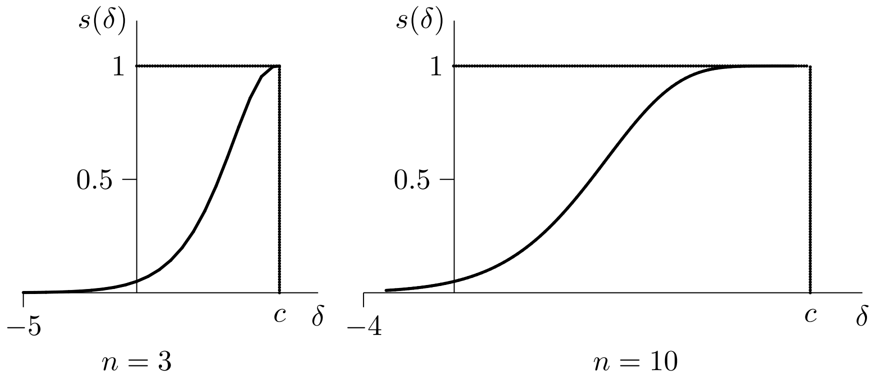

Figure 2. Variation of the size of test with δ, $\alpha = 0.05$.

The analysis of these results shows that, for a given n and α, there is a probability that we accept H_0 under model (1) and reject it under model (2). For a given α, and a given n, the probability to accept H_0 under (1) and reject H_0 under (2) is

$$\begin{aligned} p(c,\delta) &= P_{H_0}(T(\rho) < c \text{ and } T^*(\rho) > c) \\ &= P_{H_0}(c - 2\delta < T(\rho) < c) = P(\chi^2_{2n} > c - 2\delta) - \alpha. \end{aligned}$$

Hence,

$$p(c,\delta) = P_{H_0}(T(\rho) < c \text{ and } T^*(\rho) > c) = Exp\left\{\frac{c-2\delta}{2}\right\}\sum_{i=0}^{n-1}\frac{1}{i!}\left(\frac{c-2\delta}{2}\right)^i - \alpha.$$

In what follows, we present some exact values of $p(c,\delta)$.

	$\alpha = 0.1$		$\alpha = 0.05$	
n	$\delta = 0.5$	$\delta = 5$	$\delta = 0.5$	$\delta = 5$
3	0.040	0.895	0.021	0.808
20	0.017	0.292	0.010	0.195

Table 2. Exact values of $P_{H_0}(T(\rho) < c \text{ and } T^*(\rho) > c)$.

For example, we remark that, if $n = 3$ and $\delta = 5$, the probability that the acceptation of H_0 will change to its rejection is very high ($\simeq 0.8$).

4. Application

Simulated values of the probability $p(c, \delta)$ according to the situation given by Table 2 are given in the following:

	$\alpha = 0.1$		$\alpha = 0.05$	
n	$\delta = 0.5$	$\delta = 5$	$\delta = 0.5$	$\delta = 5$
3	0.040	0.895	0.020	0.811
20	0.018	0.294	0.009	0.191

Table 3. Simulated values of $p(c, \delta)$, 10000 repetitions.

First, note that these simulation results are very similar to exact values of $p(c, \delta)$ given in Table 2 which should be read as follows: for $\delta = 5$ and $n = 20$, if we repeat 10000 times a segment of process (2), the decision of acceptation of H_0 will change to rejection of the same hypothesis in 19.1 % of repetitions. We can clearly see this situation in the following example:

t	0	1	2	3	4	5	6	7	8	9	10
y_t	1.042	0.827	0.903	1.754	0.838	0.359	0.604	0.899	6.609	4.725	2.493
x_t	1.042	0.827	0.903	1.754	0.838	5.359	2.604	1.699	6.929	4.853	2.545

Table 4. $c = 31.41$, $t(\rho) = 25.174$, $t^*(\rho) = 35.174$.

The first line of the table is a segment of 10 observations of the process (Y_t) (without contamination). Here, the constant c is equal to 31.41 and $t(\rho) = 25.174$. Then, the hypothesis H_0 is accepted.

