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OPTIMAL CONTROL OF NONLINEAR EVOLUTION EQUATIONS

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Abstract

In this paper, first we consider parametric control systems driven by nonlinear evolution equations defined on an evolution triple of spaces. The parametres are time-varying probability measures (Young measures) defined on a compact metric space. The appropriate optimization problem is a minimax control problem, in which the system analyst minimizes the maximum cost (risk). Under general hypotheses on the data we establish the existence of optimal controls.

Then we pass to nonparametric systems, which are governed by nonlinear evolution equations with nonmonotone operators. We prove two existence results for such evolution inclusions, which are of independent interest and extend significantly the results existing in the literature. Then we solve time-optimal and Meyer-type optimization problems. In Section 5, we derive necessary conditions for saddle point optimality in the minimax control problem. We conclude the paper with three examples of distributed parameter control systems.

Keywords and phrases: evolution triple, compact embedding, monotone operator, pseudomonotone operator, L-generalized pseudomonotonicity, integration by parts, evolution inclusion, saddle point, necessary conditions, adjoint equation, distributed parameter systems.

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1. Introduction

In this paper, we consider optimal control systems monitored by nonlinear evolution equations. First, we examine uncertain control systems. Uncertainty can arise from errors in the measurement of the parameters of the system or from their random fluctuation. In this work, we model uncertainty by time-dependent measures on a compact measure space (transition measures). The resulting system has many solutions and the natural optimization problem to consider is a minimax control problem. Namely, the system analyst tries to minimize the maximum risk (cost). So we are in a theoretic situation similar to differential game with competing interests, where the second player is nature. We also consider an optimal control problem with no uncertainty involved and with no monotonicity conditions on the nonlinear operator of the evolution equation. We prove the existence and compactness result for the solution set of a class of related evolution inclusions. This result is actually of independent interest and is then used to solve optimal control problems. Then we derive necessary conditions for saddle point optimality of the initial minimax problem. We conclude the paper with three examples of distributed parameter, nonlinear parabolic optimal control problems.

Parametric optimal control problems were studied by Ahmed-Xiang [1], Aizicovici-Papageorgiou [2] and Papageorgiou [25], [27]. In Papageorgiou [27] the system is driven by a time dependent subdifferential evolution equation, while Ahmed-Xiang [1], Aizicovici-Papageorgiou [2] and Papageorgiou [25] work with evolution equations defined on an evolution triple. In Ahmed-Xiang the parameters are measures which are not time-dependent, while in Aizicovici-Papageorgiou and Papageorgiou the parameter belongs to a complete metric space and appears also in the nonlinear operator of the evolution equation, but this then forces stronger hypotheses on the data which are avoided here. In addition, we obtain here necessary conditions for a saddle point solution to the minimax problem (see Section 5). The existence results that we have for the nonparemetric optimal control problems (see Section 4), extend in several ways those of Cesari [8], Cesari-Hou [9], Hou [17], [18] and Papageorgiou [24]. We should also mention the related recent work of Papageorgiou [26], where a theory for optimal control problems driven by time-varying subdifferential evolution equations is developed.

2. Preliminaries

In our analysis, we will need the theory of multifunctions and the theory of evolution triples. For the reader's convenience of in this section, we recall the basic definitions and results that we will need in the sequel. More details can be found in the books of Hu-Papageorgiou [19] and Zeidler [37].

Let (Ω, Σ) be a measurable space and Y a separable Banach space. Throughout this paper we will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X : A \text{ is nonempty, closed (and convex)}\}$$

and

$$P_{(w)k(c)}(X) = \{A \subseteq X : A \text{ is nonempty, (weakly) compact (and convex)}\}.$$

A multifunction $F : \Omega \to 2^Y \setminus \{\emptyset\}$ is said to be "graph measurable", if

$$GrF = \{(\omega, y) \in \Omega \times Y : y \in F(\omega)\} \in \Sigma \times \mathcal{B}(Y)$$

with $\mathcal{B}(Y)$ being the Borel σ-field of Y. A multifunction $G : \Omega \to P_f(Y)$ is said to be measurable, if for all $y \in Y$ the distance function

$$\omega \to d(y, G(\omega)) = \inf[\| y - g \| : g \in G(\omega)]$$

is measurable. For $P_f(Y)$ -valued multifunctions measurability implies graph measurability and if there is a σ-finite measure μ on (Ω, Σ) with respect to which Σ is complete, then the two notions are equivalent.

Let (Ω, Σ, μ) be a σ-finite measure space and $F : \Omega \to 2^Y \setminus \{\emptyset\}$ a multifunction. For $1 \leq p \leq \infty$, let S_F^p be the set of all $L^p(\Omega, X)$-selectors of $F(\cdot)$. The set S_F^p is nonempty if and only if $\inf\{\| z \| : z \in F(\omega)\} \leq h(\omega)$ μ-a.e. with $h \in L^p(\Omega)$.

Let V, Z be Hausdorff topological spaces. A multifunction $F : V \to 2^Z \setminus \{\emptyset\}$ is said to be lower semicontinuous (lsc) (upper semicontinuous (usc)), if for all $C \subseteq Z$ closed the set $F^+(C) = \{v \in V : F(v) \subseteq V\}$ (resp. $F^-(C) = \{v \in V : F(v) \cap C \neq \emptyset\}$) is closed in Y.

Next let H be a separable Hilbert space and let X be a dense subspace of H carrying the structure of a separable, reflexive Banach space which is embedded continuously in H. Identifying H with its dual (pivot space), we have that $X \subseteq H \subseteq X^*$ with all embeddings being continuous and dense. Such a triple of spaces is known in the literature as "evolution triple" or "Gelfand triple". We will assume that the embedding of X into H is compact (which implies that H is embedded compactly into X^*). By $|\cdot|$ (resp.$\|\cdot\|, \|\cdot\|_*$) we denote the norm of H (resp. of X, X^*). Also by $(\cdot,\cdot)$ we denote the inner product of H and by $<\cdot,\cdot>$ the duality brackets for the pair (X^*, X). The two are compatible in the sense that $<\cdot,\cdot>|_{H\times X}=(\cdot,\cdot)$. Let $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$ and $T=[a,b]$. We define

$$W_{pq}(T)=\left\{x\in L^p(T,X): \dot{x}\in L^q(T,X^*)\right\}.$$

The time-derivative involved in this definition is understood in the sense of vector-valued distributions. Furhished with the norm

$$\| x \|_{W_{pq}(T)}=\left\{\| x \|_p^2+\| \dot{x} \|_q^2\right\}^{\frac{1}{2}}$$

the space $W_{pq}(T)$ becomes a separable reflexive Banach space. It is well-known that $W_{pq}(T)$ is continuously embedded in $C(T,H)$ (i.e. every element $x\in W_{pq}(T)\subseteq L^p(T,X)$ has a unique representative in $C(T,H)$). Moreover, since we have assumed that X is embedded compactly in H, then we have that $W_{pq}(T)$ is embedded compactly in $L^p(T,H)$ (see Zeidler [37], p. 450). For further details and additional results in this area the reader can also refer to Simon [35].

Let Y be a reflexive Banach space, $L: D(L)\subseteq Y\to Y^*$ a linear densely defined maximal monotone operator and let $K: Y\to 2^{Y^*}\setminus\{\emptyset\}$ be a multivalued operator. We say that $K(\cdot)$ is "coercive" if

$$\frac{\inf[<v,y>: v\in K(y)]}{\| y \|}\to+\infty \text{ as } \| y \|\to\infty.$$

We say that K is "L-generalized pseudomonotone", if

(i) for all $y\in Y,\ K(y)\in P_{wkc}(Y^*)$;

(ii) $K(\cdot)$ is usc from every finite dimensional subspace of $D(L)$ into Y^*_w (here by Y^*_w we denote the space Y^* equipped with the weak topology);

(iii) if $\{y_n\}_{n\geq 1}\subseteq D(L)$ with $y_n\to y$ in Y, $y\in D(L)$, $L(y_n)\to L(y)$ in Y^*, $y^*\in K(y_n)$, $n\geq 1$, $y^*_n\to y^*$ in Y^* and $\overline{\lim}(y^*_n,y_n)\leq(y^*,y)$, then $y^*\in K(y)$ and $(y^*_n,y_n)\to(y^*,y)$ as $n\to\infty$.

Let $T=[a,b]$ and V a compact metric space. Let $M^1_+(V)$ be the set of all probability measures on $(V,\mathcal{B}(V))$ (as before $\mathcal{B}(V)$ denotes the Borel σ-field of V). We endow $M^1_+(V)$ with the weak topology. This is the initial topology with respect to which the functionals $\lambda\to(f,\lambda)=\int_V f(v)\lambda(dv)$, $f\in C(V)$, are continuous. We remark that $M_+^1(V)$ topologized this way is actually a compact metrizable space (see Dellacherie-Meyer [10], p. 73 for a general version of this result). A "transition probability" or "Young measure" from T into V is defined to be a function $\lambda : T \to M_+^1(V)$ such that for every $C \in \mathcal{B}(V), t \to \lambda(t)(C)$ is measurable. In fact, this definition is equivalent to saying that the map $t \to \lambda(t)(\cdot)$ is measurable from T into $M_+^1(V)$ when the latter is endowed with the weak topology. We denote the set of all transition probabilities from T into V by $R(T, V)$. The weak topology of $M_+^1(V)$ has an obvious analog on $R(T, V)$. Let $Car(T \times V)$ denote the space of L^1-Caratheodory integrands on $T \times V$; i.e., the set of all functions $g : T \times V \to R$ such that $t \to g(t, v)$ is measurable, $v \to g(t, v)$ is continuous and for some $\psi \in L^1(T)$, $\mid g(t, v) \mid\leq \psi(t)$ a.e. for all $v \in V$. Then the "weak topology" on $R(T, V)$ is defined as the initial topology on $R(T, V)$ with respect to which the functionals $\lambda \to I_g(\lambda) = \int_T \int_V g(t, v)\lambda(t)(dv)dt, g \in Car(T \times V)$, are continuous. If instead of $g \in Car(T \times V)$, we consider a nonegative normal integrand $g(t, v)$ (i.e. $g(\cdot, \cdot)$ is jointly measurable, $v \to g(t, v)$ is lower semicontinuous and $g(t, v) \geq 0$), then $\lambda \to I_g(\lambda)$ is lower semicontinuous. Let $M(V)$ be the space of finite Borel measures on V. We know that $C(V)^* = M(V)$ (Riesz representation theorem). By $L^\infty(T, M(V)_{w^*})$ we denote the space of all $M(V)$-valued functions $\lambda(\cdot)$ such that for every $f \in C(V)$, $t \to (\lambda(t), f) = \int_V f(v)\lambda(t)(dv)$ is measurable and $\mid (\lambda(t), f) \mid\leq c \mid\mid f \mid\mid_{C(V)}$ a.e. on T (the exceptional null set depending on f). The norm of $\lambda(\cdot)$ is the infimum of all these c's. We know (see Ionescu-Tulcea [21], p. 25) that $L^1(T, C(V))^* = L^\infty(T, M(V)_{w^*})$. Identifying $Car(T \times V)$ with $L^1(T, C(V))$ and viewing $R(T, V)$ as a subset of $L^\infty(T, M(V)_{w^*})$, we see that the weak topology on $R(T, V)$ is the relative $w(L^\infty(T, M(V)_{w^*}),\ L^1(T, C(V)))$ – topology.

