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BALL INTERSECTION MODEL FOR FEJÉR ZONES OF CONVEX CLOSED SETS

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Abstract

Strongly Fejér monotone mappings are widely used to solve convex problems by corresponding iterative methods. Here the maximal of such mappings with respect to set inclusion of the images are investigated. These mappings supply restriction zones for the successors of Fejér monotone iterative methods. The basic tool is the representation of the images by intersection of certain balls.

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1. Introduction

Fejér monotone methods are used to solve problems with convex solution sets. The theory of Fejér monotone sequences and mappings has a long history. It started in the 1960s with papers by the Russian mathematician Eremin. His monograph [1] of 1979 contains the first summarized description of such methods in finite dimensional Euclidian vector spaces, with many applications to the iterative solution to convex problems. The universal character of the methods and the weak assumptions which were necessary for proving their convergence created a great interest in completing the theoretical foundations. Nowadays there exist important applications in computerized tomography and in image and signal reconstruction (see e.g. [11]).

Considering a convex problem with solution set M we can treat it iteratively by the use of so-called α-strongly M-Fejér monotone sequences (x_k) which satisfy the conditions

$$\|x_k - x\|^2 - \|x_{k+1} - x\|^2 \geq \alpha \|x_k - x_{k+1}\|^2$$

for all $x \in M$ and a fixed nonnegative number α. Generally, such sequences are generated by corresponding set-valued mappings g with fixed point set M following the procedure

$$x_{k+1} \in g(x_k), \quad x_0 \in H.$$

These mappings are defined in Section 2. Often the formulation of the problem suggests appropriate mappings g. Incidentally, an α-strongly M-Fejér monotone sequence (x_k) already converges weakly to an element $x^* \in M$ for every starting element x_o if $I - g$ is demiclosed (see [9]).

Using an inverse point of view and starting with the solution set M, we can describe the maximal sets or α-zones $G_M^\alpha(x_k)$ geometrically in which the successors x_{k+1} of the Fejér monotone iterative method have to lie according to the above given conditions. Related to g we arrive at α-zones $G_M^\alpha(y)$ which contain all possible images $z \in g(y)$ of the original element y. Studying the geometric properties of these α-zones we win new insights in the functioning of Fejér monotone methods.

2. Strongly Fejér monotone mappings

Let H be a (real) Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$. We consider a nonempty, convex and closed subset Q of H and set-valued mappings $g : Q \mapsto \mathbb{P}(Q)$, where $\mathbb{P}(Q)$ consists of all nonempty subsets of Q. We exclude the uninteresting special case $g = I$ (I the identity). First, we begin by summarizing some basic concepts given in [8].

Definition 2.1. Let M be a nonempty (proper) subset of Q and α a positive number. The mapping $g : Q \mapsto \mathbb{P}(Q)$ is said to be *α-strongly M-Fejér monotone* (in notation: $g \in \mathbb{F}^\alpha(M)$) if the conditions

$$(1) \qquad \|y - x\|^2 - \|z - x\|^2 \geq \alpha \|y - z\|^2 \quad \forall x \in M, \ \forall y \in Q, \ \forall z \in g(y)$$

and

$$y \notin g(y) \quad \forall y \in Q \backslash M \tag{2}$$

are fulfilled. Moreover, g is called *strongly M-Fejér monotone* (in notation: $g \in \mathbb{F}_s(M)$) if g is α-strongly M-Fejér monotone for some $\alpha > 0$.

Remark 2.2. If Definition 2.1 is used with the limit value $\alpha = 0$, then g is said to be *regularly M-Fejér monotone* (in notation: $g \in \mathbb{F}_r(M)$). Thus $\mathbb{F}_r(M)$ can be regarded as the limit class $\mathbb{F}^0(M)$ of $\mathbb{F}^\alpha(M)$. For regularly Fejér monotone and consequently for all the strongly Fejér monotone mappings g the set M is uniquely determined. Namely, it is the convex and closed set

$$M = M(g) := \{x \in Q : \|z - x\| \leq \|y - x\| \quad \forall y \in Q, \ \ \forall z \in g(y)\}$$

which moreover coincides with the fixed point set of g. Incidentally, the classes of strongly Fejér monotone mappings satisfy the hierarchy

$$\mathbb{F}^\beta(M) \subset \mathbb{F}^\alpha(M) \subset \mathbb{F}_s(M) \subset \mathbb{F}_r(M) \quad \text{for} \quad \beta > \alpha > 0.$$

Definition 2.3. Let be $g \in \mathbb{F}_s(M)$. Then the number

$$\alpha^* = \alpha_F^*(g) := \sup\{\alpha : g \in \mathbb{F}^\alpha(M)\}$$

is said to be the *F-index* of g. (Mappings $g \in \mathbb{F}_r(M) \setminus \mathbb{F}_s(M)$ obtain the F-index 0.)

Observe, that α^* is the maximal number for which (1) holds under condition (2). So we have $g \in \mathbb{F}^{\alpha^*}$, but $g \notin \mathbb{F}^\alpha$ for $\alpha > \alpha^*$.

Now we enlarge the concept of *Fejér zones* given in [7] to the case of strongly Fejér monotone mappings. The Fejér α-zones of M will supply *restriction sets* for the images of α-strongly M-Fejér monotone mappings.

Definition 2.4. Let M be a convex, closed, nonempty and proper subset of Q. Further, let α be a nonnegative number. Then the sets

$$G^\alpha_{Q,M}(y) = \{z \in Q : \|y - x\|^2 - \|z - x\|^2 \geq \alpha \, \|y - z\|^2 \quad \forall x \in M\} \tag{3}$$

are said to be the *Fejér α-zones* of M w.r.t. $y \in Q$. The reference to Q will be omitted if Q is the whole Hilbert space H. Besides, we call α the *index*, y the *loop* and M a *generating set* of $G^\alpha_{Q,M}(y)$.

Remark 2.5. The definition shows that the relations

$$y \in G^{\alpha}_{Q,M}(y) \quad \text{for} \quad y \in Q, \quad G^{\alpha}_{Q,M}(y) = \{y\} \quad \text{for} \quad y \in M$$

hold. By the way, in the following y turns out to play an exceptional part in $G^{\alpha}_{Q,M}(y)$ which justifies the notation loop. Besides, we have

$$G^{\beta}_{Q,M}(y) \subseteq G^{\alpha}_{Q,M}(y) \quad \text{for} \quad \beta \geq \alpha \geq 0.$$

Obviously, the mapping $G^{\alpha}_{Q,M}$ generated by the Fejér α-zones of M is defined on Q and has fixed points on M.

Now we introduce the modified mapping $\hat{G}^{\alpha}_{Q,M} : Q \mapsto \mathbb{P}(Q)$ for arbitrary $\alpha \geq 0$ by

$$\text{(4)} \qquad \hat{G}^{\alpha}_{Q,M}(y) \;:=\; \begin{cases} \{y\} & \text{if} \quad y \in M \\ G^{\alpha}_{Q,M}(y)\backslash\{y\} & \text{if} \quad y \in Q \setminus M \end{cases}$$

which is called the *Fejér α-zone mapping* of M w.r.t. Q. This mapping arises from $G^{\alpha}_{Q,M}$ by extracting the fixed points in all images $G^{\alpha}_{Q,M}(y)$ with original elements y outside of M. Because of $M \subset Q$ the case $\hat{G}^{\alpha}_{Q,M} = I$ is excluded. It is easy to see that even $G^{\alpha}_{Q,M}(y) \supset \{y\}$ holds for $y \in Q \setminus M$ (compare also Theorem 3.2). So $\hat{G}^{\alpha}_{Q,M}$ is really defined everywhere on Q.

