

A NOTE ON THE RELATIONS BETWEEN TWO TERNARY BALANCED BLOCK DESIGNS AND CHEMICAL BALANCE WEIGHING DESIGNS

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Abstract

The paper studied the problem of estimating of the weights of p objects in n weighings using a chemical balance weighing design under the restriction on the number of objects which can be placed on the right and left pans, respectively. Conditions under which the estimated weights are uncorrelated are given. The incidence matrices of two ternary balanced block designs which are used to construct chemical balance weighing designs satisfying these conditions are considered.

Keywords: chemical balance weighing design, ternary balanced block design.

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1. INTRODUCTION

Suppose we are given p objects to be weighed in n weighings on a chemical balance having no bias. The design matrix \mathbf{X} has elements $-1, 1$ or 0 if the object is placed on the right pan, left pan, or is not included in the particular weighing, respectively. The observational equations from the n weighings can be put in the form

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e},$$

where \mathbf{y} is the column vector of the observed weights, \mathbf{w} is the column vector of true weights and \mathbf{e} is the column vector of random errors with $E(\mathbf{e}) = \mathbf{0}_n$ and $E(\mathbf{e}\mathbf{e}') = \sigma^2\mathbf{I}_n$, where $\mathbf{0}_n$ is the $n \times 1$ vector with zero elements everywhere and \mathbf{I}_n denotes a unit matrix of order n , "E" stands for the expectation and \mathbf{e}' denotes the transpose of \mathbf{e} .

The normal equations estimating \mathbf{w} are of the form

$$(1.1) \quad \mathbf{X}'\mathbf{X}\hat{\mathbf{w}} = \mathbf{X}'\mathbf{y},$$

where (the solution) $\hat{\mathbf{w}}$ is the column vector of the least squares estimates of the true weights.

If the matrix \mathbf{X} is of full column rank the least squares estimates of the true weights are given by

$$\hat{\mathbf{w}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y},$$

and the variance-covariance matrix of $\hat{\mathbf{w}}$ is

$$\text{Var}(\hat{\mathbf{w}}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}.$$

The problem is to choose the matrix \mathbf{X} in such a way that the variance factors are minimized. A detailed account of weighing designs can be obtained from Raghavarao (1971) and Banerjee (1975). Hotelling (1944) have shown that the chemical balance weighing designs without any restrictions is optimal if $\mathbf{X}'\mathbf{X} = n\mathbf{I}_p$. Several methods of constructing optimal chemical balance weighing designs without any restrictions on the number of objects placed on either pan are available in the literature. Dey (1971), Saha (1975), Kageyama and Saha (1983) and others have shown how optimum chemical balance weighing designs can be constructed from the incidence matrices of balanced incomplete block designs for $p = v$ objects. Ceranka and Katulska (1988a) have constructed optimum chemical balance weighing designs from two incidence matrices of balanced incomplete block designs. Saha and Kageyama (1984) have constructed optimum chemical balance weighing designs for $v + 1$ objects in $4(r - \lambda)$ weighings from incidence matrices of balanced incomplete block designs for v treatments. For the same case, Ceranka and Katulska (1988b) have developed another method of construction. Some results of constructing chemical balance weighing designs under the restriction on the number of objects placed on either pan have been given by Swamy (1982) and Ceranka and Katulska (1999). For the same case,

Ceranka, Katulska and Mizera (1998) have constructed chemical balance weighing designs from incidence matrices of ternary balanced block designs.

In the present paper, we study another method of constructing the designs matrix \mathbf{X} for a chemical balance weighing design under the restriction on the number of objects placed on either pan. It is based on two incidence matrices of ternary balanced block designs.

2. TERNARY BALANCED BLOCK DESIGNS

A ternary balanced block design is to be a design consisting of b blocks each of size k , chosen from a set size v in such a way that each of the v elements occurs r times altogether and 0,1 or 2 times in each block, and each of the distinct $\binom{v}{2}$ pairs of elements occurs λ times. In other words, the inner product of distinct rows of the $n \times b$ incidence matrix \mathbf{N} of a ternary balanced block design is λ . It is straightforward to verify that

$$\begin{aligned} vr &= bk, \\ \lambda(v-1) &= rk - \delta, \\ \mathbf{N}\mathbf{N}' &= (\delta - \lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v', \end{aligned}$$

where $\delta = \sum_{j=1}^b n_{ij}^2$.

3. CONSTRUCTION THE DESIGN MATRIX USING TERNARY BALANCED BLOCK DESIGNS.

Consider two ternary balanced block designs with parameters $v, b_i, r_i, k_i, \lambda_i, \delta_i, i = 1, 2$. Let \mathbf{N}_i denote the incidence matrix of order $v \times b_i$. Now, we define the matrix \mathbf{X} of a chemical balance weighing design as

$$(3.1) \quad \mathbf{X} = \begin{bmatrix} \mathbf{N}'_1 - \mathbf{1}_{b_1}\mathbf{1}'_v \\ \mathbf{N}'_2 - \mathbf{1}_{b_2}\mathbf{1}'_v \end{bmatrix},$$

where $\mathbf{1}_a$ denotes the $a \times 1$ vector with the unit elements everywhere. In this design we have $p = v$ and $n = b_1 + b_2$.