3. Existence results for parametric problems

Let $T = [a, b]$, and let (X, H, X^*) be an evolution triple of spaces with all embeddings being compact, Y is a separable reflexive Banach space and V a compact metric space. In this section, we deal with the following parametric control system:

$$(1)\quad \left\{ \begin{aligned} \dot{x}(t) + A(t, x(t)) &= \int_V f(t, x(t), v)\lambda(t)(dv) + B(t)u(t) \text{ a.e. on } T \\ x(0) &= x_0 \in H,\ u \in S_U^q,\ \lambda \in S_\Sigma \end{aligned} \right\}$$

Here

$$S_U^q = \{u \in L^q(T,Y) : \ u(t) \in U(t) \text{ a.e. on } T\}$$

and

$$S_\Sigma = \{\lambda \in R(T,V) : \lambda(t) \in \Sigma(t) \text{ a.e. on } T\}.$$

The space Y models the control space and $U : T \to 2^Y \setminus \{\emptyset\}$ is the control constraint multifunction. The space V models the space of parameters and $\Sigma : T \to 2^V \setminus \{\emptyset\}$ is the parameter distribution constraint multifunction (to be defined precisely in the sequel).

Given $u \in S_U^q$ and $\lambda \in S_\Sigma$, let $x(u,\lambda)(\cdot) \in W_{pq}(T)$ be a solution to (1). Our hypotheses on the data will guarantee that $x(u,\lambda)(\cdot) \in W_{pq}(T)$ exists and is unique. Let

$$L : T \times H \times Y \to \bar{R} = R \cup \{+\infty\}$$

be an integrand representing the instantaneous cost (risk). We define

$$J(u,\lambda) = \int_0^b L(t, x(u,\lambda)(t), u(t))dt.$$

This is the total intertemporal cost when u and λ are in effect. Then our problem is the following minimax problem:

$$\beta = \inf_u \sup_\lambda [J(u,\lambda) : u \in S_U^q,\ \lambda \in S_\Sigma] \tag{2}$$

i.e the system analyst first, for a fixed control, computes the maximum cost and then he (she) minimizes these extremal costs over all admissible controls. We are looking for a control $u^* \in S_U^q$ such that

$$\beta = \sup[J(u^*,\lambda) : \lambda \in S_\Sigma].$$

We call the control $u^* \in S_U^q$ "optimal".

Now we can introduce our hypotheses on the data of problem (1):

$\mathbf{H(A)}$: $A : T \times X \to X^*$ is an operator such that

(i) for all $x \in X,\ t \to A(t,x)$ is measurable;

(ii) for every $t \in T,\ x \to A(t,x)$ is demicontinuous and monotone;

(iii) for almost all $t \in T$ and all $x \in X$, $\| A(t,x) \|_* \le a_1(t) + c_1 \| x \|^{p-1}$ with $a_1 \in L^q(T)$, $c_1 > 0, 2 \le p < \infty, \frac{1}{p} + \frac{1}{q} = 1$;

(iv) for almost all $t \in T$ and all $x \in X$, $< A(t,x), x > \geq c \mid\mid x \mid\mid^p - a(t)$ with $c > 0$, $a \in L^1(T)_+$.

$\mathbf{H(f)}$: $f : T \times H \times V \to H$ is a function such that

(i) for every $(x,v) \in H \times V$, $t \to f(t,x,v)$ is measurable;

(ii) for almost all $t \in T$, all $x, y \in H$ and all $v \in V$ we have

$$|f(t,x,v) - f(t,y,v)| \leq k(t)|x-y|$$

with $k \in L^1(T)$;

(iii) for all $t \in T$ and all $x \in H$, $v \to f(t,x,v)$ is continuous;

(iv) for almost all $t \in T$, all $x \in H$ and all $v \in V$,

$$|f(t,x,v)| \leq a_2(t) + c_2(x)^{2/q} \text{ with } a_2 \in L^q(T),\ c_2 > 0.$$

$\mathbf{H(B)}$: $B \in L^\infty(T, \mathcal{L}(Y,H))$ (by $\mathcal{L}(Y,H)$ we denote the Banach space of bounded linear operators from Y into H).

$\mathbf{H(U)}$: $U : T \to P_{f_c}(Y)$ is a measurable multifunction such that

$$t \to |U(t)| = \sup\{||u|| : u \in U(t)\} \in L^q(T)_+;$$

$\mathbf{H(\Sigma)}$: $\Gamma : T \to P_f(V)$ is a measurable multifunction
and $\Sigma(t) = \{\lambda \in M_+^1(V) :\ \lambda(\Gamma(t)) = 1\}$.

$\mathbf{H(L)}$: $L : T \times H \times Y \to \bar{R} = R \cup \{+\infty\}$ is an integrand such that

(i) $(t,x,u) \to L(t,x,u)$ is measurable;

(ii) for all $t \in T$, $(x,u) \to L(t,x,u)$ is lower semicontinuous;

(iii) for all $t \in T$ and all $x \in H$, $u \to L(t,x,u)$ is convex;

(iv) for almost all $t \epsilon T$ and all $x \in H$, $u \in Y$ we have

$$\varphi(t) - c_3(|x| + ||u||) \leq L(t,x,u)$$

with $\varphi \in L^1(T)$, $c_3 > 0$.

Let $G_1 : T \times H \to 2^H \setminus \{\emptyset\}$ be the multifunction defined by

$$G_1(t,x) = \int_V f(t,x,v)\Sigma(t)(dv) = \left\{ \int_V f(t,x,v)\lambda(dv) : \lambda \in \Sigma(t) \right\}.$$

Proposition 1. *If hypotheses $H(f)$ and $H(\Sigma)$ hold, then $G_1 : T \times H \to P_{f_c}(H)$, for all $x \in H$, $t \to G_1(t,x)$ is Lebesgue measurable and for almost all $t \in T$, $GrG_1(t,\cdot)$ is sequentially closed in $H \times H_w$ (by H_w denote the Hilbert space H furnished with the weak topology).*

Proof. First we show that $G_1(\cdot,\cdot)$ has values in $P_{f_c}(H)$. Convexity is clear. So we only need to show that $G_1(t,x)$ is closed. To this end, let $y_n \in G_1(t,x), n \geq 1$, and assume that $y_n \to y$ in H as $n \to \infty$. By definition we have

$$y_n = \int_V f(t,x(t),v)\lambda_n(dv),\ \lambda_n \in \Sigma(t), n \geq 1.$$

Recall that $M_+^1(V)$ furnished with the weak topology is a compact metrizable space (see Section 2). So by passing to a subsequence if necessary, we may assume that $\lambda_n \to \lambda$ in $M_+^1(V)$ as $n \to \infty$.

From the "Portmanteau Theorem" (see Parthasarathy [34], Theorem 6.1, p. 40), we have that $\lambda(\Gamma(t)) = 1$ and so $\lambda \in \Sigma(t)$. Moreover, from the definition of the weak topology, we have $\int_V f(t,x,v)\lambda_n(dv) \to \int_V f(t,x,v)\lambda(dv)$ in H as $n \to \infty$ (cf. hypothesis H(f) (iii)). Thus

$$y = \int_V f(t,x,v)\lambda(dv)$$

with $\lambda \in \Sigma(t)$ and so $y \in G_1(t,x)$. Therefore $G_1(t,x) \in P_{f_c}(H)$.

Next note that

$$\begin{aligned} GrG_1(t,\cdot) &= \{(t,y) \in T \times H :\ y \in G_1(t,x)\} \\ &= \{(t,y) \in T \times H : (h,y) \leq \sigma(h, G_1(t,x)) \text{ for all } h \in H\} \end{aligned}$$

Here $\sigma(\cdot, G_1(t,x))$ is the support function of the set $G_1(t,x)$; i.e. $\sigma(h, G_1(t,x)) = \sup[(h,y) : y \in G_1(t,x)]$. We have:

$$\begin{aligned} \sigma(h, G_1(t,x)) &= \sup\left[(h, \int_V f(t,x,v)\lambda(dv) : \lambda \in \Sigma(t)\right] \\ &= \sup\left[\int_V (h, f(t,x,v)\lambda(dv) : \lambda \in \Sigma(t)\right]. \end{aligned}$$

We will show that $Gr\Sigma \in \mathcal{L}(T) \times B(M_+^1(V))$, with $\mathcal{L}(T)$ being the Lebesque σ-field of T. Indeed let $g \in C(V)$. Using the fact that discrete measures are dense in $M_+^1(V)$ for the weak topology (see Parthasarathy [34], Theorem 6.3, p. 44), we have

$$\sigma(g, \Sigma(t)) = \sup[(\lambda, g) : \lambda \in \Sigma(t)] = \sup[g(v) : v \in \Gamma(t)].$$

Let $\gamma_n : T \to V, n \geq 1$, be Lebesgue measurable functions such that $\Gamma(t) = \overline{\{\gamma_n(t)\}}_{n\geq 1}$ for all $t \in T$ (see Hu-Papageorgiou [19], Theorem 2.5, p. 156). So we have

$$\sigma(g, \Sigma(t)) = \sup[g(v) : v \in \Gamma(t)] = \sup_{n\geq 1} g(\gamma_n(t))$$
$$\Rightarrow t \to \sigma(g, \Sigma(t)) \text{ is Lebesgue measurable.}$$

Now let $\{g_m\}_{m\geq 1}$ be dense in the Banach space $C(V)$. Since $\sigma(\cdot, \Sigma(t))$ is continuous, we have

$$\begin{aligned} Gr\Sigma &= \cap_{m\geq 1}\{(t,\lambda) \in T \times M_+^1(V) : (\lambda, g_m) \leq \sigma(g_m, \Sigma(t))\} \\ &\in \mathcal{L}(T) \times \mathcal{B}(M_+^1(V)). \end{aligned}$$

Using this fact, we see that for every $\theta \in \mathbb{R}$

$$\begin{aligned} E_\theta &= \{t \in T : \sigma(h, G_1(t,x)) > \theta\} \\ &= proj_T\left\{(t,\lambda) \in Gr\Sigma : \int_V (h, f(t,x,v)\lambda(dv) > \theta\right\}. \end{aligned}$$