Theorem 2.6. *The Fejér α-zone mapping $\hat{G}^{\alpha}_{Q,M}$ is α-strongly M-Fejér monotone* (i.e. $\hat{G}^{\alpha}_{Q,M} \in \mathbb{F}^{\alpha}(M)$) *and moreover the maximal such mapping with respect to set inclusion of the images.*

Proof. By Definition 2.4 the mapping $G^{\alpha}_{Q,M}$ fulfils condition (1) of α-strongly M-Fejér monotone mappings in Definition 2.1 and is just that mapping whose images are maximally chosen with respect to set inclusion. On the other hand, $\hat{G}^{\alpha}_{Q,M}$ modifies $G^{\alpha}_{Q,M}$ in such a way that condition (2) of α-strongly M-Fejér monotone mappings is additionally satisfied by conserving the property of maximality. ∎

The geometric characterization of the Fejér α-zone mappings $\hat{G}^{\alpha}_{Q,M}$ will give us an idea how Fejér monotone methods work. Remembering the trivial difference between $\hat{G}^{\alpha}_{Q,M}$ and $G^{\alpha}_{Q,M}$ it suffices to study the Fejér α-zones of M w.r.t. all $y \in Q$. Since the relation

$$G^{\alpha}_{Q,M}(y) = G^{\alpha}_{H,M}(y) \cap Q = G^{\alpha}_{M}(y) \cap Q, \quad y \in Q$$

holds, we can, moreover, restrict ourselves to the case $Q = H$, i.e. to the Fejér α-zones $G^\alpha_M(y)$. Introducing the sets

$$G^\alpha_x(y) = \{z \in H : \|y-x\|^2 - \|z-x\|^2 - \alpha\,\|y-z\|^2 \geq 0\} \tag{5}$$

we obtain in view of (3) the representation

$$G^\alpha_M(y) = \bigcap_{x \in M} G^\alpha_x(y) \tag{6}$$

for the Fejér α-zones of M. Obviously, the translation property

$$G^\alpha_x(y) = y + G^\alpha_{x-y}(0), \quad G^\alpha_M(y) = y + G^\alpha_{M-y}(0)$$

holds. Further, observe that from the inverse point of view the Fejér α-zone does not uniquely determine M. So we can look for minimal generating sets M. This aspect is investigated in Section 6. Finally, the sets $G^\alpha_x(y)$ in (5) can be regarded as the Fejér α-zones of the singletons $\{x\}$ w.r.t. $y \in H$.

The following two statements are easy consequences of (6). The first is useful to determine outer approximations of Fejér zones.

Lemma 2.7. *Let $\alpha \geq 0$ and $y \in H$ be given. If $N \subseteq M$, then*

$$G^\alpha_M(y) \subseteq G^\alpha_N(y).$$

Finally, knowing the Fejér zones belonging to decompositions of M, we can construct the Fejér zones of M by intersection.

Lemma 2.8. *Let $\alpha \geq 0$ and $y \in H$ be given. If $\{M_e\}_{e\in E}$ is a family of subsets with $M = \bigcup_{e\in E} M_e$, then*

$$G^\alpha_M(y) = \bigcap_{e\in E} G^\alpha_{M_e}(y).$$

3. Ball intersection model of Fejér zones

We use the notation

$$B(x_0, r) := \{x \in H : \|x - x_0\| \leq r\}$$

for the *ball* with *midpoint* $x_0 \in H$ and *radius* r. We introduce *intervals* (line-segments) in their closed and open forms, namely

$$[x_1, x_2] = \operatorname{conv}\{x_1, x_2\} = \{\mu\, x_1 + \nu\, x_2 : \mu \geq 0, \;\; \nu \geq 0, \;\; \mu + \nu = 1\}$$

and $(x_1, x_2) = [x_1, x_2] \setminus \{x_1, x_2\}$. The *affine hull* of a set L, that means the smallest affine set containing L, is abbreviated by $\operatorname{aff} L$. Besides, $\operatorname{ri} L$ and $\operatorname{rbd} L$ represent the *relative interior* and the *relative boundary* of L (with respect to $\operatorname{aff} L$). First, we prepare a geometric interpretation of the Fejér α-zones $G_M^\alpha(y)$ by intersection of certain balls (*ball intersection model*). We consider an arbitrary, but fixed element $y \in H$ and introduce for $\alpha \geq 0$ and elements $x \in H$ the balls

$$\mathbb{B}^\alpha(y, x) = B(z_m^\alpha(y, x), r_m^\alpha(y, x))$$

with midpoints

$$z_m^\alpha(y, x) = y + \frac{1}{1+\alpha}\,(x - y) = \frac{1}{1+\alpha}\, x + \frac{\alpha}{1+\alpha}\, y \tag{7}$$

in $[y, x]$ and radii

$$r_m^\alpha(y, x) = \frac{1}{1+\alpha}\, \|x - y\|. \tag{8}$$

All balls contain y on the boundary. The balls degenerate for $x = y$ to the singleton $\{y\}$. The opposite boundary elements of y in the balls are

$$z_s^\alpha(y, x) := y + \frac{2}{1+\alpha}\,(x - y) = \frac{2}{1+\alpha}\, x - \frac{1-\alpha}{1+\alpha}\, y \tag{9}$$

lying in $[y, 2x - y]$. Besides, we have the diameter intervals

$$[y, z_s^\alpha(y, x)] = \operatorname{aff}\{y, x\} \cap \mathbb{B}^\alpha(y, x), \quad y \neq z_s^\alpha(y, x) \quad \text{for} \quad x \neq y$$

with midpoints $z_m^\alpha(y, x)$ and lengths $2\, r_m^\alpha(y, x)$.

In the following, M is supposed to be a fixed convex, closed, nonempty and proper subset of H. It is well-known that the metric projector P_M onto M is well-defined.

Lemma 3.1. *The sets $G_x^\alpha(y)$ in* (5) *are balls, namely it is*

$$G_x^\alpha(y) = \mathbb{B}^\alpha(y, x) = B(z_m^\alpha(y, x), r_m^\alpha(y, x)), \quad y \in H$$

for all $x \in H$. Further, the inclusions

$$[y, z_s^\alpha(y, P_M y)] \subseteq G_x^\alpha(y) \subseteq B(y, 2\, r_m^\alpha(y, x))$$

hold for all $x \in M$.

Proof. For arbitrary elements x and z in H the relation

$$\|x\|^2 - \|z - x\|^2 - \alpha\|z\|^2 = 2\,\langle x, z\rangle - (1+\alpha)\,\|z\|^2 \geq 0$$

is equivalent to

$$\|z\|^2 - \frac{2}{1+\alpha}\,\langle z, x\rangle = \|z - \frac{1}{1+\alpha}\,x\|^2 - \frac{1}{(1+\alpha)^2}\,\|x\|^2 \leq 0.$$

By a transformation, replacing x by $x - y$ and z by $z - y$, we find that the inequality

$$\|y - x\|^2 - \|z - x\|^2 - \alpha\,\|y - z\|^2 \geq 0$$

corresponds to

$$\|z - y - \frac{1}{1+\alpha}(x - y)\|^2 - \frac{1}{(1+\alpha)^2}\|y - x\|^2 \leq 0$$

or in view of (7) and (8) to

$$\|z - z_m^\alpha(y, x)\| \leq r_m^\alpha(y, x).$$

Considering (5) this just proves the first assertion $G_x^\alpha(y) = \mathbb{B}^\alpha(y, x)$. According to the above deduction the characterizing inequality for $z \in G_x^\alpha(y)$ is also equivalent to

$$2\,\langle y - x, y - z\rangle - (1+\alpha)\,\|y - z\|^2 \geq 0. \tag{10}$$

If we choose now $z = z_s^\alpha(y, P_M y)$ in the left-hand side we obtain, in view of (9) and the immediate consequence $y - z_s^\alpha(y, P_M y) = \frac{2}{1+\alpha}\,(y - P_M y)$, the expression

$$\begin{aligned} p_s^\alpha &= 2\langle y - z_s^\alpha(y, P_M y), y - x\rangle - (1+\alpha)\|y - z_s^\alpha(y, P_M y)\|^2 \\ &= \frac{4}{1+\alpha}\,\langle y - P_M y, y - x\rangle - \frac{4}{1+\alpha}\,\|y - P_M y\|^2 \\ &= \frac{4}{1+\alpha}\,\langle y - P_M y, P_M y - x\rangle. \end{aligned}$$

Since the metric projector P_M satisfies the inequality

$$\langle y - P_M y, P_M y - x \rangle \geq 0 \quad \forall x \in M$$

(see e.g. [2, p. 117]), we get $p_s^\alpha \geq 0$. Hence $z_s^\alpha(y, P_M y)$ fulfils inequality (10) and belongs consequently to $G_x^\alpha(y)$. Besides, $y \in G_x^\alpha(y)$. As the ball $G_x^\alpha(y)$ is convex, it contains also the interval $[y, z_s^\alpha(y, P_M y)]$. Thus the lower interval inclusion holds for all $x \in M$. Finally, we get from inequality (10) the estimate

$$2\,\|y - x\| \geq (1 + \alpha)\,\|y - z\|$$

if we apply Schwarz's inequality. But this means by (8) in other notation that $z \in G_x^\alpha(y)$ implies $z \in B(y, 2\,r_m^\alpha(y, x))$. Hence, the upper ball inclusion is also true, even for all $x \in H$. ■

