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APPENDIX TO THE PAPER "OSGOOD TYPE CONDITIONS FOR AN mTH ORDER DIFFERENTIAL EQUATION"

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Assume that $I = [0, a]$, E is a Banach space, $B = \{x \in E : \ \|x\| \leq b\}$ and $f : I \times B \mapsto E$ is a bounded continuous function. Let $M = \sup\{\|f(t, x)\| : t \in I, \ x \in B\}$. We choose a positive number d such that $d \leq a$ and

$$\sum_{j=1}^{m-1} \| \eta_j \| \frac{d^j}{j!} + M\frac{d^m}{m!} \leq b$$

for given $\eta_1, \ldots, \eta_{m-1} \in E$. Put $J = [0, d]$ and

$$F(x)(t) = p(t) + \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1} f(s, x(s))ds \quad (t \in J).$$

Denote by $C(J, E)$ the Banach space of continuous functions $x : J \mapsto E$ with usual supremum norm. Let $\tilde{B} \subset C(J, E)$ be the subset of those functions with values in B. It is known that F is a continuous mapping $\tilde{B} \mapsto \tilde{B}$ and

$$\|F(x)(t) - F(x)(\tau)\| \leq K|t - \tau| \quad \text{for } \ t, \tau \in J \ \text{ and } \ x \in \tilde{B}, \tag{1}$$

where $K = \sum\limits_{j=1}^{m-1} \|\eta_j\| \frac{d^{j-1}}{(j-1)!} + M\frac{d^{m-1}}{(m-1)!}$. Moreover, a continuous function $x : J \mapsto B$ is a solution to the Cauchy problem

$$x^{(m)} = f(t,x) \tag{2}$$
$$x(0) = 0, x'(0) = \eta_1, \dots, x^{(m-1)}(0) = \eta_{m-1}$$

if x is a fixed point of F.

The purpose of this paper is to show that the following theorem is true:

Theorem. *Let $w : [0,2b] \mapsto \mathbb{R}_+$ be a continuous nondecreasing function such that $w(0) = 0$, $w(r) > 0$ for $r > 0$ and*

$$\int_{0+} \frac{dr}{\sqrt[m]{r^{m-1}w(r)}} = \infty.$$

If

$$\|f(t,x) - f(t,y)\| \le w(\|x-y\|) \quad \text{for } t \in I, \quad x,y \in B, \tag{3}$$

then the successive approximations u_n, defined by

$$u_0 = 0, u_{n+1} = F(u_n) \quad \text{for } n \in N, \tag{4}$$

converge uniformly on J to the unique solution u of (2).

Proof. First, similarly as in the proof of Theorem III. 9.1 in [1], we shall show that

$$\lim_{n\to\infty} \|u_n(t) - u_{n-1}(t)\| = 0 \quad \text{for } t \in J. \tag{5}$$

Put $\lambda(t) = \overline{\lim}_{n\to\infty}\|u_n(t) - u_{n-1}(t)\|$. From (1) and (2) it is clear that

$$\|u_n(t_1) - u_{n-1}(t_1)\| \le \|u_n(t_2) - u_{n-1}(t_2)\| + 2K|t_1 - t_2|.$$

For any $\varepsilon > 0$ there is $n_0 \in N$ such that

$$\|u_n(t_2) - u_{n-1}(t_2)\| \le \lambda(t_2) + \varepsilon \quad \text{for } n \ge n_0.$$

Therefore

$$\|u_n(t_1) - u_{n-1}(t_1)\| \le \lambda(t_2) + \varepsilon + 2K|t_1 - t_2| \quad \text{for } n \ge n_0$$

and consequently,

$$\lambda(t_1) \le \lambda(t_2) + \varepsilon + 2K|t_1 - t_2|.$$

As ε is arbitrary, we get

$$\lambda(t_1) \leq \lambda(t_2) + 2K|t_1 - t_2|.$$

This implies

$$|\lambda(t_1) - \lambda(t_2)| \leq 2K|t_1 - t_2| \quad \text{for} \quad t_1, t_2 \in J,$$

which proves the continuity of $\lambda(\cdot)$.