But by the Yankov-von Neumann-Aumann projection theorem (see Hu-Papageorgiou [19], Theorem 1.33, p. 149), we have that $proj_T\{(t,\lambda) \in Gr\Sigma : \int_V (h, f(t,x,v)\lambda(dv) > \theta\} \in \mathcal{L}(T)$. Since $\theta \in \mathbb{R}$ was arbitary we deduce that for every $h, x \in H, t \to \sigma(h, G_1(t,x))$ is Lebesgue measurable. Hence if $\{h_m\}_{m\geq 1}$ is dense in H, we have

$$\begin{aligned} &GrG_1(\cdot, x) \\ &= \cap_{m\geq 1}\{(t,y) \in T \times H : (h_m, y) \leq \sigma(h_m, G_1(t,x))\} \in \mathcal{L}(T) \times \mathcal{B}(H) \\ &\Rightarrow t \to G_1(t,x) \text{ is a Lebesgue measurable multifunction (see Section 2).} \end{aligned}$$

Next we will show that for every $t \in T,\ G_1(t, \cdot)$ has a graph which is sequentially closed in $H \times H_w$. To this end, let $(x_n, y_n) \in GrG_1(t, \cdot),\ n \geq 1$, and assume that $x_n \to x,\ y_n \to y$ in H as $n \to \infty$. We have

$$y_n = \int_V f(t, x_n, v)\lambda_n(dv),\ \lambda_n \in \Sigma(t),\ n \geq 1.$$

By passing to a subsequence if necessary, we may assume that $\lambda_n \xrightarrow{w} \lambda$ in $M_+^1(V)$ as $n \to \infty,\ \lambda \in \Sigma(t)$. Note that by virtue of hypothesis $H(f)$ (ii), for almost all $t \in T$, we have $f(t, x_n, \cdot) \xrightarrow{c} f(t, x, \cdot)$ as $n \to \infty$, where $\xrightarrow{c}$ denotes continuous convergence. So for almost all $t \in T$, $f(t, x_n, \cdot) \to f(t, x, \cdot)$ in $C(V)$ as $n \to \infty$ (see Dugundji [12], Remark 7.5, p. 268) and by Rao's theorem (see Parthasarathy [34], Theorem 6.8, p. 51) we have

$$\begin{aligned} y_n &= \int_V f(t, x_n, v)\lambda_v(dv) \to y \\ &= \int_V f(t, x, v)\lambda(dv) \text{ in } H \text{ as } n \to \infty,\ \lambda \in \Sigma(t) \\ &\Rightarrow y \in G_1(t, x). \end{aligned}$$

This proves that the graph of $G_1(t, \cdot)$ is sequentially closed in $H \times H_w$. ■

Let $F : T \times H \to P_{f_c}(H)$ be the multifunction defined by $F(t, x) = G_1(t, x) + G_2(t)$ with $G_2(t) = B(t)U(t)$. From Proposition 1 we know that for all $x \in H,\ t \to F(t, x)$ is Lebesgue measurable, for almost all $t \in T$, $GrF(t, \cdot)$ is sequentially closed in $H \times H_w$ and

$$|F(t, x)| = \sup[|y| : y \in F(t, x)] \le \hat{a}_2(t) + \hat{c}_2|x|^{2/q}$$

a.e. on T for all $x \in H$ with $\hat{a}_2 \in L^q(T)$, $c_2 > 0$. We consider the following evolution inclusion:

$$\text{(3)} \qquad \left\{ \begin{aligned} \dot{x}(t) + A(t, x(t)) &\in F(t, x(t)) \text{ a.e on } T \\ x(0) &= x_0 \end{aligned} \right\}$$

Let $R \subseteq W_{pq}(T)$ be the solution set of (3). Hypotheses $H(A)$ and the properties of the multifunction $F(\cdot, \cdot)$, allow us to use the results of Papageorgiou [30] (see also Papageorgiou-Shahzad [32], [33]) and have that R is weakly compact in $W_{pq}(T)$ and compact in $L^p(T, H)$.

Now consider the map $(u, \lambda) \to x(u, \lambda)$, which to a given control-parameter pair $(u, \lambda) \in L^q(T, Y) \times R(T, V)$ assigns the unique solution $x(u, \lambda)(\cdot) \in W_{pq}$ of (1) (see Aizicovici-Papageorgiou [2]). On $R(T, V)$ we consider the weak topology defined in Section 2.

Proposition 2. *If hypotheses $H(A), H(f), H(B), H(\Sigma)$ hold, then $(u, \lambda) \to x(u, \lambda)$ is sequentially continuous from $L^q(T, Y)_w \times R(T, V)$ into $L^p(T, H)$ (by $L^q(T, Y)_w$ we denote the Lebesgue-Bochner space $L^q(T, Y)$ endowed with the weak topology).*

Proof. We need to show that if $u_n \to u$ in $L^q(T, Y)$ and $\lambda_n \to \lambda$ in $R(T, V)$ as $n \to \infty$, then $x(u_n, \lambda_n) \to x(u, \lambda)$ in $L^p(T, H)$ as $n \to \infty$. In what follows, we set $x_n = x(u_n, \lambda_n), n \geq 1$ and $x = x(u, \lambda)$. Note that $\{x_n\}_{n\geq 1} \subseteq R \subseteq W_{pq}(T) \subseteq L^p(T, H)$. So by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} y$ in $W_{pq}(T)$, $x_n \to y$ in $L^p(T, H)$ and $x_n(t) \to x(t)$ in H for all $t \in T \setminus N,\ |N| = 0$. We have:

$$\left\{\begin{aligned} \dot{x}_n(t) + A(t, x(t)) \;&=\; \int_V f(t, x_n(t), v)\lambda_n(t)(dv) + B(t)u_n(t) \text{ a.e on } T \\ x_n(0) \;&=\; x_0,\ \lambda_n \in S_\Sigma. \end{aligned}\right\} \tag{4}$$

Denote by $((\cdot,\cdot))$ the duality brackets for the pair $(L^p(T, X), L^q(T, X^*))$ and by $(\cdot,\cdot)_{pq}$ the duality brackets for the pair $(L^p(T, H), L^q(T, H))$ (recall that if Z is a reflexive Banach space or more generally if Z is a Banach space and Z^* has the Radon-Nikodym property with respect to the Lebesgue measure on T, then $L^p(T, Z)^* = L^q(T, Z^*)$; see Ionescu-Tulcea [21], Theorem 10, p. 99 and Diestel-Uhl [11], Theorem 1, p. 98). Also let

$$\hat{A} : L^p(T, X) \to L^q(T, X^*), \hat{f} : L^p(T, H) \times R(T, V) \to L^q(T, H)$$
$$\text{and } \hat{B} : L^q(T, Y) \to L^q(T, H)$$

be defined by

$$\begin{aligned} \hat{A}(w)(\cdot) &= A(\cdot, w(\cdot)), \hat{f}(w, \lambda)(\cdot) \\ &= \int_V f(\cdot, w(\cdot), v)\lambda(\cdot)(dv) \;\text{ and }\; \hat{B}(u)(\cdot) = B(\cdot)u(\cdot) \end{aligned}$$

Set $\hat{f}_n(w)(\cdot) = \hat{f}(w, \lambda_n)(\cdot)$ for all $w \in L^p(T, H), n \geq 1$. We have

$$((\dot{x}_n, x_n - y)) + ((\hat{A}(x_n), x_n - y)) = (\hat{f}(x_n), x_n - y)_{pq} + (\hat{B}u_n, x_n - y)_{pq}.$$

We know that $x_n \xrightarrow{w} y$ in $W_{pq}(T),\ x_n \to y$ in $L^p(T, H)$ and $x_n(t) \to y(t)$ in H for all $t \in T \setminus N,\ |N| = 0$. The sequence

$$\{< \dot{x}_n(\cdot), x_n(\cdot) - x(\cdot) >\}_{n\geq 1} \subseteq L^1(T)$$

is uniformly integrable. So given $\varepsilon > 0$, we can find $s, t \in T \setminus N,\ s \leq t$, such that

$$\int_t^b | < \dot{x}_n(\tau), x_n(\tau) - x(\tau) > | d\tau \leq \frac{\varepsilon}{2} \text{ and } \int_0^s | < \dot{x}_n(\tau), x_n(\tau) - x(\tau) > | d\tau \leq \frac{\varepsilon}{2}$$

In what follows by $((\cdot,\cdot))_{st}$ we denote the duality brackets for the pair $(L^p([s,t],X), L^q([s,t],X^*)$. Using the integration by parts formula for functions in $W_{pq}(T)$ (see Zeidler [37], Proposition 23.23, p. 423) we have

$$\begin{aligned}&((\dot{x}_n, x_n - x))_{st}\\ &= \frac{1}{2}|x_n(t)-x(t)|^2 - \frac{1}{2}|x_n(s)-x(s)|^2 + ((\dot{x}, x_n - x))_{st} \to 0 \text{ as } n \to \infty\end{aligned}$$

So we have

$$\begin{aligned}((\dot{x}_n, x_n - x)) &= \int_0^b < \dot{x}_n(\tau), x_n(\tau) - x(\tau) > d\tau\\ &= \int_0^s < \dot{x}_n(\tau), x_n(\tau) - x(\tau) > d\tau\\ &+ \int_t^b < \dot{x}_n(\tau), x_n(\tau) - x(\tau) > d\tau + ((\dot{x}, x_n - x))_{st}\\ &\geq -\varepsilon + ((\dot{x}, x_n - x))_{st}\\ &\Rightarrow \underline{\lim}((\dot{x}_n, x_n - x)) \geq -\varepsilon.\end{aligned} \tag{5}$$

Similarly we obtain that

$$\overline{\lim}((\dot{x}_n, x_n - x)) \leq \varepsilon. \tag{6}$$

From (5) and (6) we infer that $((\dot{x}_n, x_n - x)) \to 0$ as $n \to \infty$. Also we have

$$\int_0^b \left(\int_V f(t, x_n(t), v)\lambda_n(t)(dv), x_n(t) - y(t) \right) dt \to 0$$

and $(\hat{B}u_n, x_n - y)_{pq} \to 0$ as $n \to \infty$.