Theorem 3.2 (Ball intersection model)**.** *The Fejér α-zones $G_M^\alpha(y)$ of M are nonempty, convex, closed and bounded for all $y \in H$ and all $\alpha \geq 0$. More precisely, they have the ball intersection form*

$$G_M^\alpha(y) = \bigcap_{x \in M} \mathbb{B}^\alpha(y, x).$$

Besides, they satisfy the interval ball inclusions

$$[y, z_s^\alpha(y, P_M y)] \subseteq G_M^\alpha(y) \subseteq \mathbb{B}^\alpha(y, P_M y),$$

where $\{y,\ z_s^\alpha(y, P_M y)\} \subseteq \operatorname{rbd} G_M^\alpha(y)$ *and*

$$\begin{aligned} &\operatorname{diam}\,[y, z_s^\alpha(y, P_M y)] = \operatorname{diam}\, G_M^\alpha(y) = \operatorname{diam}\, \mathbb{B}^\alpha(y, P_M y) \\ &= \lambda_1^\alpha \|y - P_M y\|, \quad \lambda_1^\alpha = \frac{2}{1 + \alpha}. \end{aligned}$$

Proof. The representation of $G_M^\alpha(y)$ by intersection of the given balls uses (6) and Lemma 3.1. Since these balls are convex, closed and bounded, the same holds for the intersection set. Obviously, $G_M^\alpha(y)$ is nonempty, since the loop y is an element. The set inclusions are simple consequences of Lemma 3.1 and the ball intersection form of $G_M^\alpha(y)$. Namely, $[y, z_s^\alpha(y, P_M y)]$ lies in all balls $\mathbb{B}^\alpha(y, x)$ with $x \in M$, and one of the balls is $\mathbb{B}^\alpha(y, P_M y)$. Further, because of (8) and (9) we get the relations

$$\begin{aligned} \operatorname{diam}\,[y, z_s^\alpha(y, P_M y)] &= \|y - z_s^\alpha(y, P_M y)\| = \lambda_1^\alpha \|y - P_M y\|, \\ \operatorname{diam}\, \mathbb{B}^\alpha(y, P_M y) &= 2\, r_m^\alpha(y, P_M y) = \lambda_1^\alpha \|y - P_M y\|. \end{aligned}$$

By the interval ball inclusion this is also the diameter of $G_M^\alpha(y)$. But the result concerning the diameters shows also that the elements of the set

$$R = \operatorname{aff}\{y, z_s^\alpha(y, P_M y)\} \setminus [y, z_s^\alpha(y, P_M y)]$$

do not belong to $G_M^\alpha(y)$. Therefore y and $z_s^\alpha(y, P_M y)$ are even on the relative boundary of $G_M^\alpha(y)$. ∎

The subset $[y, z_s^\alpha(y, P_M y)]$ of $G_M^\alpha(y)$ can be considered as its *skeleton*. This interval is generated by $G_M^\alpha(y) \cap \operatorname{aff}\{y, P_M y\}$ and connects the loop with its counter point. The set $\mathbb{B}^\alpha(y, P_M y)$ is the ball hull of $G_M^\alpha(y)$, the so-called *Chebyshev ball*. If Lemma 3.1 is used with $x = P_M y$, then

$$\mathbb{B}^\alpha(y, P_M y) \subseteq B(y, 2\, r_m^\alpha(y, P_M y)) \tag{11}$$

arises. The latter is the smallest ball with centre y containing $G_M^\alpha(y)$. In Section 7 we will show by examples that the interval ball inclusions in Theorem 3.2 are sharp. The examples will illustrate also that the relative interior of the skeleton can belong to the interior, the relative interior or the relative boundary of the Fejér α-zone (compare Examples 7.1, 7.4 and 7.6).

Corollary 3.3. *The Fejér α-zone mapping $\hat{G} = \hat{G}_M^\alpha$ has just the F-index $\alpha^* = \alpha_F^*(\hat{G}) = \alpha$.*

Proof. By Theorem 2.6 we get $\hat{G}_M^\alpha \in \mathbb{F}^\alpha$. Hence, $\alpha^* \geq \alpha$ by Definition 2.3 of the F-index. Suppose $\alpha^* > \alpha$. Then we have $\hat{G}_M^\alpha \in \mathbb{F}^{\alpha^*}$. Moreover, $\hat{G}_M^{\alpha^*} \in \mathbb{F}^{\alpha^*}$ and $\hat{G}_M^\alpha(y) \subseteq \hat{G}_M^{\alpha^*}(y)$ for all y (see Theorem 2.6 and the remark after Definition 2.3). Now consider some $y \notin M$. Then $z_s^\alpha(y, P_M y) \in \hat{G}_M^\alpha(y)$ by Theorem 3.2 and (4), where

$$0 < \|z_s^\alpha(y, P_M y) - y\| = \lambda_1^\alpha \|P_M y - y\|, \quad \lambda_1^\alpha = \frac{2}{1+\alpha}$$

while

$$\|z - y\| \leq \lambda_1^{\alpha^*} \|P_M y - y\| < \lambda_1^\alpha \|P_M y - y\|$$

is fulfilled for all $z \in \hat{G}_M^{\alpha^*}(y)$. This is a contradiction. ∎

So taking the parameter α in $G_M^\alpha(y)$ as an index is in accordance with the above result.

4. Similarity relations between Fejér zones

First, we define the operator $S = S_y^\lambda : H \mapsto H$ with fixed $y \in H$ and $\lambda > 0$ by

$$Sx = y + \lambda\,(x - y) = (1 - \lambda)\,y + \lambda\,x \tag{12}$$

This operator is a *similarity transformation*. S acts as a *stretching* with *ratio* λ relative to the *centre* y. Moreover, S is a so-called *affine* operator. Obviously, S is invertible. The inverse turns out to be the central stretching

$$S^{-1} = S_y^{1/\lambda} \tag{13}$$

with ratio $1/\lambda$. The next statement shows that S maps balls onto balls. Here and in the following we use the convention $S\,L = \{Sx : x \in L\}$ for sets L.

Lemma 4.1. *The operator S translates and stretches balls according to*

$$S\,B(x_0, r) = B(Sx_0, \lambda\,r).$$

If C_0 is a convex cone and $C(x_0) = C_0 + x_0$ its translated form, then

$$S\,C(x_0) = C(S\,x_0), \quad S\,C(y) = C(y). \tag{14}$$

Further, it is evident that S is compatible with set operations, for instance we conclude for two sets L_1 and L_2

$$S(L_1 \cap L_2) = S\,L_1 \cap S\,L_2.$$

Observe that this property is preserved if a whole family of sets participates.

Lemma 4.2. *For an arbitrary subset M and an arbitrary family of subsets $\{L_x\}_{x \in M}$ indexed by M we have the relation*

$$S \bigcap_{x \in M} L_x = \bigcap_{x \in M} S\,L_x.$$

Finally, the family $\{S_y^\lambda\}_{\lambda > 0}$ is a multiplicative group with the product

$$S_y^\lambda\,S_y^\mu = S_y^{\lambda\,\mu} \quad (\textit{multiplicative rule}). \tag{15}$$

For a mapping $g : Q \mapsto \mathbb{P}(Q)$ we call the affine combination

$$g_\lambda := (1-\lambda)\, I + \lambda\, g = I + \lambda\,(g - I) \tag{16}$$

of I and g a *relaxation* of g with (real) parameter $\lambda > 0$.

Lemma 4.3. *For relaxations g_λ we get*

$$g_\lambda(y) = S_y^\lambda\, g(y).$$

So a relaxation of a mapping is a stretching of the image sets relative to the original elements.

We start with the observation that the sets $G_x^\alpha(y) = \mathbb{B}^\alpha(y,x)$ are, for fixed y and x, mutually similar relative to the parameter α. Comparing two such sets with parameters α and β the reference number

$$\mu = \mu(\alpha,\beta) = \frac{1+\beta}{1+\alpha}, \tag{17}$$

will be decisive.