Further, from (4) it follows that

$$\|u_{n+1}(t) - u_n(t)\| = \|F(u_n)(t) - F(u_{n-1})(t)\|$$

$$\leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} \|f(s, u_n(s)) - f(s, u_{n-1}(s))\| ds.$$

By (3) this implies

$$(6) \quad \|u_{n+1}(t) - u_n(t)\| \leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} w(\|u_n(s) - u_{n-1}(s)\|) ds.$$

Since the sequence $(\|u_n(\cdot) - u_{n-1}(\cdot)\|)$ is equicontinuous and uniformly bounded, from the definition of $\lambda(\cdot)$ and Arzela's Lemma we deduce that for fixed $t \in J$ there exists a subsequence (n_k) such that $\lim_{k\to\infty} \|u_{n_k+1}(t) - u_{n_k}(t)\| = \lambda(t)$ and $\|u_{n_k}(s) - u_{n_k-1}(s)\| \to \lambda_1(s)$ uniformly in $s \in J$. Replacing n by n_k in (6) and passing to the limit as $k \to \infty$, we obtain the inequality

$$\lambda(t) \leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} w(\lambda_1(s)) ds.$$

As $\lambda_1(s) \leq \overline{\lim} \|u_n(s) - u_{n-1}(s)\| = \lambda(s)$ and $w(r)$ is nondecreasing, we see that

$$0 \leq \lambda(t) \leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} w(\lambda(s)) ds \quad \text{for} \quad t \in J.$$

Applying now Theorem 2 of [3], we conclude that $\lambda(t) \equiv 0$ on J, which proves (5).

On the other hand, (3) implies that

$$\alpha\left(f(t,X)\right) \le w(\alpha(X)) \quad \text{for } t \in J \text{ and } X \subset B, \tag{7}$$

where α is the Kuratowski measure of noncompactness. Now we shall show that the sequence (u_n) has a limit point.

Let $V = \{u_n : n \in N\}$. Then, by (1), V is a bounded equicontinuous subset of $\tilde{B}$. Denote by v the function defined by $v(t) = \alpha(V(t))$ for $t \in J$, where $V(t) = \{u_n(t) : n \in N\}$. It is well known that the function v is continuous. As $V = F(V) \cup \{0\}$, we have

$$V(t) \subset F(V)(t) \cup \{0\}$$

and consequently $\alpha(V(t)) \le \alpha(F(V)(t))$. Since

$$F(V)(t) \subset \frac{1}{(m-1)!}\left\{\int_0^t (t-s)^{m-1} f(s,u_n(s))ds : n \in N\right\},$$

then Heinz's lemma [2] proves that

$$\begin{aligned}\alpha(F(V)(t)) &\le \frac{1}{(m-1)!}\alpha\left(\left\{\int_0^t (t-s)^{m-1} f(s,u_n(s))ds : n \in N\right\}\right)\\ &\le \frac{1}{(m-1)!}\int_0^t \alpha(\{(t-s)^{m-1} f(s,u_n(s)) : n \in N\})ds\\ &\le \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}\alpha(\{f(s,u_n(s)) : n \in N\})ds.\end{aligned}$$

Moreover, in view of (7), we have

$$\alpha(\{f(s,u_n(s)) : n \in N\}) \le w(\alpha(V(s))).$$

Hence

$$v(t) \le \alpha(F(V)(t)) \le \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1} w(v(s))ds \quad \text{for } t \in J.$$

Applying now Theorem 2 from [3], we deduce that $v(t) \equiv 0$ on J. This proves that $V(t)$ is relatively compact for $t \in J$ and consequently, by Ascoli's Theorem, V is relatively compact in $C(J,E)$. Hence the sequence (u_n) has a subsequence (u_{n_k}) which converges to a limit u. This fact, together with

(5) and (4), implies that $u = F(u)$, i.e. u is a solution of (2). Suppose that $\overline{u}$ is another solution of (2). Then

$$\|u(t) - \overline{u}(t)\| = \|F(u)(t) - F(\overline{u})(t)\|$$

$$\leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} w(\|u(s) - \overline{u}(s)\|) ds \quad \text{for} \ \ t \in J,$$

and therefore by Theorem 2 of [3] we get $\|u(t) - \overline{u}(t)\| \equiv 0$ on J. Thus $u = \overline{u}$. From the above considerations it is clear that the sequence (u_n) has a unique limit point u, and hence $\lim_{n\to\infty} u_n(t) = u(t)$ uniformly on J.

References

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