Therefore, finally we have $\overline{\lim}((\hat{A}(x_n), x_n - y)) = 0$. But $\hat{A}$ is clearly monotone demicontinuous, hence maximal monotone. In particular then $\hat{A}$ is generalized pseudomonotone (see Hu-Papageorgiou [19]) and so $\hat{A}(x_n) \xrightarrow{w} \hat{A}(y)$ in $L^q(T, X^*)$ and $((\hat{A}(x_n), x_n)) \to ((\hat{A}(y), y))$ as $n \to \infty$. Recall that $\lambda_n \to \lambda$ in $R(T,V)$ is equivalent to saying that $\lambda_n \xrightarrow{w^*} \lambda$ in $L^\infty(T, M(V)_{w^*})$ (see Section 2). For every $h \in G$ we have $g_n(h)(t,\cdot) = (h, f(t, x_n(t), \cdot) \in C(V)$, $n \geq 1$, and $g(h)(t,\cdot) = (h, f(t, y(t), \cdot)) \in C(V)$. Evidently for almost all $t \in T$ we have $g_n(h)(t,\cdot) \to g(h)(t,\cdot)$ in C(V) as $n \to \infty$ (cf. hypotheses $H(f)$ (ii) and (iii)) and so $g_n(h) \to g(h)$ in $L^1(T, C(V))$ as $n \to \infty$ (dominated convergence theorem). Denote by $(\cdot,\cdot)_0$ the duality brackets for the pair $(L^1(T, C(V)), L^\infty(T, M(V)_{w^*}))$. For every $A \in \mathcal{L}(T)$ we have

$$(\chi_A g_n(h), \lambda_n)_0 \to (\chi_A g(h), \lambda)_0 \text{ as } n \to \infty.$$

So for every $(h, A) \in H \times \mathcal{L}(T)$, if $\hat{f}_n(t) = \int_V f(t, x_n(t), v)\lambda_n(t)(dv),\ n \geq 1$, $\hat{f}(t) = \int_V f(t, y(t), v)\lambda(t)(dv), \hat{f}_n, \hat{f} \in L^q(T, H), n \geq 1$, we have

$$(\chi_A h, \hat{f}_n)_{pq} \to (\chi_A h, \hat{f})_{pq} \text{ as } n \to \infty$$

$$\Rightarrow (s, \hat{f}_n)_{pq} \to (s, \hat{f})_{pq} \text{ as } n \to \infty \text{ for all simple functions } s \in L^p(T, H).$$

But simple functions are dense in $L^p(T, H)$. So we have

$$(g, \hat{f}_n) \to (g, \hat{f}) \text{ as } n \to \infty \text{ for all } g \in L^p(T, H),$$

$$\Rightarrow \hat{f}_n \xrightarrow{w} \hat{f} \text{ in } L^q(T, H) \text{ as } n \to \infty.$$

Thus by passing to the limit as $n \to \infty$ in (4) we obtain

$$\begin{aligned}
\dot{y} + \hat{A}(y) &= \hat{f} + \hat{B}u \\
\Rightarrow \dot{y}(t) + A(t, y(t)) &= \int_V f(t, y(t), v)\lambda(t)(dv) + B(t)u(t) \text{ a.e. on } T \\
y(0) &= x_0, \lambda \in S_\Sigma \\
\Rightarrow y &= x(u, \lambda) = x.
\end{aligned}$$

This proves the desired continuity of $(u, \lambda) \to x(u, \lambda)$. ■

Let $\eta(u) = \sup[J(u, \lambda) : \lambda \in S_\Sigma]$.

Proposition 3. *If hypotheses $H(A), H(f), H(B), H(U), H(\Sigma)$ and $H(L)$ hold, then $\eta : L^q(T, Y)_w \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is sequentially lower semicontinuous.*

Proof. We need to show that for every $\theta \in \mathbb{R}$ the lower level set

$$\Delta_\theta = \{u \in L^q(T, Y) : \eta(u) \leq \theta\}$$

is sequentially closed in $L^q(T, Y)_w$. So let $u_n \in \Delta_\theta, n \geq 1$, and assume that $u_n \to u$ in $L^q(T, Y)$ as $n \to \infty$. Given $\varepsilon > 0$ we can find $\lambda \in S_\Sigma$ such that $\eta(u) - \varepsilon \leq J(u, \lambda)$. By virtue of proposition 2, we have that $x(u_n, \lambda) \to x(u, \lambda)$ in $L^p(T, H)$ as $n \to \infty$. So invoking Theorem 2.1 of Balder [5], we obtain

$$J(u, \lambda) \leq \underline{\lim} J(u_n, \lambda) \leq \underline{\lim} \eta(u_n) \leq \theta$$

$$\Rightarrow \eta(u) - \varepsilon \leq \theta.$$

Taking $\varepsilon \downarrow 0$ we conclude that $u \in \Delta_\theta$. This proves the desired sequential lower semicontinuity of η. ■

Now we are ready for the first existence result concerning problem (2).

Theorem 1. *If hypotheses $H(A), H(f), H(B), H(U), H(\Sigma)$ and $H(L)$ hold, then problem (2) admits an optimal control $u^* \in S_\Sigma$.*

Proof. From Proposition 3 we know that $\eta(\cdot)$ is sequentially lower semicontinuous on $L^q(T,Y)_w$, while S^q_U is weakly compact, thus sequentially weakly compact (Eberlein-Smulian theorem). So by the Weirstrass theorem there exists $u^* \in S^q_U$ such that $\eta(u^*) = \beta$. Evidently u^* is the desired optimal control. ∎

If we split the cost integrand $L(t,x,u)$, we can say more. So suppose that

$$L(t,x,u) = L_1(t,x) + L_2(t,u)$$

We make the following hypotheses:

$\mathbf{H(L)_1}$: $L_1 : T \times H \to \mathbb{R}$ and $L_2 : T \times Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ are two integrands such that

(i) for every $x \in H$ and $u \in Y$, $t \to L_1(t,x)$ and $t \to L_2(t,u)$ are measurable;

(ii) for every $t \in T$, $x \to L_1(t,x)$ is continuous and $u \to L_2(t,u)$ is lower semicontinuous and convex;

(iii) for almost all $t \in T$ and all $|x| \leq M$, we have $|L_1(t,x)| \leq \gamma_M(t)$ with $\gamma_M \in L^1(T)$ while for almost all $t \in T$ and all $u \in Y$, we have

$$\varphi(t) - c_3||u|| \leq L(t,u) \text{ with } \varphi \in L^1(T),\ c_3 > 0.$$

Theorem 2. *If hypotheses $H(A)$, $H(f)$, $H(B)$, $H(U)$, $H(\Sigma)$ and $H(L)_1$ hold, then problem (2) admits an optimal control-parameter pair $[u^*, \lambda^*] \in S^q_U \times S_\Sigma$; i.e., $J(u^*, \lambda^*) = \beta$.*

Proof. From Theorem 1 we know that there exists $u^* \in S^q_U$ such that $\eta(u^*) = \beta$. Then

$$\beta = \sup[J_1(u^*,\lambda) + J_2(u^*) : \lambda \in S_\Sigma] = \sup[J_1(u^*,\lambda) : \lambda \in S_\Sigma] + J_2(u^*).$$

By virtue of Proposition 2 and hypothesis $H(L)_1, \lambda \to J_1(u^*,\lambda)$ is sequentially continuous from $R(T,V)$ (with the weak topology as always) into $\mathbb{R}$. We claim that S_Σ furnished with the relative weak topology as a subset of $R(T,V)$, is compact. Recall that the weak topology of $R(T,V)$ coincides with the relative weak*-topology of $L^\infty(T, M(V)_{w^*})$ (see Section 2).

Since S_Σ is relatively w^*-compact in $L^\infty(T, M(V)_{w^*})$ (Alaoglu's theorem), it remains to show that S_Σ is sequentially w^*-closed in $L^\infty(T, M(V))$.

So let $\lambda_n \in S_\Sigma, n \geq 1$, and assume that $\lambda_n \xrightarrow{w^*} \lambda$ in $L^\infty(T, M(V)_{w^*})$ as $n \to \infty$. Then $\lambda \in R(T, V)$. Let $g \in L^1(T, C(V))$. For every $n \geq 1$ we have

$$\begin{aligned}(g, \lambda_n)_0 \leq \sigma(g, S_\Sigma) &= \sup[(g,\lambda)_0 : \lambda \in S_\Sigma] = \sup[\int_0^b (g(t), \lambda(t))dt : \lambda \in S_\Sigma] \\ &= \int_0^b \sup[(g(t), \lambda) : \lambda \in \Sigma(t)]dt \\ &\quad \text{(see Hu-Papageorgiou [19], Theorem 3.24, p. 183)} \\ &= \int_0^b \sigma(g(t), \Sigma(t))dt.\end{aligned}$$

Take $g(t) = \chi_A(t)w$ with $(A, w) \in \mathcal{L}(T) \times C(V)$. We have

$$\begin{aligned}&\int_A (w, \lambda(t))dt \leq \int_A \sigma(w, \Sigma(t))dt \\ &\quad \Rightarrow (w, \lambda(t)) \leq \sigma(w, \Sigma(t)) \text{ for all } t \in T \setminus N(w),\ |N(w)| = 0.\end{aligned}$$

Let $\{w_m\}_{m \geq 1}$ be dense in $C(V)$. Since $\sigma(\cdot, \Sigma(t))$ is continuous, it follows that

$$(w, \lambda(t)) \leq \sigma(w, \Sigma(t)) \text{ for all } t \in T \setminus N,\ N = \cap_{m\geq 1} N(w_m),\ |N| = 0$$

$$\Rightarrow \lambda(t) \in \Sigma(t) \text{ a.e. on } T.$$

Hence $S_\Sigma \subseteq R(T, V)$ is compact. Once again via the Weirstrass theorem, we obtain $\lambda^* \in S_\Sigma$ such that $\beta = \sup[J_1(u^*, \lambda) : \lambda \in S_\Sigma] + J_2(u^*) = J_1(u^*, \lambda^*) + J_2(u^*) = J(u^*, \lambda^*)$. ■

4. Existence results for nonparametric problems

In this section, we turn our attention to nonparametric optimal control systems. To solve the optimal control problems, we prove an existence theorem for evolution inclusions which is of independent interest and extends previous such results existing in the literature.

Let T, (X, H, X^*) and Y be as in the previous section. The system under consideration is described by the following nonlinear evolution equation:

$$(7) \qquad \left\{ \begin{aligned} \dot{x}(t) + A(t, x(t)) &= f(t, x(t))u(t) \text{ a.e. on } T \\ x(0) &= x_0 \in H,\ u \in S_U^q \end{aligned} \right\}$$

We start with the study of a time-optimal control problem. So let $K(t)$ be the time-varying target-set and for a given control function $u \in S_U^q$, let $R(u)$ be the set of all trajectories of (7) generated by the control u. Let $Q(u) = \{t \in T : x(t) \in K(t), x \in R(u)\}$. Under general hypotheses on the data we will show that for all $u \in S_U^q$, $R(u) \neq \emptyset$ and we will assume that $\cup_{u \in S_U^q} Q(u) \neq \emptyset$. Let $J(u) = \inf_u Q(u)$ (we make the usual convention that $\inf \emptyset = +\infty$). Then the " time-optimal control problem" is the following:

$$\inf[J(u) : u \in S_U^q] = t^*. \tag{8}$$

A control $u^* \in S_U^q$ such that $J(u^*) = t^*$ is said to be optimal. We look for the existence of optimal controls. Our approach will use an existence theorem for evolution inclusions, which is of idependent interest, since it generalizes earlier results in this direction, which assumed that $A(t, \cdot)$ is monotone (see Attouch-Damlamian [4], Papageorgiou [30], Papageorgiou-Shahzad [32], [33] and the references therein).