Lemma 4.4. *Let $\alpha \geq 0$, $\beta \geq 0$ and $\mu > 0$ be numbers satisfying*

$$(1+\alpha)\,\mu = 1+\beta.$$

Then we get with the stretching operator $S = S_y^\mu$ the relations

$$G_x^\alpha(y) = S\, G_x^\beta(y) = G_{Sx}^\beta(y), \quad y \in H.$$

Proof. We introduce the notation $\lambda(\alpha) = \frac{1}{1+\alpha}$. Then

$$\lambda(\alpha) = \mu\,\lambda(\beta).$$

From (7), (8) and the multiplicative rule of central stretchings we obtain, on the one hand,

$$\begin{aligned} z_m^\alpha(y,x) &= S_y^{\lambda(\alpha)}\, x = S_y^\mu\, S_y^{\lambda(\beta)}\, x = S_y^\mu\, z_m^\beta(y,x), \\ r_m^\alpha(y,x) &= \lambda(\alpha)\, \|x-y\| = \mu\,\lambda(\beta)\, \|x-y\| = \mu\, r_m^\beta(y,x). \end{aligned}$$

So, by Lemma 3.1 and Lemma 4.1 we conclude

$$\begin{aligned} S_y^\mu\, G_x^\beta(y) &= S_y^\mu\, B(z_m^\beta(y,x), r_m^\beta(y,x)) = B(S_y^\mu\, z_m^\beta(y,x), \mu\, r_m^\beta(y,x)) \\ &= B(z_m^\alpha(y,x), r_m^\alpha(y,x)) = G_x^\alpha(y). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} z_m^\alpha(y,x) &= S_y^{\lambda(\alpha)}\, x = S_y^{\lambda(\beta)}\, S_y^\mu\, x = z_m^\beta(y, S_y^\mu\, x), \\ r_m^\alpha(y,x) &= \lambda(\alpha)\,\|x-y\| = \lambda(\beta)\,\mu\,\|x-y\| \\ &= \lambda(\beta)\|S_y^\mu\, x - y\| = r_m^\beta(y, S_y^\mu\, x). \end{aligned}$$

This implies with $S = S_y^\mu$ also

$$\begin{aligned} G_x^\alpha(y) &= B(z_m^\alpha(y,x), r_m^\alpha(y,x)) \\ &= B(z_m^\beta(y, S_y^\mu\, x), r_m^\beta(y, S_y^\mu\, x)) = G_{Sx}^\beta(y). \end{aligned}$$

■

We see that $G_x^\alpha(y)$ can be obtained from $G_x^\beta(y)$ by central stretching with ratio $\mu = \mu(\alpha,\beta)$ in (17) relative to y. Hence, for fixed elements x and $y \neq x$ we get a chain of stretched balls $G_x^\alpha(y) = \mathbb{B}^\alpha(y,x)$ with the common boundary element y expanding for α decreasing to 0 along aff $\{y,x\}$. Further, stretching of $G_x^\beta(y)$ can also be achieved by stretching on the level of the element x. These properties transfer to the corresponding Fejér zones.

Theorem 4.5. *Let $\alpha \geq 0$, $\beta \geq 0$ and $\mu > 0$ be numbers satisfying*

$$(1+\alpha)\,\mu = 1 + \beta.$$

Then we get with the stretching operator $S = S_y^\mu$ the relations

$$G_M^\alpha(y) = S G_M^\beta(y) = G_{S\,M}^\beta(y), \quad y \in H.$$

Proof. The assertion follows from the intersection representation (6) of Fejér zones, Lemma 4.4 and Lemma 4.2. So we obtain, on the one hand,

$$G_M^\alpha(y) = \bigcap_{x\in M} G_x^\alpha(y) = \bigcap_{x\in M} S\,G_x^\beta(y) = S \bigcap_{x\in M} G_x^\beta(y) = S\,G_M^\beta(y)$$

and on the other hand,

$$G_M^\alpha(y) = \bigcap_{x\in M} G_x^\alpha(y) = \bigcap_{x\in M} G_{Sx}^\beta(y) = \bigcap_{u\in S\,M} G_u^\beta(y) = G_{S\,M}^\beta(y).$$

■

Since $S = S_y^{\mu(\alpha,\beta)}$ is invertible by (13) and the relation $\mu(\alpha,\beta)\,\mu(\beta,\alpha) = 1$ is fulfilled by (17), the formula $S^{-1} = S_y^{\mu(\beta,\alpha)}$ holds.

Remark 4.6. The results of Theorem 4.5 can also be expressed in terms of Fejér zone mappings defined in (4). Namely, observing the definition of relaxations in (16) and Lemma 4.3 we find $(\hat{G}^\beta_M)_\mu(y) = S^\mu_y\, \hat{G}^\beta_M(y)$ and

$$\hat{G}^\alpha_M = (\hat{G}^\beta_M)_\mu = \hat{G}^\beta_{S\,M}.$$

So Fejér zone mappings with the same generating set M but different indices are transfered into each other by relaxations, and relaxations of Fejér zone mappings are again Fejér zone mappings with another index. Besides, relaxation can be reached by suitable stretching of the generating set. The stretching of the zones recovers also in the corresponding boundaries and intervals. For instance, rays $r(y,d)$ induce intersections

$$G^\alpha_M(y) \cap r(y,d) = S^\mu_y\,(G^\beta_M(y) \cap r(y,d))$$

such that intervals starting from the loop y and ending on the relative boundary of the zones are also connected by central stretching. In particular, this applies to the skeletons. But this aspect will be more illuminated in [10].

The Fejér zones form for fixed $y \notin M$ in view of Corollary 3.3 a chain with

$$\{y\} \subset G^\beta_M(y) \subset G^\alpha_M(y) \subset G^0_M(y) \quad \text{for} \quad \beta > \alpha > 0.$$

The similarity properties which have been discussed can be exploited to simplify the proofs. For instance, choose $\beta = 0$ and $\beta = 1$ in Theorem 4.5, respectively. Then indexing by α means central stretching relative to y with the ratios $\mu = \lambda_0 = \frac{1}{1+\alpha}$ and $\mu = \lambda_1 = \frac{2}{1+\alpha}$ depending on wether the starting point is the Fejér 0-zone or 1-zone, respectively. So it suffices to restrict the investigations concerning Fejér α-zones to one of the standard indices $\alpha = 0$ or $\alpha = 1$, followed by the hint that the general case arises from the properties of stretching. This would have been possible already in Section 3. But we will practice it from the next section using $\alpha = 0$. The other case $\alpha = 1$ will be more convenient for the interval union model of Fejér zones (see [10]).

5. Relations for balls

Besides, we consider *rays* (half-lines)

$$r(x_0,d) = \{x_0 + \nu\, d, \ \ \nu \geq 0\}$$

in the direction $d \in H$ starting from x_0 and *lines*

$$l(x_0, d) = \{x_0 + \nu\, d, \;\; \nu \in \mathbb{R}\}$$

in the direction $d \in H$ passing through x_0. Further, there will occur *hyperplanes*

$$H(x_0, d) = \{z \in H : \langle d, z - x_0\rangle = 0\}$$

with normal $d \neq 0$ containing x_0 as well as *halfspaces*

$$H_-(x_0, d) = \{z \in H : \langle d, z - x_0\rangle \leq 0\}, \quad H_+(x_0, d) = H_-(x_0, -d)$$

with x_0 on the boundary and outer and inner normal d, respectively. The following identity is less known but easy to check. It serves to prove some ball relations which are used in Section 6 to reduce the generating set M in the representation (6) of Fejér α-zones.

Lemma 5.1. *For all elements u, v in H and for arbitrary $\mu \in \mathbb{R}$ we have*

$$\|\mu\, u + (1-\mu)\, v\|^2 = \mu\, \|u\|^2 + (1-\mu)\|v\|^2 - \mu\,(1-\mu)\, \|u - v\|^2.$$

Lemma 5.2. *Let x_1 and x_2 be fixed elements of H. Then*

$$B(x, \|x_2 - x\|) \subseteq B(x_1, \|x_2 - x_1\|) \quad \forall x \in [x_1, x_2].$$

Proof. Obviously, the assertion is true for $x = x_1$ and $x = x_2$. So we choose an element $x \in (x_1, x_2)$ which has the representation

$$x = \mu\, x_1 + (1-\mu)\, x_2$$

for some $\mu \in (0, 1)$. We consider an arbitrary element z of the ball $B(x, \|x_2 - x\|)$. Thus z satisfies $\|z - x\| \leq \|x_2 - x\|$ and consequently, replacing x by the above representation,

$$\|z - x_2 + \mu\,(x_2 - x_1)\| \leq \mu\, \|x_2 - x_1\|.$$

Now we get in view of Lemma 5.1 the estimates

$$\begin{aligned} \mu\, \|z - x_1\|^2 &\leq \mu\, \|z - x_1\|^2 + (1-\mu)\, \|z - x_2\|^2 \\ &= \|\mu\,(z - x_1) + (1-\mu)\,(z - x_2)\|^2 + \mu\,(1-\mu)\, \|x_2 - x_1\|^2 \\ &= \|z - x_2 + \mu\,(x_2 - x_1)\|^2 + \mu\,(1-\mu)\, \|x_2 - x_1\|^2 \\ &\leq \mu^2\, \|x_2 - x_1\|^2 + \mu\,(1-\mu)\, \|x_2 - x_1\|^2 = \mu\, \|x_2 - x_1\|^2. \end{aligned}$$

Hence, we have $z \in B(x_1, \|x_2 - x_1\|)$. ■

Lemma 5.3. *Let x_1, x_2 and y be fixed elements of H. Then*

$$B(x_1, \|y - x_1\|) \cap B(x_2, \|y - x_2\|) \subseteq B(x, \|y - x\|) \quad \forall x \in [x_1, x_2].$$