Lemma 2.1. *The design \mathbf{X} given by (3.1) is nonsingular if and only if $v \neq k_1$ or $v \neq k_2$.*

Proof. For the design matrix \mathbf{X} given by (3.1), we have

$$\mathbf{X}'\mathbf{X} = [\delta_1 + \delta_2 - (\lambda_1 + \lambda_2)]\mathbf{I}_v + [b_1 + b_2 + \lambda_1 + \lambda_2 - 2(r_1 + r_2)]\mathbf{1}_v\mathbf{1}'_v.$$

It is easy to see that

$$\det(\mathbf{X}'\mathbf{X}) = \frac{1}{v}[\delta_1 + \delta_2 - (\lambda_1 + \lambda_2)]^{v-1}[b_1(v - k_1)^2 + b_2(v - k_2)^2].$$

Evidently $\delta_1 \neq \lambda_1$ and $\delta_2 \neq \lambda_2$, and $\det(\mathbf{X}'\mathbf{X}) = 0$ if and only if $v = k_1$ and $v = k_2$. So the lemma is proved.

If the design \mathbf{X} given by (3.1) is nonsingular, then

$$(3.2) \quad (\mathbf{X}'\mathbf{X})^{-1} = \frac{1}{\delta_1 + \delta_2 - \lambda_1 - \lambda_2} \left(\mathbf{I}_v - \frac{v(b_1 + b_2 + \lambda_1 + \lambda_2 - 2r_1 - 2r_2)}{b_1(v - k_1)^2 + b_2(v - k_2)^2} \mathbf{1}_v\mathbf{1}'_v \right).$$

Theorem 3.1. *For a nonsingular chemical balance weighing design with \mathbf{X} given by (3.1) the estimated weights are uncorrelated if and only if*

$$(3.3) \quad b_1 + b_2 = 2r_1 + 2r_2 - \lambda_1 - \lambda_2.$$

Proof. From (3.2) we have that $Cov(\hat{w}_i, \hat{w}_{i'}) = 0$, $i, i' = 1, 2, \dots, p$, $i \neq i'$ if and only if $b_1 + b_2 + \lambda_1 + \lambda_2 - 2r_1 - 2r_2 = 0$. Hence the theorem.

If the estimated weights are uncorelated, then

$$Var(\hat{w}_i) = \frac{\sigma^2}{\delta_1 + \delta_2 - \lambda_1 - \lambda_2}.$$

In a special case, when \mathbf{N}_1 is the incidence matrix of a ternary balanced block design with parameters $v, b_1, r_1, k_1, \delta_1$ and \mathbf{N}_2 is the incidence matrix of complementary design, i.e., $\mathbf{N}_2 = 2\mathbf{1}_v\mathbf{1}'_{b_1} - \mathbf{N}_1$ with parameters $v, b_2 = b_1, r_2 = 2b_1 - r_1, k_2 = 2v - k_1, \lambda_2 = \lambda_1 + 4b_1 - 4r_1, \delta_2 = \delta_1 + 4b_1 - 4r_1$, condition (3.3) is reduced to the form

$$(3.4) \quad b_1 = 2r_1 - \lambda_1.$$

Corollary 3.1. *If condition (3.4) holds, then a chemical balanced weighing design with*

$$(3.5) \quad \mathbf{X} = \begin{bmatrix} \mathbf{N}'_1 - \mathbf{1}_{b_1} \mathbf{1}'_v \\ \mathbf{1}_{b_1} \mathbf{1}'_v - \mathbf{N}'_1 \end{bmatrix},$$

gives uncorrelated estimated weights.

Corollary 3.2. *A chemical balanced weighing design with \mathbf{X} given by (3.5) based on ternary balanced block designs with parameters*

- (i) $v, b_1 = v, r_1 = v - 2, k_1 = v - 2, \lambda_1 = v - 4, \delta_1 = v,$
- (ii) $v, b_1 = v, r_1 = v - 3, k_1 = v - 3, \lambda_1 = v - 4, \delta_1 = v + 6$

gives uncorrelated estimated weights.

Consider now the design matrix \mathbf{X} given by (3.1) in the case $\mathbf{N}_2 = 2\mathbf{1}_v \mathbf{1}'_{b_2}$, i.e., $r_2 = 2b_2, k_2 = 2v, \lambda_2 = \delta_2 = 4b_2$, condition (3.3) is reduced to the form

$$(3.6) \quad b_1 + b_2 = 2r_1 - \lambda_1.$$

Corollary 3.3. *If condition (3.6) holds, then a chemical balance weighing design with*

$$(3.7) \quad \mathbf{X} = \begin{bmatrix} \mathbf{N}'_1 - \mathbf{1}_{b_1} \mathbf{1}'_v \\ \mathbf{1}_{b_2} \mathbf{1}'_v \end{bmatrix}$$

gives uncorrelated estimated weights.

Corollary 3.4. *The existence of a ternary balanced block design with the parameters $v, b_1, r_1, k_1, \lambda_1, \delta_1$, satisfying $b_1 < 2r_1 - \lambda_1$ implies the existence of chemical balance weighing design with \mathbf{X} given by (3.7), where $b_2 = 2r_1 - \lambda_1 - b_1$.*

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