So consider the following evolution inclusion:

$$\left\{ \begin{array}{rcl} \dot{x}(t) + A(t, x(t)) & \in & F(t, x(t)) \text{ a.e. on } T \\ x(0) & = & x_0 \end{array} \right\} \tag{9}$$

Our hypotheses on the data of (9) are as follows:

$\mathbf{H(A)_1}$: $A : T \times X \to X^*$ is an operator such that

(i) for every $x \in X, t \to A(t, x)$ is measurable;

(ii) for all $t \in T, x \to A(t, x)$ is demicontinuous and pseudomonotone (see Hu-Papageorgiou [19], Definition 6.1, p. 365 or Zeidler [37], pp. 585–586);

(iii) for almost all $t \in T$ and all $x \in X$, $||A(t, x)||_* \leq a(t) + c||x||^{p-1}$ with $a \in L^q(T)_1, c > 0, 2 \leq p < \infty, \frac{1}{p} + \frac{1}{q} = 1$;

(iv) for almost all $t \in T$ and all $x \in X$,

$< A(t, x), x > \geq c_1||x||^p - c_0||x||^{p-1} - a_1(t)$ with $c_1,\ c_0 > 0$ and $a_1 \in L^1(T)$.

$\mathbf{H(F)}$: $F : T \times H \to P_{f_c}(H)$ is a multifunction such that

(i) for every $x \in H,\ t \to F(t, x)$ is measurable;

(ii) for all $t \in T,\ GrF(t, \cdot)$ is sequentially closed in $H \times H_w$;

(iii) for almost all $t \in T$ and all $x \in H$, we have

$$|F(t,x)| = \sup[|y| : y \in F(t,x)] \le a_2(t) + c_2|x|^{2/q}$$

with $a_2 \in L^q(T)_+,\ c_2 > 0$ and if $p = 2$, for almost all $t \in T$ and all $x \in H$, we have

$$|F(t,x)| \le a_2(t) + c_2|x|$$

$a_2 \in L^q(T)_+, c_2 > 0$ and $(y,x) \le \gamma$ for some $\gamma > 0$ and all $y \in F(t,x)$.

The proof of our existence theorem is based on the following surjectivity result for L-generalized pseudomonotone operators (see Section 2) due to Papageorgiou-Papalini-Renzacci [31]. This result was first proved for single-valued operators by Lions [22] and B.A. Ton [36].

Proposition 4. *If Y is a reflexive Banach space, $L : D(L) \subseteq Y \to Y^*$ is a linear densely defined maximal monotone operator and $K : Y \to 2^{Y^*} \setminus \{\emptyset\}$ is a bounded (i.e maps bounded sets to bounded sets), L-generalized pseudomonotone, coercive operator, then $R(L+K) = Y^*$.*

Using this proposition we can have the following existence result for problem (9).

Proposition 5. *If hypotheses $H(A)_1, H(F)$ hold and $x_0 \in H$, then the solution set $S(x_0)$ of 9 is nonempty, weakly compact in $W_{pq}(T)$ and compact in $C(T,H)$.*

Proof. First, assume $x_0 \epsilon X$. We introduce the operator $A_1 : T \times X \to X^*$ defined by $A_1(t,x) = A(t, x + x_0)$. Evidently, $t \to A_1(t,x)$ is measurable, $x \to A_1(t,x)$ is demicontinuous, pseudomonotone, $||A_1(t,x)||_* \le \hat{a}(t) + \hat{c}||x||^{p-1}$ a.e on T for all $x \in X$, with $\hat{a} \in L^q(T)_+,\ \hat{c} > 0$ and $< A_1(t,x), x > \ge \hat{c}_1||x||^p - \hat{c}_0||x||^{p-1} - \hat{a}_1(t)$ a.e on T for all $x \in X$, with $\hat{c}_1,\ \hat{c}_0,\ \hat{a}_1 \in L^1(T)_+$. Thus all the properties of $A(t,x)$ are passed to $A_1(t,x)$.

Similarly, let $F_1 : T \times H \to P_{f_c}(H)$ be defined by $F_1(t,x) = F(t, x+x_0)$. We see that $t \to F_1(t,x)$ is measurable, $GrF_1(t,\cdot)$ is sequentially closed in $H \times H_w$ and

$$|F_1(t,x)| = \sup[|y| : y \in F_1(t,x)] \le \hat{a}_2(t) + \hat{c}_2(t)|x|^{2/q}$$

a.e on T for all $x \in H$, with $\hat{a}_2 \in L^q(T)_+,\ \hat{c}_2 > 0$. Consider the following evolution inclusion

$$(10) \qquad \left\{ \begin{array}{c} \dot{x}(t) + A_1(t, x(t)) \in F_1(t, x(t)) \text{ a.e on } T \\ x(0) = 0 \end{array} \right\}$$

Note that $x \in W_{pq}(T)$ is a solution to (9) if and only if $\hat{x}(*) = x(*) - x_0$ is a solution to (10). Hence it suffices to prove the proposition for problem (10).

To this end, let $L : D(L) \subseteq L^p(T, X) \to L^q(T, X^*)$ be the linear operator defined by $Lx = \dot{x}$ for $x \in D(L) = \{x \in W_{pq}(T) : x(0) = 0\}$ (the time derivative is defined in the sense of vector-valued distributions). As in the proof of Theorem 3.1 Papageorgiou-Papalini-Renzacci [31], we can easily check that L is a maximal monotone linear operator. Let

$$\hat{A}_1 : L^p(T, X) \to L^q(T, X^*)$$

be defined by $\hat{A}_1(x)(\cdot) = A_1(\cdot, x(\cdot))$ and

$$G_1 : L^p(T, X) \to P_{f_c}(L^q(T, X^*))$$

by

$$G_1(x) = S^q_{-F_1(\cdot, x(\cdot))} = \{g \in L^q(T, X^*) : g(t) \in -F_1(t, x(t)) \text{ a.e. on } T\}.$$

Then introduce the multivalued operator

$$K : L^p(T, X) \to 2^{L^q(T, X^*)}$$

defined by $K(x) = \hat{A}_1(x) + G_1(x)$. Since $G_1(\cdot)$ has nonempty values (see for example Hu-Papageorgiou [19]), so does $K(\cdot)$. Moreover, it is easy to see that for all $x \in L^p(T, X), K(x) \in P_{wkc}(L^q(T, X^*))$ and that $K(\cdot)$ is bounded.

Claim 1. K is L-generalized pseudomonotone.

First, we show that $K(\cdot)$ is usc from $L^p(T, X)$ into $L^q(T, X^*)_w$. So let $C \subseteq L^q(T, X^*)$ be a nonempty and weakly colsed set. We need to show that

$$K^-(C) = \{x \in L^p(T, X) : K(x) \cap C \neq \emptyset\}$$

is closed. For this purpose, consider $\{x_n\}_{n \geq 1} \subseteq K^-(C)$ such that $x_n \to x$ in $L^p(T, X)$ as $n \to \infty$. Let $v_n \epsilon K(x_n) \cap C, n \geq 1$. By virtue of the growth conditions $H(A)_1$ (iii) and $H(F)$ (iii), we have that $\{v_n\}_{n \geq 1} \subseteq L^q(T, X^*)$ is bounded. Thus we may assume that $v_n \to v$ in $L^q(T, X^*)$ as $n \to \infty$. Let $g_n \in G_1(x_n)$ such that $v_n = \hat{A}_1(x_n) + g_n, n \geq 1$. Because of hypothesis $H(F)$ (iii) we may assume that

$$-g(t) \subseteq \overline{\text{conv}} w - \overline{\lim} F_1(t, x_n(t)) \subseteq F_1(t, x(t))$$

a.e on T, with the last inclusion being a consequence of the fact that $GrF_1(t,\cdot)$ is sequentially closed in $H \times H_w$. Since $x_n \to x$ in $L^p(T,X)$ as $n \to \infty$, we may also assume that $x_n(t) \to x(t)$ a.e on T in X as $n \to \infty$. We have $v_n(t) = A_1(t, x_n(t)) + g_n(t)$ a.e on T, $n \geq 1$. Note that

$$| < v_n(t), x_n(t) - x(t) > | \leq ||v_n(t)||_* ||x_n(t) - x(t)|| \leq \varphi_1(t)||x_n(t) - x(t)||$$

a.e on T with $\varphi_1 \in L^q(T)_+$ and

$$\begin{aligned} &| < g_n(t), x_n(t) - x(t) > | \\ &= |(g_n(t), x_n(t) - x(t))| \leq |g_n(t)||x_n(t) - x(t)| \leq \varphi_2(t)|x_n(t) - x(t)| \end{aligned}$$

a.e on T with $\varphi_2 \in L^q(T)_+$. Hence $< v_n(t), x_n(t) - x(t) >, < g_n(t), x_n(t) - x(t) > \to 0$ a.e on T as $n \to \infty$ and so $< A_1(t, x_n(t)), x_n(t) - x(t) > \to 0$ a.e on T as $n \to \infty$. Because $A_1(t,\cdot)$ is pseudomonotone we have that $A_1(t, x_n(t)) \xrightarrow{w} A_1(t, x(t))$ a.e on T in X^* as $n \to \infty$. Then via the generalized dominated convergence theorem (see for example Ash [3], Theorem 7.5.2, p. 295), we have that $\hat{A}_1(x_n) \xrightarrow{w} \hat{A}_1(x)$ in $L^q(T, X^*)$. Thus in the limit as $n \to \infty$ we obtain $v = \hat{A}_1(x) + g$ with $g \in G_1(x)$ and $x \in C$. So $x \in K^-(C)$ which proves the upper semicontinuinty of $K(\cdot)$ from $L^p(T,X)$ into $L^q(T,X^*)_w$.

Next let $\{x_n\}_{n\geq 1} \subseteq D(L)$ and assume that $x_n \to x$ in $L^p(T,X)$, $Lx_n \to Lx$ in $L^q(T,X^*)$ (hence $x_n \to x$ in $W_{pq}(T)$), $x_n^* \in K(x_n)$, $n \geq 1$, $x_n^* \to x^*$ in $L^q(T,X^*)$ and $\overline{\lim}((x_n^*, x_n - x)) \leq 0$. We have $x^* = \hat{A}_1(x_n) + g_n$ with $g_n \in G_1(x_n)$, $n \geq 1$. As above we may assume $g_n \to g$ in $L^q(T,H)$. Moreover, since $W_{pq}(T)$ is embedded compactly in $L^p(T,H)$, we also have that $x_n \to x$ in $L^p(T,H)$ as $n \to \infty$. Thus we obtain

$$\begin{aligned} &\overline{\lim}((\hat{A}_1(x_n), x_n - x)) = \overline{\lim}((x_n^* - g_n, x_n - x)) \\ &\leq \overline{\lim}((x_n^*, x_n - x)) - lim(g_n, x_n - x)_{pq} \leq 0. \end{aligned}$$

But from Proposition 1 of Papageorgiou [29] we know that $\hat{A}_1$ is L-generalized pseudomonotone. Hence $((\hat{A}_1(x_n), x_n)) \to ((\hat{A}_1(x), x))$ and so $((x_n^*, x_n)) \to ((x_n^*, x))$ as $n \to \infty$ and this proves the claim.