Proof. We choose $z \in B(x_1, \|y - x_1\|) \cap B(x_2, \|y - x_2\|)$ and $x \in [x_1, x_2]$. Then we have for some $\mu \in [0, 1]$ the representation $x = \mu x_1 + \nu x_2$ with the abbreviation $\nu = 1 - \mu$. Exploiting the condition for z and the identity in Lemma 5.1 we can estimate

$$\begin{aligned} \|z - x\|^2 &= \|\mu(z - x_1) + \nu(z - x_2\|^2 \\ &= \mu\|z - x_1\|^2 + \nu\|z - x_2\|^2 - \mu\nu\|x_1 - x_2\|^2 \\ &\leq \mu\,\|y - x_1\|^2 + \nu\,\|y - x_2\|^2 - \mu\nu\,\|x_1 - x_2\|^2 \\ &= \|\mu(y - x_1) + \nu(y - x_2)\|^2 = \|y - x\|^2. \end{aligned}$$

This shows $z \in B(x, \|y - x\|)$. ∎

Proposition 5.4. *Let be $L \subset H$ and $y \in H$. Then*

$$\bigcap_{x \in L} B(x, \|y - x\|) = \bigcap_{x \in \overline{\operatorname{conv}} L} B(x, \|y - x\|).$$

Proof. The inclusion $\supseteq$ is obvious. So we prove only the reverse inclusion. First, we consider the relation

$$\bigcap_{x \in L} B(x, \|y - x\|) \subseteq B(z, \|y - z\|) \quad \forall z \in K$$

with $K = \operatorname{conv} L$. It holds for $L = \{x_1, x_2\}$ in view of Lemma 5.3. By induction this special case can be generalized to $L = \{x_1, x_2, \cdots, x_n\}$. Moreover, the above relation is also true for an arbitrary set L using the representation of elements in the convex hull of L. Now, obviously the systems of inequalities $\{\|z - x\| \leq r_x \quad \forall x \in N\}$ and $\{\|z - x\| \leq r_x \quad \forall x \in \overline{N}\}$ are equivalent for each index set $N \subseteq H$ if the functional r_x depends continuously on x. This corresponds to

$$\bigcap_{x \in N} B(x, r_x) = \bigcap_{x \in \overline{N}} B(x, r_x).$$

Using this equation for $N = \operatorname{conv} L$ and $r_x = \|y - x\|$ the relation is even shown for $K = \overline{\operatorname{conv}} L$. But then the assertion is an immediate consequence. ∎

The next statements refer to the intersection of balls and halfspaces. They are used to describe the Fejér α-zones of rays in Section 7.

Lemma 5.5. *Let x_1, x_2 and y be fixed elements of H. Then*

$$B(x_1, \|y - x_1\|) \cap H_-(y, x_1 - x_2) \subseteq B(x_2, \|y - x_2\|) \cap H_-(y, x_1 - x_2).$$

Proof. Let be $d = x_1 - x_2$ the outer normal of the above halfspace. Then we have for $z \in H$

$$\begin{aligned} p &= \|y - x_2\|^2 - \|z - x_2\|^2 = \|y - x_1 + d\|^2 - \|z - x_1 + d\|^2 \\ &= \|y - x_1\|^2 - \|z - x_1\|^2 + 2\,\langle y - x_1, d\rangle - 2\,\langle z - x_1, d\rangle \\ &= \|y - x_1\|^2 - \|z - x_1\|^2 - 2\,\langle z - y, d\rangle. \end{aligned}$$

Now assume $z \in B(x_1, \|y - x_1\|) \cap H_-(y, d)$. Then $\|y - x_1\|^2 - \|z - x_1\|^2 \geq 0$ and $\langle z - y, d\rangle \leq 0$ such that $p \geq 0$. But this simply means $z \in B(x_2, \|y - x_2\|)$ and by assumption also $z \in B(x_2, \|y - x_2\|) \cap H_-(y, d)$. ■

Corollary 5.6. *Let y be a fixed element of H and $r(x_0, d)$ a ray in H. Then*

$$\bigcap_{x \in r(x_0, d)} B(x, \|y - x\|) = B(x_0, \|y - x_0\|) \cap H_+(y, d).$$

Proof. The elements x in $r(x_0, d)$ have the representation $x = x_0 + \nu\, d$, where $\nu \geq 0$. Hence

$$H_-(y, x_0 - x) = H_-(y, -\nu\, d) = H_+(y, d)$$

holds for all such x. Consequently, Lemma 5.5 supplies for $x_1 = x_0$ and each $x_2 = x \in r(x_0, d)$ the relations

$$B(x_0, \|y - x_0\|) \cap H_+(y, d) \subseteq B(x, \|y - x\|) \cap H_+(y, d).$$

But this means

$$\begin{aligned} & B(x_0, \|y - x_0\|) \cap H_+(y, d) \subseteq \bigcap_{x \in r(x_0, d)} B(x, \|y - x\|) \cap H_+(y, d) \\ & \subseteq B(x_0, \|y - x_0\|) \cap H_+(y, d). \end{aligned}$$

Correspondingly, we have for elements $x_\lambda = x_0 + \lambda\, d \in r(x_0, d)$ the relation

$$\bigcap_{0\le\mu\le\lambda} B(x_\mu, \|y - x_\mu\|) \cap H_-(y, d) = B(x_\lambda, \|y - x_\lambda\|) \cap H_-(y, d). \tag{18}$$

If $y^* = H(y, d) \cap l(x_0, d)$ denotes the projection of y onto the line $l(x_0, d)$, then the length of the interval

$$[y_\lambda, y^*] = B(x_\lambda, \|y - x_\lambda\|) \cap r(y^*, -d)$$

is the height of the ballcap in (18) with basic area in $H(y, d)$. Obviously, we have $\lim_{\lambda\to+\infty} y_\lambda = y^*$. Hence, (18) supplies for $\lambda \to +\infty$ the relation

$$\bigcap_{x\in r(x_0,d)} B(x, \|y - x\|) \cap H_-(y, d) \subseteq H(y, d)\ .$$

This shows the assertion. ■

6. Reduced ball intersection models

Obviously M is not the only set whose elements x can constitute the Fejér α-zones $G^\alpha_M(y)$ by intersection of correponding sets $G^\alpha_x(y)$ (compare (6)).

Definition 6.1. The set N in H is said to be a *generating set* for $G^\alpha_M(y)$ if

$$G^\alpha_M(y) = \bigcap_{x\in N} G^\alpha_x(y).$$

The aim is to reduce the ball intersection model in Theorem 3.2 by using only a part of the sets $G^\alpha_x(y)$. Trivially, M itself is a generating set for $G^\alpha_M(y)$ (in accordance with the notation in Definition 2.4).

A first observation is that N can generally depend on y, but not on α.

Lemma 6.2. *Let be $\alpha_1 \ge 0$, $\alpha_2 \ge 0$ and $y \in H$. Then N is a generating set for $G^{\alpha_1}_M(y)$ iff N is a generating set for $G^{\alpha_2}_M(y)$.*

Proof. Let N be a generating set for $G^{\alpha_1}_M(y)$. Let $S = S^\mu_y$ be the central stretching with $\mu = \mu(\alpha_2, \alpha_1)$ according to (17). By Theorem 4.5, Lemma 4.2 and Lemma 4.4, we have

$$G^{\alpha_2}_M(y) = S\, G^{\alpha_1}_M(y) = \bigcap_{x\in N} S\, G^{\alpha_1}_x(y) = \bigcap_{x\in N} G^{\alpha_2}_x(y),$$

i.e., N is a generating set for $G_M^{\alpha_2}(y)$. The inverse implication holds now by the symmetry of α_1 and α_2 in the Lemma. ■

By the above result the proofs can be restricted to a fixed α. In the following, the case $\alpha = 0$ is used. The next simple result supplies a tool to construct new generating sets. Remember that M is supposed to be nonempty, convex and closed.

Lemma 6.3. *Let $\alpha \geq 0$ and $y \in H$ be given. If N is a generating set for $G_M^\alpha(y)$ and L satisfies*

$$L \subseteq M, \quad N \subseteq \overline{\operatorname{conv}}\, L,$$

then L is also a generating set for $G_M^\alpha(y)$.