The second line contains observations obtained when the same segment is contaminated at $t = 5$ with $\delta = 5$. The value of $t^*(\rho) = 35.174$. This leads to rejection of H_0.

5. Simulation power study

In what follows, we propose to study the effect of an innovation outlier on the power of the test for a given δ and a given significance level α. The basic method is simulation procedure. To illustrate numerically this effect, Table 5 presents the values of the power of the test for $n = 3, 20, 50$ and for some values of δ. The true value of ρ used in this example is 0.4 and the significance level is $\alpha = 0.05$.

		ρ_0				
n	δ	0.0	0.2	0.4	0.6	0.8
3	0.00	0.251	0.130	0.050	0.012	0.003
	2.00	0.703	0.447	0.198	0.056	0.010
	4.00	0.999	0.944	0.603	0.201	0.036
	6.00	1.000	1.000	0.997	0.580	0.112
20	0.00	0.774	0.395	0.050	0.000	0.000
	2.00	0.900	0.574	0.103	0.000	0.000
	4.00	0.971	0.755	0.192	0.001	0.000
	6.00	0.995	0.890	0.316	0.002	0.000
50	0.00	0.973	0.679	0.050	0.000	0.000
	2.00	0.988	0.783	0.084	0.000	0.000
	4.00	0.996	0.859	0.129	0.000	0.000
	6.00	0.998	0.919	0.191	0.000	0.000

Table 5. Variation of the power of the test, $\alpha = 0.05$, 10000 runs.

We can remark that, for all values of n, - when $\rho_0 = 0.4$, we obatin exactly the size of the testand - when $\rho_0 = 0.4$ and $\delta = 0$, we find ourselves in the original model. Then, we obtain exactly the significance level 0.05 since

$\rho = 0.4$ is the true value used in these simulations. Also, for a given ρ_0, when δ grows, the power increases. If ρ_0 is close to zero, the power can reach quickly the maximal value one. But, if ρ_0 is close to one, the power is zero except for very small samples. Then, one can say that, in the presence of a single innovation outlier, the size and the power of the test change.

6. Conclusions

The contaminant δ can increase the size of the test up to 1 (when δ approaches $c\delta/2$) and can decrease this size down to 0 (when δ tends to $-\infty$). Also, the power of the test is influenced by this contaminant. Then, one can affirm that the statistic used in this test is sensitive to changes of the magnitude of innovations. This makes the criterion test rather useless and a robust statistic is needed. For further investigations, one can generalize the problem by considering innovations distributed according to exponential distribution with unknown parameter θ which should be estimated. In this case, it would be interesting to make adaptive procedures for estimation of the parameters and then testing the hypothesis $H_0 : \rho = \rho_0$. On the other hand, if a detection procedure allows us to decide that there is a contaminant at a position t_0, the problem is to estimate the magnitude δ of this contaminant.

Acknowledgements

The author is thankful to the referee for helpful comments and for some useful suggestions which enhanced the presentation.

References

[1] C.B. Bell and E.P. Smith, *Inference for non-negative autoregressive schemes*, Communication in Statistics, Theory and Methods, **15** (8) (1986), 2267–2293.

[2] Y. Berkoun, H. Fellag, M. Ibazizen and R. Zieliński, *Maximal size of the student and the Anova tests under exactly one contaminant*, Journal of Mathematical Sciences **81**, (5) (1996), 2900–2904.

[3] A.J. Fox, *Outliers in time series*, J. Roy. Stat. Soc. **34** (B) (1972), 350–363.

[4] D.P. Gaver and P.A.W. Lewis, *First-order autoregressive Gamma sequences and point process*, Adv. Appl. Prob. **12** (1980), 727–745.

[5] G. Saporta, *Probabilités, Analyses des données et Statistique*, Technip Ed. (1990).

[6] M.A.A. Turkman, *Bayesian analysis of an autoregressive process with exponential white noise*, Statistics **21** (4) (1990), 601–608.

Received 10 April 2000
Revised 7 May 2001