Claim 2. $K(\cdot)$ is coercive.

$$\left(\text{i.e. } \lim_{||x||_p \to \infty} \frac{\inf[((x^*, x)) : x^* \epsilon K(x)]}{||x||_p} = +\infty \right).$$

Let $x \in L^p(T,X)$ and $x^* \in K(x)$. We have $x^* = \hat{A}_1(x) + g$ with $g \in G_1(x)$ and so

$$((x^*,x)) = ((\hat{A}_1(x),x)) + ((g,x)).$$

First, assume that $p > 2$. We have

$$((\hat{A}_1(x),x)) \leq \hat{c}_1 ||x||_p^p - \hat{k}_0 ||x||_p^{p-1} - ||\hat{a}||_1 \text{ for some } \hat{k}_0 > 0.$$

Also via Young's inequality with $\varepsilon > 0$, we obtain

$$((g,x)) = (g,x)_{pq} \geq -\beta \frac{1}{\varepsilon^q q} 2^q ||\hat{a}||_q^q - \frac{\theta}{\varepsilon^q q} 2^q \hat{c}_2^q ||x||_p^2 - \beta \frac{\varepsilon^p}{p} ||x||_p^p,\ \theta > 0.$$

Thus finally we have

$$((x^*,x)) \geq (\hat{c}_1 - \beta \frac{\varepsilon^p}{p}) ||x||_p^p - \hat{k}_0 ||x||_p^{p-1} - \theta_1(\varepsilon) ||x||_p^2 - \theta_2(\varepsilon),\ \theta_1(\varepsilon),\ \theta_2(\varepsilon) > 0.$$

Choose $\varepsilon > 0$ so that $\hat{c}_1 > \beta \frac{\varepsilon^p}{p}$. Then since $p > 2$, we see that

$$\frac{((x^*,x))}{||x||_p} \to +\infty \text{ as } ||x||_p \to +\infty$$
$$\Rightarrow K(\cdot) \text{ is coercive as claimed (for } p > 2).$$

If p=2, then we have

$$((x^*,x)) \geq \hat{c}_1 ||x||_2^2 - \theta_3 ||x||_2 - \theta_4, \text{ with } \theta_3,\ \theta_4 > 0$$

(see hypothesis $H(F)$ (iii)). So again we have coercivity of $K(\cdot)$.

Because of Claims 1 and 2 we can apply Proposition 4 and have that $R(L+K) = L^q(T,X^*)$ hence problem (10) and equivalently problem (9) has a solution $x \in W_{pq}(T)$ (provided $x_0 \in X$).

Next we remove the extra condition that $x_0 \in X$. So let $x_0 \in H$. Then we can find $\{x_{on}\}_{n \geq 1} \subseteq X$ such that $x_{on} \to x_0$ in H as $n \to \infty$. From the first part of the proof we know that the multivalued Cauchy problem:

$$\dot{x}(t) + A(t,x(t)) \in F(t,x(t)) \text{ a.e. on } T$$
$$x(0) = x_{0n}$$

has a solution $x_n \in W_{pq}(T), n \geq 1$. Then

$$\dot{x}_n(t) + A(t,x_n(t)) = h_n(t) \text{ a.e on } T,$$

$$x_n(0) = x_{0n},\ n \geq 1$$

with $h_n(t) \in F(t, x_n(t))$ a.e on T, $n \geq 1$. With the same a priori estimation as in the proof of Theorem 1 of Aizicovici-Papageorgiou [2], we can check that $\{x_n\}_{n\geq 1} \subseteq W_{pq}(T)$ is bounded. Thus we may assume that $x_n \to x$ in $W_{pq}(T)$ as $n \to \infty$. Also we can say that $g_n \to g$ in $L^q(T, H)$. We have

$$\overline{\lim}((\hat{A}(x_n) - h_n, x_n - x)) = \overline{\lim}((\dot{x}_n, x - x_n)).$$

The integration by parts formula for functions in $W_{pq}(T)$ gives us

$$((\dot{x}_n, x - x_n)) \leq \frac{1}{2}|x_{0n} - x(0)|^2 + ((\dot{x}, x - x_n)) \to 0 \text{ as } n \to \infty.$$

Also $((g_n, x_n - x)) = (g_n, x_n - x)_{pq} \to 0$ as $n \to \infty$. So we conclude that

$$\overline{\lim}((\hat{A}(x_n), x_n - x)) \leq 0$$

$$\Rightarrow \quad \hat{A}(x_n) \xrightarrow{w} \hat{A}(x) \text{ in } L^q(T, X^*) \text{ as } n \to \infty$$

(by the L-generalized pseudomonotonicity of A). Therefore in the limit as $n \to \infty$, we have

$$\dot{x} + \hat{A}(x) = h$$

with

$$h(t) \in \overline{\text{conv}} w - \overline{\lim} F(t, x_n(t)) \subseteq F(t, x(t)) \text{ a.e on } T \text{ (as before)}.$$

Thus $x \in S(x_0)$. This proves the existence part of the proposition.

Now we will establish the compactness properties of the solution set $S(x_0) \subseteq W_{pq}(T)$. From standard a priori estimation (see for example Aizicovici-Papageorgiou [2] and Papageorgiou-Shahzad [32]), we know that there exist $M_1, M_2, M_3 > 0$ such that for all $x \in S(x_0)$ we have

$$|x(t)| \leq M_1 \text{ for all } t \in T,\ ||x||_{L^p(T,X)} \leq M_2 \text{ and } ||\dot{x}||_{L^q(T,X^*)} \leq M_3.$$

Thus $S(x_0)$ is bounded hence relatively weakly compact in $W_{pq}(T)$. Set $\psi(t) = a_2(t) + c_2 M_2^{2/q}$ for all $t \in T$. Then $\psi \in L^q(T)_+$. Moreover, by replacing $F(t, x)$ by

$$\hat{F}(t, x) = \begin{cases} F(t, x) & \text{if } |x| \leq M_1 \\ F(t, \frac{M_1 x}{|x|}) & \text{if } |x| > M_1 \end{cases}$$

we may assume without any loss of generality that $|F(t,x)| = \sup[|y| : y \in F(t,x)] \leq \psi(t)$ a.e on T for all $x \in H$. Then we introduce the set

$$V = \{g\epsilon L^q(T,H) : |g(t)| \leq \psi(t) \text{ a.e. on } T\}.$$

On V we consider the relative weak $L^q(T,H)$-topology. Furnished with this topology, V is a compact metrizable space. Then let $R : V \to 2^{C(T,H)}$ be the multifunction which to every $g \in V$ assigns the set of solutions to the following Cauchy problem:

$$\begin{cases} \dot{x}(t) + A(t,x(t)) = g(t) \text{ a.e. on } T \\ x(0) = x_0. \end{cases}$$

For every $g \in V$, we have $R(g) \neq \emptyset$.

Claim 3. $R(V)$ is compact in $C(T,H)$.

Let $\{x_n\}_{n\geq 1} \subseteq R(V)$. Then by definition $x_n \in R(g_n), g_n \in V, n \geq 1$. We may assume that $g_n \to g$ in $L^q(T,H)$. From the a priori estimation mentioned above, we have that $\{x_n\}_{n\geq 1} \subseteq W_{pq}(T)$ is bounded and so we may assume that $x_n \to x$ in $W_{pq}(T), x_n \to x$ in $L^p(T,H)$ and $x_n(t) \to x(t)$ for all $t \in T \setminus N$, $|N| = 0$ as $n \to \infty$.

Observe that $\{< \hat{x}_n(\cdot), x_n(\cdot) - x(\cdot) >\}_{n\geq 1} \subseteq L^1(T)$ in uniformly integrable. So given $\varepsilon > 0$, we can find $t \in T \setminus N$ such that

$$\int_t^b | < \dot{x}_n(s), x_n(s) - x(s) > | ds < \varepsilon. \tag{11}$$

As before let $((\cdot,\cdot))_t$ denote the duality brackets for the pair $(L^p([0,t],X), L^q([0,t],X^*))$. From the integration by parts formula for functions in $W_{pq}(T)$, we have

$$((\dot{x}_n, x_n - x))_t = \frac{1}{2}|x_n(t) - x(t)|^2 + ((\dot{x}, x_n - x))_t \to 0 \text{ as } n \to \infty.$$

We note that

$$\begin{aligned} &((\dot{x}_n, x_n - x)) = ((\dot{x}_n, x_n - x))_t + \textstyle\int_t^b < \dot{x}_n(s), x(s) - x(s) > ds \\ &\Rightarrow ((\dot{x}_n, x_n - x)) \geq ((\dot{x}_n, x_n - x))_t - \varepsilon \quad \text{(see (11) above)} \\ &\Rightarrow \underline{\lim}((\dot{x}_n, x_n - x)) \geq -\varepsilon. \end{aligned} \tag{12}$$

Taking $\varepsilon \downarrow 0$ we conclude that $\overline{\lim}((\dot{x}_n, x_n - x)) \geq 0$. Also from (11) and (12) we have

$$((\dot{x}_n, x_n - x)) \leq ((\dot{x}_n, x_n - x))_t + \varepsilon$$
$$\overline{\lim}((\dot{x}_n, x_n - x)) \leq \varepsilon.$$

Let $\varepsilon \downarrow 0$ to conclude that $\overline{\lim}((\dot{x}_n, x_n - x)) \leq 0$. Therefore we can say that $((\dot{x}_n, x_n - x)) \to 0$ as $n \to \infty$. Since $((\dot{x}_n, x_n - x)) + ((\hat{A}(x_n), x_n - x)) = (g_n, x_n - x)_{pq} \to 0$ as $n \to \infty \Rightarrow ((\hat{A}(x_n), x_n - x)) \to 0$ as $n \to \infty$. But recall that $\hat{A}$ is L-generalized pseudomonotone (see Papageorgiou [29], Proposition 1). So $\hat{A}(x_n) \xrightarrow{w} \hat{A}(x)$ in $L^q(T, X^*)$. Thus in the limit as $n \to \infty$, we obtain

$$\begin{aligned} \dot{x} + \hat{A}(x) &= g, \ g \in V \\ \Rightarrow \dot{x} + A(t, x(t)) &= g(t) \text{ a.e. on } T \\ x(0) &= x_0, g \in V \\ \Rightarrow x &\in R(V). \end{aligned}$$

From the integration by parts formula we have

$$\begin{aligned} &\frac{1}{2}|x_n(t) - x(t)|^2 = ((\dot{x}_n - \dot{x}, x_n - x))_t \\ &= ((g_n - g, x_n - x)_t - ((\hat{A}(x_n) - \hat{A}(x), x_n - x))_t \\ &\Rightarrow \frac{1}{2}|x_n(t) - x(t)|^2 \leq \int_0^b |(g_n(s) - g(s), x_n(s) - x(s))| ds \end{aligned}$$

$$\int_0^b | < A(s, x_n(s)), x_n(s) - x(s) > | ds + ((\hat{A}(x), x_n - x))_t. \tag{13}$$