Proof. By Lemma 6.2 we can choose $\alpha = 0$. Then we have

$$G_x^0(y) = \mathbb{B}^0(y, x) = B(x, \|y - x\|).$$

Since N is a generating set for $G_M^\alpha(y)$ and satisfies $N \subseteq \overline{\operatorname{conv}}\, L \subseteq M$, Proposition 5.4 leads to the assertion

$$\bigcap_{x \in L} B(x, \|y - x\|) = \bigcap_{x \in \overline{\operatorname{conv}}\, L} B(x, \|y - x\|) = \bigcap_{x \in M} B(x, \|y - x\|).$$ ■

Corollary 6.4. *The Fejér α-zones can be obtained for arbitrary loop $y \in H$ and arbitrary index $\alpha \geq 0$ by reducing the generating set M to any subset L satisfying $M = \overline{\operatorname{conv}}\, L$, i.e. for such a set L we obtain*

$$G_M^\alpha(y) = \bigcap_{x \in L} \mathbb{B}^\alpha(y, x).$$

Proof. We use Lemma 6.3 with $N = M$ and the ball intersection model in Theorem 3.2. ■

The following concept allows the elimination of further sets $G_x^\alpha(y)$ in the intersection.

Definition 6.5. The set

$$M(y) = \{x \in M : (y, x) \cap M = \emptyset\}$$

is said to be the *visible part* of M w.r.t. $y \in H$. Moreover, the set

$$M_s(y) = \{x \in H : [y, x] \cap M(y) \neq \emptyset\}$$

is called the *shadow extension* of M w.r.t. $y \in H$.

For $y \in M$ we have the trivial cases $M(y) = \{y\}$ and $M_s(y) = H$. Obviously, $M(y)$ is a nonempty closed subset of the closed set $\operatorname{rbd} M$. Further, $M_s(y)$ is a closed convex but unbounded set containing M. Finally, M and $M_s(y)$ have the same visible part w.r.t. y.

Theorem 6.6. *For a fixed loop $y \in H$ and each index $\alpha \geq 0$ the Fejér α-zones can be obtained by reducing the generating set M to the visible part $M(y)$, namely it is*

$$G_M^\alpha(y) = \bigcap_{x \in M(y)} \mathbb{B}^\alpha(y, x).$$

Proof. By Lemma 6.2 we can restrict ourselves to the case $\alpha = 0$. Observe $\mathbb{B}^0(y, x) = B(x, \|y - x\|)$ and therefore

$$G_M^0(y) = \bigcap_{x \in M} B(x, \|y - x\|).$$

For any $y \in H$ and any $x \in M$ there is a unique $x' \in M(y) \cap [y, x]$ (see e.g. [4, p. 45]). By Lemma 5.2 we get $B(x', \|y - x'\|) \subseteq B(x, \|y - x\|)$. So the balls with midpoints $x \in M \setminus M(y)$ do not really contribute to the intersection and can be cancelled. Hence, the visible part $M(y)$ of M is a generating set for $G_M^0(y)$. ■

The next statement shows that innumerable closed convex sets are generating sets for the same Fejér zone $G_M^\alpha(y)$.

Corollary 6.7. *Let $\alpha \geq 0$ and $y \in H$ be given. Then*

$$G_M^\alpha(y) = G_{\overline{\operatorname{conv}}\, M(y)}^\alpha(y) = G_{M_s(y)}^\alpha(y).$$

Proof. It is easy to see that $\overline{\operatorname{conv}}\, M(y)$ and $M_s(y)$ have the same visible part $M(y)$ w.r.t. y as M. Hence, the assertion follows by Theorem 6.6 and Lemma 6.3. ■

In view of $M(y) \subseteq \operatorname{rbd} M \subseteq M$ the set $\operatorname{rbd} M$ is a generating set for $G_M^\alpha(y)$ for all $\alpha \geq 0$ and independently on y. A question arises if there are other efficient generating sets $N(y)$ beside $M(y)$ and $\operatorname{rbd} M$. Following Corollary 6.4, we are confronted with the concepts of extremal and exposed elements which is crucial in characterizing convex sets L. An element $x \in L$ is called *extremal* if $L \setminus \{x\}$ is also convex. The set of these elements is denoted by $\operatorname{ext} L$. An element $x \in L$ is said to be *exposed* if there exists a supporting hyperplane K of L with $L \cap K = \{x\}$. The set of exposed elements is abbreviated by $\exp L$ (see e.g. [2] or [3]). There are convex sets L without extremal or exposed elements. But $\operatorname{ext} L$ and $\exp L$ are nonempty if L is additionally compact. The following well-known result shows that in this case the closed convex hull of both $\operatorname{ext} L$ and $\exp L$ generates already the whole set L (see [3, p. 41, p. 130]).

Lemma 6.8. *For compact convex sets L we have the representation*

$$L = \overline{\operatorname{conv}} \operatorname{ext} L = \overline{\operatorname{conv}} \exp L.$$

Observe the relation $\exp L \subseteq \operatorname{ext} L \subseteq \overline{\exp L}$ for compact sets L. (see [3, p. 125]). It is worth mentioning that $\overline{\operatorname{ext} L}$ is the minimal closed set with the above property. Namely, if L is compact, then for subsets N of L the representation $L = \overline{\operatorname{conv}}\, N$ is equivalent to $\operatorname{ext} L \subseteq \overline{N}$ (see [3, p. 42]). Now we turn again to the closed convex set M.

Theorem 6.9. *Let M be compact. Then the Fejér α-zones can be obtained by reducing the generating set M to its sets of extreme and exposed elements, respectively. This means*

$$G_M^\alpha(y) = \bigcap_{x \in \operatorname{ext} M} \mathbb{B}^\alpha(y,x) = \bigcap_{x \in \exp M} \mathbb{B}^\alpha(y,x).$$

Proof. Corollary 6.4 supplies the assertion if L is there replaced by $\operatorname{ext} M$ and $\exp M$, respectively. ■

The latter result can be improved by combining it with the result in Theorem 6.6. We start with an auxiliary statement.

Lemma 6.10. *Let $y \in H$ and $M(y)$ be compact. Then*

$$\operatorname{ext} \overline{\operatorname{conv}}\, M(y) = \operatorname{ext} M \cap M(y).$$

Proof. By [2, p. 112] we have $\operatorname{ext} M' \supseteq \operatorname{ext} M \cap M'$ for each convex subset M' of M. Further, the relation $\operatorname{ext} \overline{\operatorname{conv}}\, N \subseteq N$ is fulfilled for each compact set N (see [3, p. 38]). Observing these results for $M' = \overline{\operatorname{conv}}\, M(y)$ and $N = M(y)$ we get

$$M(y) \supseteq \operatorname{ext} \overline{\operatorname{conv}}\, M(y) \supseteq \operatorname{ext} M \cap \overline{\operatorname{conv}}\, M(y) \supseteq \operatorname{ext} M \cap M(y).$$

The proof is ready if we show that $\operatorname{ext}\overline{\operatorname{conv}}\, M(y) \subseteq \operatorname{ext} M$. We assume the contrary. Then there is an element x with $x \in \operatorname{ext}\overline{\operatorname{conv}}\, M(y) \setminus \operatorname{ext} M$. This x lies in an open interval (x_1, x_2) with $\{x_1, x_2\} \subseteq M \setminus M(y)$. But then we have elements $x_1' \in (y, x_1) \cap M$ and $x_2' \in (y, x_2) \cap M$. So we can find an interval $[z_1, z_2]$ in M which meets $[y, x]$ in the interior. This is a contradiction to $x \in M(y)$. Hence, the assertion follows. ■

For compact sets their convex hull is also compact (see [3, p. 22]). So we have $M' = \overline{\operatorname{conv}}\, M(y) = \operatorname{conv} M(y)$ under the assumptions of Lemma 6.10. Further, the set $\operatorname{ext} M' = \operatorname{ext} M \cap M(y)$ is nonempty.

Theorem 6.11. *Let $M(y)$ be compact for a fixed loop $y \in H$. Then the Fejér α-zones can be obtained for each index $\alpha \geq 0$ by reducing the generating set M to $\operatorname{ext} M \cap M(y)$. This means*

$$G_M^\alpha(y) = \bigcap_{x \in \operatorname{ext} M \cap M(y)} \mathbb{B}^\alpha(y, x).$$

Proof. We suppose $M(y)$ to be compact. As just mentioned, $\overline{\operatorname{conv}}\, M(y)$ is compact, too. But then Corollary 6.7 and Theorem 6.9 supply

$$G_M^\alpha(y) = G_{\overline{\operatorname{conv}}\, M(y)}(y) = G_L^\alpha(y),$$

where $L = \operatorname{ext}\overline{\operatorname{conv}}\, M(y)$. In view of Lemma 6.10, this is the assertion. ■

Observe that $M_e(y) = \operatorname{ext} M \cap M(y)$ does not have to be the minimal generating set for $G_M^\alpha(y)$, not even for sets M with compact visible part $M(y)$. For instance, let M be a ball which does not contain y. Then all boundary elements are extreme and even exposed (see [3, p. 35, p. 90–91]). Thus $M_e(y)$ coincides with the visible part $M(y)$ which is the surface of a ball cap. But, considering Lemma 6.3 for $N = M(y)$, a subset L whose closed convex hull includes this ball cap is already a generating set. So a dense subset L of $M_e(y)$ suffices.