Note that $\int_0^b |(g_n(s) - g(s), x_n(s) - x(s))| ds \to 0$ as $n \to \infty$. Moreover, from the proof of Proposition 1 of Papageorgiou [29], we know that if $\xi_n(s) = < A(s, x_n(s), x_n(s) - x(s) >, n \geq 1$, then $\xi_n \to 0$ in $L^1(T)$ as $n \to \infty$. So $\int_0^b | < A(s, x_n(s), x_n(s) - x(s) > | ds \to 0$ as $n \to \infty$. Finally, consider

$$\sup_{t \in T} ((\hat{A}(x), x_n - x))_t = \sup_{t \in T} \int_0^t < A(s, x(s), x_n(s) - x(s) > ds.$$

Let $d_n(t) = \int_0^t < A(s, x(s), x_n(s) - x(s) > ds$. Evidenlty $d_n(\cdot) \in AC(T)$. Let $t_n \in T$ such that $d_n(t_n) = \sup_{t \in T} d_n(t), n \geq 1$. We may say that $t_n \to t$ in T. Then

$$d_n(t_n) = ((\hat{A}(x), x_n - x))_{t_n} = ((\chi_{[0,t_n]} \hat{A}(x), x_n - x)).$$

Observe that

$$\int_0^b ||\chi_{[0,t_n]}(s)A(s,x(s)) - \chi_{[0,t]}(s)A(s,x(s))||_*^q ds$$

$$\int_{t_n\wedge t}^{t_n\vee t} ||A(s,x(s))||_*^q ds \text{ (with } t_n\vee t = \max\{t_n,t\}\,,\ t_n\wedge t = \min\{t_n,t\})$$

and $\int_{t_n\wedge t}^{t_n\vee t} ||A(s,x(s))||_*^q ds \to 0$ as $n\to\infty$. So $\chi_{[0,t_n]}\hat{A}(x) \to \chi_{[0,t]}\hat{A}(x)$ in $L^q(T,X^*)$ as $n\to\infty$. Hence $d_n(t_n)\to 0$ as $n\to\infty$ and so $d_n\to 0$ in $C(T)$ as $n\to\infty$. Therefore $\sup((\hat{A}(x), x_n - x))_t \to 0$ as $n\to\infty$.

Returnig to (13) and using these convergences, we infer that $x_n \to x$ in C(T,H) as $n\to\infty$ and so $R(V)$ is indeed compact in $C(T,H)$.

Since $S(x_0)\subseteq R(V)$, to finish the proof it suffices to show that $S(x_0)$ is weakly closed in $W_{pq}(T)$. So let $\{x_n\}_{n\geq 1}\subseteq S(x_0)$ and assume that $x_n\to x$ in $W_{pq}(T)$ as $n\to\infty$ and since $x_n(t)\to x(t)$ in H for all $t\in T$ as $n\to\infty$ (recall that $W_{pq}(T)$ is embedded continuously in $C(T,H)$), we have that $g\in S^q_{F(\cdot,x(\cdot))}$ (see Hu-Papageorgiou [19], Proposition 3.9, p. 694). As before, in the limit as $n\to\infty$, we can check that $\dot{x}+\hat{A}(x)=g$ and so $S(x_0)$ is weakly closed in $W_{pq}(T)$. Therefore we conclude that $S(x_0)$ is nonempty, weakly compact in $W_{pq}(T)$ and compact in $C(T,H)$. ∎

Now we are ready to deal with the time-optimal problem (8). We introduce the following hypotheses on the data of the problem:

$\mathbf{H(f)_1}$: $f : T\times H\to \mathcal{L}(Y,H)$ is a map such that

(i) for every $x\in H$ and $u\in Y$, $t\to f(t,x)u$ is measurable;

(ii) for every $t\in T$ and every $h\in H$, $x\to f(t,x)^*h$ is continuous;

(iii) for almost all $t\in T$ and all $x\in H$,

$$||f(t,x)||_{\mathcal{L}} \leq a_2(t) + c_2|x|^{p-1}$$

with $a_2\in L^q(T)_+,\ c_2>0,\ 2<p<\infty,\ \frac{1}{p}+\frac{1}{q}$ and if $p=2$, then for almost all $t\in T$, all $x\in H$ and all $u\in U(t)$, we have

$$(f(t,x)u,x)\leq \gamma \text{ with } \gamma>0.$$

$\mathbf{H(U)_1}$: $U : T\to P_{f_c}(H)$ is measurable and for almost all $t\in T$

$$|U(t)| = \sup[||u||_Y : u\in U(t)] \leq M,\ M>0.$$

H(K) : $K : T \to P_{f_c}(H)$ is usc.

$\mathbf{H_0}$: $\cup_{t\in T}(\Gamma(t) \cap K(t)) \neq \emptyset$, where $\Gamma(t) = \{x(u)(t) : u \in S_u^q\}$.

Remark. Hypothesis H_0 is a controllability condition and is equivalent to saying that $\cup_{u\in S_U^q} Q(u) \neq \emptyset$ where $Q(u) = \{t \in T : x(t) \in K(t), x \in R(u)\}$.

Theorem 3. *If hypothesis $H(A), H(f)_1, H(U)_1, H(K)$ and H_0 hold, then the time-optimal control problem* (8) *admits a solution $u^* \in S_U^q$.*

Proof. Let $F : T \times H \to P_{f_c}(H)$ be the multifunction defined by $F(t,x) = f(t,x)U(t)$. Since $U(\cdot)$ is measurable (cf. hypothesis $H(U)_1$), we can find $u_n : T \to Y, n \geq 1$, Lebesgue measurable functions such that $U(t) = \overline{\{u_n(t)\}}_{n\geq 1}$ for all $t \in T$ (see Hu-Papageorgiou [19], Theorem 2.4, p. 156). Then

$$F(t,x) = \overline{\{f(t,x)u_n(t)\}}_{n\geq 1}$$

$$\Rightarrow t \to F(t,x) \text{ is measurable.}$$

Also let $y_n \in F(t,x_n), n \geq 1$ and assume that $x_n \to x, y_n \to y$ in H as $n \to \infty$. We have $y_n = f(t,x_n)u_n,\ u_n \in U(t), n \geq 1$. By passing to a subsequence if necessary, we may assume that $u_n \xrightarrow{w} u$ in Y as $n \to \infty$ and $u \in U(t)$. Then by virtue of hypothesis $H(f)_1$ (ii) for every $h \in H$ we have

$$(f(t,x_n)u_n, h) = (u_n, f(t,x_n)^*h) \to (u, f(t,x)^*h) = (f(t,x)u, h) \text{ as } n \to \infty$$

$$\Rightarrow y_n \xrightarrow{w} f(t,x)u \ \text{ in } H \text{ as } n \to \infty \text{ with } u \in U(t),$$

$$\Rightarrow y = f(t,x)u,\ u \in U(t),\ \text{i.e. } y \in F(t,x).$$

So $GrF(t,\cdot)$ is sequentially closed in $H \times H_w$. In addition, we have

$$|F(t,x)| = \sup[|y| : y \in F(t,x)] \leq Ma_2(t) + Mc_2|x|^{p-1}$$

$$\text{a.e. on } T \text{ for all } x \in H$$

Thus if we consider problem (9) with $F(t,x)$ as above, we have that $S(x_0)$ is nonempty weakly compact in $W_{pq}(T)$ and compact in $C(T,H)$ (see Proposition 4).

Now let $\{u_n\}_{n\geq 1} \subseteq S_U^q$ be a minimizing sequence for the time-optimal control Problem 8.

Then by definition we have

$$x_n(t_n) \in K(t_n) \text{ with } x_n \in R(u_n), n \geq 1, t_n \downarrow t^* \text{ as } n \to \infty.$$

Note that $\{x_n\}_{n\geq 1} \subseteq S(x_0)$. So we may assume that $u_n \to u$ in $L^q(T,Y), u \in S_U^q, x_n \xrightarrow{w} x$ in $W_{pq}(T)$ and $x_n \to x$ in $C(T,H)$ as $n \to \infty$. Hence we have $x_n(t_n) \to x(t^*)$ in H as $n \to \infty$. Also by virtue of hypothesis $H(K)$, we have $x(t^*) \in \overline{\lim} K(t_n) \subseteq K(t^*)$. To finish the proof of the theorem it remains to show that $x \in R(u)$. To this end, let $h \in L^p(T,H)$. We have

$$\begin{aligned}
&(f(\cdot,x_n(\cdot))u_n(\cdot),h)_{pq}\\
&= \int_0^b (f(t,x_n(t))u_n(t),h(t))dt = \int_0^b (u_n(t), f(t,x_n(t))^* h(t))_{Y,Y^*}dt\\
&\Rightarrow \int_0^b (u(t), f(t,x(t))^* h(t))_{Y,Y^*}dt = \int_0^b (f(t,x(t))u(t),h(t))dt.
\end{aligned}$$

Since $h \in L^p(T,H)$ was arbitary, we infer that

$$f(\cdot,x_n(\cdot))u_n(\cdot) \xrightarrow{w} f(\cdot,x(\cdot))u(\cdot) \text{ in } L^q(T,H) \text{ as } n \to \infty.$$

We have

$$\begin{aligned}
((\hat{A}(x_n), x_n - x)) &= ((-\dot{x}_n + f(\cdot,x_n(\cdot))u_n(\cdot), x_n - x))\\
&= ((-\dot{x}_n, x_n - x)) + (f(\cdot,x_n(\cdot))u_n(\cdot), x_n - x)).
\end{aligned}$$

From the proof of Proposition 4 we know that $((-\dot{x}_n, x_n - x)) \to 0$ as $n \to \infty$, while from the previous considerations we have that

$$(f(\cdot,x_n(\cdot))u_n(\cdot), x_n - x)_{pq} \to 0 \text{ as } n \to \infty.$$

Thus we obtain

$$\lim((\hat{A}(x_n), x_n - x)) = 0.$$

As before via the L-generalized pseudomonotonicity of $\hat{A}(\cdot)$, we have that $\hat{A}(x_n) \to \hat{A}(x)$ in $L^q(T,X^*)$ as $n \to \infty$. So in the limit as $n \to \infty$, we obtain

$$\begin{aligned}
\dot{x} + \hat{A}(x) &= f(\cdot,x(\cdot))u(\cdot)\\
\Rightarrow \dot{x}(t) + \hat{A}(t,x(t)) &= f(t,x(t))u(t) \text{ a.e. on } T\\
x(0) &= x_0,\ u \in S_U^q\\
\Rightarrow x &\in R(u).
\end{aligned}$$

■

We can also deal with optimal problems of the Meyer type. So now our cost functional has the form

$$J(u) = \int_0^b L(t,x(u)(t),u(t))dt + \varphi(x(u)(b)) \to \inf_{u\in S_U^q} = \beta. \tag{14}$$

For the cost integrand $L(t,x,u)$ we assume hypotheses $H(L)$ (see Section 3). For $\varphi(\cdot)$ we make the following hypothesis:

$\mathbf{H}(\varphi):\ \varphi : H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous.

Then working as in the proof of Theorem 3 with minimizing sequences (direct method), we can have the following theorem.