Now we turn to the question which elements have necessarily to be in the generating set. We start with a modification of Lemma 2.7 which reflects the proper subset relation.

Lemma 6.12. *Let $y \in H \setminus M$ and $z \in H \setminus M_s(y)$ be given. Then*

$$G_{M_z}^\alpha(y) \subset G_M^\alpha(y)$$

holds, where $M_z = \overline{\operatorname{conv}}\,(M \cup \{z\})$.

Proof. Again we can restrict ourselves to the case $\alpha = 0$. By Lemma 2.7 we see because of $M \subseteq M_z$ that

$$G^0_{M_z}(y) = \bigcap_{x \in M_z} B(x, \|y - x\|) \subseteq \bigcap_{x \in M} B(x, \|y - x\|) = G^0_M(y)$$

is true. Since $z \notin M_s(y)$ is supposed, we have $[y, z] \cap M = \emptyset$. Observing that $[y, z]$ is compact, the well-known (second) separating theorem implies the existence of a hyperplane $H(x^*, e)$ which strictly separates $[y, z]$ and M (see e.g. [3, p. 25, p. 73]). Thus we obtain for the corresponding halfspace $H_+ = H_+(x^*, e)$ with inner normal e the relations

$$M \subseteq H_+, \quad [y, z] \cap H_+ = \emptyset.$$

Now we consider the ray $r(y, e) \perp H_+$. Obviously, the length of the intervals

$$[y, x_r] = B(x, \|y - x\|) \cap r(y, e), \quad x_r = y + \lambda_x\, e, \quad x \in H$$

increases monotone with $\lambda_x \geq 0$. If $x \in M$ is arbitrary, then the relations $z \notin H_+$, $x^* \in H(x^*, e)$ and $x \in H_+$ imply $0 \leq \lambda_z < \lambda_{x^*} \leq \lambda_x$ and therefore

$$[y, z_r] \subset [y, x^*_r] \subseteq [y, x_r].$$

Since $z \in M_z$, we have, on the one hand

$$G^0_{M_z}(y) \cap r(y, e) \subseteq B(z, \|y - z\|) \cap r(y, e) = [y, z_r].$$

On the other hand, we find

$$[y, z_r] \subset [y, x^*_r] \subseteq \bigcap_{x \in M} [y, x_r] = G^0_M(y) \cap r(y, e).$$

Hence, $G^0_{M_z}(y) \subset G^0_M(y)$ follows. ■

Let $M(y)$ be compact. Then $\operatorname{ext} M \cap M(y)$ generates the Fejér zones of M by Theorem 6.11. But now we will state that it loses its property of a generating set, if isolated elements are cancelled.

Proposition 6.13. *Let $y \in H \setminus M$. If the set $M_e(y) = \operatorname{ext} M \cap M(y)$ has an isolated element z, then*

$$\bigcap_{x \in M_e(y) \setminus \{z\}} \mathbb{B}^\alpha(y, x) \supset G^\alpha_M(y).$$

Proof. For an isolated element z of $M_e(y)$ we consider $N(y) = M_e(y) \setminus \{z\}$. Further, we introduce the set $N' = \overline{\text{conv}}\, N(y)$. Then $y \notin N'$. The properties of z guarantee that this element cannot belong to the shadow extension $N'_s(y)$ of N' w.r.t. y. Since M contains the visible part $N'_z(y)$ of $N'_z = \overline{\text{conv}}\,(N' \cup \{z\})$, we get by Theorem 3.2, Theorem 6.6, Lemma 6.12 and Corollary 6.4 the relations

$$\begin{aligned} G^\alpha_M(y) &= \bigcap_{x \in M} \mathbb{B}^\alpha(y,x) \subseteq \bigcap_{x \in N'_z(y)} \mathbb{B}^\alpha(y,x) \\ &= G^\alpha_{N'_z}(y) \subset G^\alpha_{N'}(y) = \bigcap_{x \in N(y)} \mathbb{B}^\alpha(y,x). \end{aligned}$$

This is the assertion. ■

7. Examples and Conclusions

In the following, we use the balls

$$G^\alpha_x(y) = \mathbb{B}^\alpha(y,x) = B(z^\alpha_m(y,x), r^\alpha_m(y,x))$$

with midpoints (7) and radii (8). In particular, we have

$$\mathbb{B}^0(y,x) = B(x, \|y-x\|), \quad \mathbb{B}^1(y,x) = B(\frac{y+x}{2}, \frac{\|y-x\|}{2}).$$

Further, the set M is always supposed to be convex and closed such that the metric projector P_M onto M exists. Besides, remember the notations at the beginning of Section 5.

Example 7.1 (One-ball Fejér zone)**.** Let $M = \{x_1\}$ be a singleton and $y \neq x_1$. Then we get by Theorem 3.2 the ball Fejér zone

$$G^\alpha_M(y) = \mathbb{B}^\alpha(y, x_1),$$

where trivially $x_1 = P_M y$ holds. But $G^\alpha_M(y)$ remains the same ball if M is extended within the limits

$$\{x_1\} \subseteq M \subseteq r(x_1, x_1 - y).$$

Namely, then the visible part $M(y)$ of M reduces to $\{x_1\}$, and the assertion follows by Theorem 6.6.

Example 7.2 (Fejér zones with two-ball overlaps)**.** We consider the set $M = [x_1, x_2]$ with $x_1 \neq x_2$ and $y \notin [x_1, x_2]$. Then M is compact and $\operatorname{ext} M = \exp M = \{x_1, x_2\}$. Hence, we have by Theorem 6.9 the Fejér zone

$$G_M^\alpha(y) = \mathbb{B}^\alpha(y, x_1) \cap \mathbb{B}^\alpha(y, x_2).$$

If $y \in \operatorname{aff} M$, then one ball is contained in the other (see also Example 7.1). Otherwise it is easy to see, that both balls overlap. But $G_M^\alpha(y)$ does not change if we have more general

$$[x_1, x_2] \subseteq M \subseteq [x_1, x_2]_s(y), \quad y \notin \operatorname{aff}[x_1, x_2].$$

Then there holds $M(y) = [x_1, x_2]$ and $\operatorname{ext} M \cap M(y) = \{x_1, x_2\}$.

Example 7.3 (Fejér zones with k-ball overlaps)**.** Generalizing the foregoing two examples we assume a set X_k of k affinely independent elements $x_1, x_2, \cdots, x_k$. Then $M = \operatorname{conv} X_k$ is a polyhedron with the proper vertices $x_i \in X_k$ $(i = 1, 2, \cdots, k)$. Obviously, M is compact and $\operatorname{ext} M = \exp M = X_k$. Now let $y \notin M$. Assume that the first l vertices belong to the visible part $M(y)$ of M and the other $k - l$ do not. Then Theorem 6.11 supplies

$$G_M^\alpha(y) = \bigcap_{i=1}^{l} \mathbb{B}^\alpha(y, x_i).$$

For $k = 2$ we have $l = 1$ in the case $y \in \operatorname{aff} M$ and $l = 2$ otherwise. The above ball representation is minimal, that is, none of the l balls is superfluous (compare Proposition 6.13). Again M can be extended up to the shadow extension of the polyhedron w.r.t. y without changing the Fejér zones.

Example 7.4 (Ball cap Fejér zones)**.** Let $M = r(x_0, d)$ be a ray. Then

$$G_M^\alpha(y) = \mathbb{B}^\alpha(y, x_0) \cap H_+(y, d).$$

For $y \notin l(x_0, d)$ this is a proper ball cap. Otherwise, the halfspace contains the ball and can be omitted. We start with the case $\alpha = 0$. By Theorem 3.2 and Corollary 5.6 we see

$$G_M^0(y) = \bigcap_{x \in r(x_0, d)} B(x, \|y - x\|) = B(x_0, \|y - x_0\|) \cap H_+(y, d).$$

This is already the assertion. The case for general α arises if the central stretching $S_y^{\lambda_1}$ with ratio $\lambda_1 = \frac{1}{1+\alpha}$ is applied to the above identity and Theorem 4.5, Lemma 4.2 and (14) are observed.

Example 7.5 (Hyperplane ball Fejér zones)**.** Let $M = l = l(x_0, d)$ be a line which does not contain y. Then

$$G^\alpha_M(y) = \mathbb{B}^\alpha(y, x_0) \cap H(y, d) = \mathbb{B}^\alpha(y, P_l y) \cap H(y, d)$$

is a ball in the hyperplane $H(y, d)$. Namely, $l(x_0, d) = r(x_0, d) \cup r(x_0, -d)$. Hence, by Example 7.4, Lemma 2.8 and the equality $H_-(y, d) = H_+(y, -d)$ we find

$$G^\alpha_M(y) = \mathbb{B}^\alpha(y, x_0) \cap H_+(y, d) \cap H_-(y, d).$$

Because of $H(y, d) = H_+(y, d) \cap H_-(y, d)$ this is the first assertion. But, we have also $l = l(P_l y, d)$. So the second assertion follows analogously.

Example 7.6 (Flat ball Fejér zones)**.** Let $M = L = L(x_0)$ be a flat (affine subspace) containing x_0. Then $L = L_0 + x_0$, where L_0 is the linear part of L. We state

$$G^\alpha_M(y) = \mathbb{B}^\alpha(y, x_0) \cap L^\perp(y) = \mathbb{B}^\alpha(y, P_L y) \cap L^\perp(y).$$

For $y \notin L$ this is a ball in the orthogonal flat $L^\perp(y) = L_0^\perp + y$ through y. Namely, introducing the abbreviation $E = \operatorname{bd} B(0,1) \cap L_0$, there holds $M = \bigcup_{e \in E} l(x_0, e)$. This means by Lemma 2.8 and by Example 7.5

$$\begin{aligned} G^\alpha_M(y) &= \bigcap_{e \in E} G^\alpha_{l(x_0,e)}(y) = \bigcap_{e \in E} \big(\mathbb{B}^\alpha(y, x_0) \cap H(y, e)\big) \\ &= \mathbb{B}^\alpha(y, x_0) \cap \bigcap_{e \in E} H(y, e). \end{aligned}$$

This is the assertion if

$$L_0^\perp = \bigcap_{e \in E} H(0, e), \quad L^\perp(y) = y + \bigcap_{e \in E} H(0, e) = \bigcap_{e \in E} H(y, e)$$

is observed. The same Fejér α-zone arises if M is the shadow extension of a flat L w.r.t. y. This follows by Corollary 6.7. In particular, if M is a hyperplane $H = H(y^*, d)$, then $H^\perp(y) = l(y, d) = l(P_H y, y - P_H y)$ and therefore

$$G^\alpha_M(y) = [y, z^\alpha_s(y, P_H y)]$$

coincides with its skeleton (see Section 3). If M is a halfspace $H_+(y^*, d)$, the same result is obtained. Namely, the visible part of $H_+(y^*, d)$ w.r.t. y is just $H(y^*, d)$.

Example 7.7 (Cone limited ball part Fejér zones)**.** Let C_0 be a closed convex cone and $M = C = C(x_0) = C_0 + x_0$ its translate with vertex x_0. If

$$C_0^o = \{z \in H : \langle z, x \rangle \leq 0 \quad \forall x \in C_0\}$$

denotes the *polar cone* and $C_0' = -C_0^o$ the *cocone* of C_0, then the Fejér zone

$$G_M^\alpha(y) = \mathbb{B}^\alpha(y, x_0) \cap C'(y)$$

is a ball part in the translated cocone $C'(y) = y + C_0'$ of C_0 with vertex y. Namely, the representation $M = \bigcup_{e \in E} r(x_0, e)$ holds with the abbreviation $E = \operatorname{bd} B(0,1) \cap C_0$. This means by Lemma 2.8 and Example 7.4

$$G_M^\alpha(y) = \bigcap_{e \in E} G_{r(x_0,e)}^\alpha(y) = \mathbb{B}^\alpha(y, x_0) \cap \bigcap_{e \in E} H_+(y, e).$$

This is the assertion, since we have

$$C_0^o = \bigcap_{e \in E} H_-(0, e), \quad C'(y) = y + \bigcap_{e \in E} H_+(0, e) = \bigcap_{e \in E} H_+(y, e).$$

Example 7.8 (Fejér zones of a ball)**.** If $M = B(0,1)$ is the unit ball, then in view of Section 6 the representation

$$G_M^\alpha(y) = \bigcap_{x \in \operatorname{bd} B(0,1)} \mathbb{B}^\alpha(y, x)$$

holds. By Theorem 6.6 it suffices to choose x in the visible part of $B(0,1)$ w.r.t. y and as already mentioned even in a dense subset of it. For $H = \mathbb{R}^2$ the Fejér zones of the unit circle have the form of a rose leaf. The precise description in polar form is

$$r = r(\varphi) = \frac{2}{1+\alpha}\,(x_0 \cos\,\varphi - 1), \quad |\varphi| \leq \arccos\frac{1}{x_0}$$

if without loss of generality $y = (x_0, 0)^T$ is assumed.

Applying Lemma 2.7 we are able to construct inner and outer approximations of Fejér zones. By inner approximations we get conditions for Fejér zones with inner points. Outer approximations are suitable to characterize Fejér zones without inner points.

Definition 7.9. The element $d \in H$ is said to be a *recession direction* of the set M if M has an element x_0 such that the whole ray $r(x_0, d)$ is contained in M.

The recession direction d does not depend on the special element x_0 of M. The set of all recession directions is a closed convex cone, the so-called *recession cone* $C_\infty(M)$ of M (see [2, p. 108–110]). If both d and $-d$ are recession directions of M, then M contains a whole line $l = l(x_0, d)$ for some $x_0 \in M$. Such a line is said to be a *recession line* of M.

Theorem 7.10. *Let $C_0 = C_\infty(M)$ be the recession cone of M and let $C'(y) = C_0' + y$ be its translated cocone with vertex $y \in H$. Then the Fejér zones $G_M^\alpha(y)$ are contained in $C'(y)$.*

Proof. For an arbitrary $x_0 \in M$ we consider the translated cone $N = C(x_0) = C_0 + x_0 \subseteq M$. By Example 7.7 we get $G_N^\alpha(y) = \mathbb{B}^\alpha(y, x_0) \cap C'(y)$. So Lemma 2.7 shows $G_M^\alpha(y) \subseteq G_N^\alpha(y) \subseteq C'(y)$. ∎

Corollary 7.11. *Let be $y \in H$. If d is a recession direction of M, then $G_M^\alpha(y)$ is contained in the halfspace $H_+(y, d)$.*

Proof. Obviously, $r(0, d)$ belongs to the recession cone $C_0 = C_\infty(M)$. Hence, by Theorem 7.10, we find for the translated cocone $C'(y)$ of C_0 the inclusion $G_M^\alpha(y) \subseteq C'(y)$. Finally, we use the cocone property

$$C'(y) \subseteq r'(y) = r(0, d)' + y = H_+(0, d) + y = H_+(y, d).$$ ∎

Lemma 7.12. *Let $y \in H$. If M contains a recession line l, then $G_M^\alpha(y)$ has no inner points.*

Proof. For $y \in M$ the assertion is trivial. Hence, suppose $y \notin M$. In view of $l \subseteq M$ and Theorem 2.7 we see that $G_M^\alpha(y) \subseteq G_l^\alpha(y)$. Since $l = l(x_0, d)$ for some $x_0 \in M$ and some $d \neq 0$, we have $G_l^\alpha(y) = \mathbb{B}^\alpha(y, x_0) \cap H(y, d)$ and consequently $G_l^\alpha(y) \subseteq H(y, d)$ by Example 7.5. But, the hyperplane $H(y, d)$ has no inner points. So the same holds for $G_M^\alpha(y)$. ∎

Theorem 7.13. *Let C_0 be a closed convex cone without any recession line, $C(x_0)$ the translate with vertex $x_0 \in H$ and $C'(y)$ the translate of the cocone with vertex $y \in H$. If M is contained in $C(x_0)$ and if the best approximation of x_0 in $C'(y)$ is different from y, then $G_M^\alpha(y)$ has inner points.*

Proof. Observing $M \subseteq C(x_0)$, Lemma 2.7 and Example 7.7 we find

$$G_M^\alpha(y) \supseteq \mathbb{B}^\alpha(y, x_0) \cap C'(y).$$

Since C_0 contains no recession line, both cones C_0' and $C'(y)$ have full dimension. Besides, by assumption we have $P_{C'(y)} x_0 \neq y$. This guarantees that the ball $\mathbb{B}^\alpha(y, x_0)$ and the cone $C'(y)$ have more than the element y in common. So the intersection of both sets and consequently $G_M^\alpha(y)$ itself possess inner points. ∎

Corollary 7.14. *Let $y \in H \setminus M$. If M has a bounded visible part $M(y)$, then $G_M^\alpha(y)$ has inner points.*

Proof. If $M(y)$ is bounded and $y \notin M$, then there is an element x_0 different from y and outside of M and a closed convex cone C_0 without recession line with the following two properties: $C(x_0) = C_0 + x_0$ is the closed convex cone hull of M w.r.t. x_0 and $C'(y) = C_0' + y$ satisfies $P_{C'(y)} x_0 \neq y$. So Theorem 7.13 can be applied. ∎

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