Theorem 4. *If hypotheses $H(A)_1, H(f)_1, H(U)_1, H(L)$ and $H(\varphi)$ hold, then problem* (14) *admits an optimal state-control pair $(x^*, u^*) \in W_{pq}(T) \times S_U^q$.*

We close this section with a partial generalization of Proposition 4 in which the multifunction $F(t,x)$ is defined only on $T \times X$. For this purpose our hypotheses on F are now the following:

$\mathbf{H(F)_1}:\ F : T \times X \to P_{f_c}(H)$ is a multifunction such that

(i) for every $x \in X$, $t \to F(t,x)$ is measurable;

(ii) for all $t \in T$, $GrF(t,\cdot)$ is sequentially closed in $X \times H_w$;

(iii) for almost all $t \in T$ and all $x \in X$, we have

$$|F(t,x)| = \sup[|y| : y \in F(t,x)] \le a_2(t) + c_2||x||^{p-1} \text{ with } a_2 \in L^q(T),\ c_2 > 0;$$

(iv) there exists $\gamma > 0$ such that for almost all $t \in T$, all $x \in X$ and all $y \in F(t,x)$ we have $(y,x) \le \gamma$.

Proposition 6. *If hypotheses $H(A)_1, H(F)_1$ hold and $x_0 \in H$, then the solution set $S(x_0)$ of problem* (9) *is nonempty, weakly compact in $W_{pq}(T)$ and compact in $C(T,H)$.*

Proof. As in the proof of Proposition 4, first we treat the case $x_0 \in X$. We introduce $A_1(t,x) = A(t, x + x_0)$, $F_1(t,x) = F_1(t, x + x_0)$ and consider Problem (10). Let $K = \hat{A}_1 + G_1 : L^p(T,X) \to P_{wkc}(L^q(T,X^*))$, where $\hat{A}_1$ and G_1 are as in the proof of Proposition 4. Finally let $L : D(L) \subseteq L^p(T,X) \to P_{wkc}(L^q(T,X^*))$ be the linear maximal monotne operator defined by $L(x) = \dot{x}$ for all $x \in D(L) = \{x \in W_{pq}(T) : x(0) = 0\}$

Claim 1. $K(\cdot)$ is L-generalized pseudomonotone and bounded.

Boundedness is clear. So we need to show the L-generalized pseudomonotonicity. First, as in the proof of Proposition 6, we have that $K(\cdot)$ is usc from $L^p(T,X)$ into $L^q(T,X^*)_w$. Next let $\{x_n\} \subseteq D(L)$ such that $x_n \to x$ in $L^p(T,X), Lx_n \to Lx$ in $L^q(T,X^*)$ (hence $x_n \to x$ in $W_{pq}(T)$). $x_n^* \in K(x_n), n \geq 1, x_n^* \to x^*$ in $L^q(T,X^*)$ as $n \to \infty$ and $\overline{\lim}((x_n^*, x_n - x)) \leq 0$. Since $x_n \to x$ in $W_{pq}(T)$, we also have that $x_n \to x$ in $L^p(T,H)$ as $n \to \infty$. For every $n \geq 1$ we can write that $x_n^* = \hat{A}_1(x_n) + g_n$ with $g_n \in G_1(x_n)$ and we may assume that $g_n \to g$ in $L^q(T,H)$ as $n \to \infty$. Then we have $\overline{\lim}((\hat{A}_1(x_n), x_n - x)) \leq 0$ since $\hat{A}_1$ is L-generalized pseudomonotone, $\hat{A}_1(x_n) \xrightarrow{w} \hat{A}_1(x)$ and $((\hat{A}_1(x_n), x_n)) \to ((\hat{A}_1(x), x))$. Moreover, from the proof of Proposition 1, p. 440 of Papageorgiou [29] we know that $x_n(t) \to x(t)$ a.e. on T in X as $n \to \infty$. So using hypothesis $H(F)_1$ (ii) and Proposition 3.9 p. 694 of Hu-Papageorgiou [19], we obtain that $g \in G_1(x)$. Therefore in the limit as $n \to \infty$ we have that

$$\begin{aligned} x^* &= \hat{A}_1(x) + g, \text{ with } g \in G_1(x) \\ \Rightarrow x^* &\in K(x). \end{aligned}$$

Also $((x_n^*, x_n)) \to ((x^*, x))$. Thus K is L-generalized pseudomonotone as claimed.

Claim 2. $K(\cdot)$ is coercive.

Let $x \in L^p(T,X)$ and $x^* \in K(x)$. Using hypotheses $H(A)_1$ (iv) and $H(F)_1$ (iv), we have

$$((x^*, x)) \geq \hat{c}_1 ||x||_p^p - k_1 ||x||_p^{p-1} - k_2 \text{ for some } \hat{c}_1, k_1, k_2 > 0.$$

This proves the coercivity of $K(\cdot)$.

Apply Proposition (5) to obtain $\hat{x} \in W_{pq}(T)$ a solution to (10). Then $x(\cdot) = \hat{x}(\cdot) + x_0 \in W_{pq}(T)$ is a solution to (9). The existence of a solution when $x_0 \in H$ and the compactness of the solution set $S(x_0)$, are proved as in Proposition 4. ∎

Remark. Proposition 6 extends the work of Hirano [16] where $A(t,\cdot)$ is monotone, F is single-valued and $F(t,\cdot)$ is both continuous and sequentially weakly continuous.

5. Necessary conditions for saddle point optimality

In this section, we derive necessary conditions for saddle point optimality in the parametric optimal control problem of Section 3.

So T, (X, H, X^*), and V are as in Section 3 and the control space Y is a separable Banach space. Let $[u^*, \lambda^*] \in S_U^q \times S_\Sigma$ be a saddle point for the cost functional $J(u,\lambda) = \int_0^b L(t, x(u,\lambda)(t), u(t))dt$. Hence we have

$$J(u^*, \lambda) \leq J(u^*, \lambda^*) \leq J(u, \lambda^*), \text{ for all } u \in S_U^q \text{ and all } \lambda \in S_\Sigma.$$

To derive necessary conditions for this kind of optimality, we will need stronger hypotheses on the data. In what follows $x^* = x(u^*, \lambda^*)$.

$\mathbf{H(A)_2}:$ $A : T \times X \to X^*$ is an operator such that

(i) for all $x \in X, t \to A(t,x)$ is measurable;

(ii) for all $t \in T, x \to A(t,x)$ is continuously Frechet differentiable and for almost all $t \in T$ all $x, y \in X$ we have

$$< A(t,x) - A(t,y),\ x - y >\geq \beta||x-y||^p \text{ with } \beta > 0, 2 \leq p < \infty;$$

(iii) for almost all $t \in T$ and all $x \in X$ we have $||a(t,x)||_* \leq a_1(t) + c_1||x||^{p-1}$ with $a_1 \in L^q(T)_+, c_1 > 0, \frac{1}{p} + \frac{1}{q} = 1$ and $A_x^{'}(\cdot, x^*(\cdot))y \in L^q(T, X^*)$ for all $y \in X$;

(iv) for almost all $t \in T$ and all $y \in X$ we have

$$< A(t,y), y > \geq c||y||^p - a(t) \quad \text{and} \quad < A_x^{'}(t, x^*(t))y, y >$$

$$\geq \hat{c}||y||^p - \hat{a}(t) \text{ with } c, \hat{c} > 0 \text{ and } \hat{a} \in L^1(T)_+.$$

$\mathbf{H(f)_3}:$ $f : T \times H \times V \to H$ is a function such that

(i) for all $x \in H,\ v \in V,\ t \to f(t,x,v)$ is measurable;

(ii) for almost all $t \in T$, all $x, y \in H$ and all $v \in V$

$$|f(t,x,v) - f(t,y,v)| \leq k(t)|x-y| \text{ with } k \in L^1(T)_+,$$

for all $t \in T$ and all $x \in\ v \to f(t,x,v)$ is continuous and for all $t \in T$ and all $v \in V$ $x \to f(t,x,v)$ is continuously Frechet differentiable with $f_x^{'}(\cdot, x^*(\cdot), v)h \in L^q(T,H)$ for all $h \in H$;

(iii) for almost all $t \in T$ all $x \in H$ and all $v \in V$ we have

$$|f(t,x,v)| \leq a_2(t) + c_2|x|^{2/q} \text{ with } a_2 \in L^q(T)_+, c_2 > 0.$$

Let $U \in P_{f_c}(Y)$. The set of admissible controls is given by

$$\hat{U}_{ad} = \{u : T \to Y \text{ is measurable and } u(t) \in U \text{ a.e. on } T\}.$$

The topology on $\hat{U}_{ad}$ is the metric topology induced by the metric.

$$\hat{d}(u_1, u_2) = |\{t \in T : u_1(t) \neq u_2(t)\}|$$

(recall that $|\cdot|$ denotes the Lebesgue measure on T). It is well-known that $(\hat{U}_{ad}, \hat{d})$ is a complete metric space (see Ekeland [13]).

$\mathbf{H(L)_2}$: $L : T \times H \times Y \to \mathbb{R}$ is an integrand such that

(i) for all $x \in H,\ u \in Y,\ t \to L(t, x, u)$ is measurable;

(ii) for all $t \in T,\ (x, u) \to L(t, x, u)$ is continuous;

(iii) for all $t \in T$ and all $u \in Y,\ x \to L(t, x, u)$ in continuously Frechet differentiable and $L'_x(\cdot, x^*(\cdot), u^*(\cdot)) \in L^1(T, H)$.

We start with a simple lemma that we will need in the sequel.

Lemma 1. *Given $h \in L^1(T, H)$ and $\delta > 0$, we can find $C_\delta \subseteq T$ such that $|C_\delta| = \delta|T|$ and*

$$\sup_{t\in T}\Big|\delta\int_o^t h(s)ds - \int_{C_\delta\cap[0,t]} h(s)ds\Big| = o(\delta)\ \left(\frac{o(\delta)}{\delta} \to 0 \text{ as } \delta \downarrow 0\right).$$

Proof. Consider the map $t \to \chi_{[0,t]}(\cdot)h(\cdot)$. This map is continuous from T into $L^1(T, H)$. Invoking Proposition 1 of Fryszkowski-Rzeżuchowski [14], we deduce that given $0 < \delta < 1$, we can find $C_\delta \subseteq T$ with

$$|C_\delta| = \delta|T| \text{ such that } \sup_{t\in T}\Big|\delta\int_0^t h(s)ds - \int_{C_\delta\cap[0,t]} h(s)ds\Big| = o(\delta).$$

■

Now we are ready to state and prove our necessary conditions for saddle point optimality.

Theorem 5. *If hypotheses $H(A)_2$, $H(f)_3$, $H(B)$, $H(\Sigma)$, $H(L)_2$ hold and $(u^*, \lambda^*) \in \hat{U}_{ad} \times S_\Sigma$ is a saddle point for the cost functional $J(\cdot,\cdot)$ then there exists $\varphi \in W_{pq}(T)$ such that*

(a) $-\dot{\varphi}(t) + A'_x(t, x^*(t))^*(t) - \int_V f'_x(t, x^*(t), v)\varphi(t)\lambda^*(t)(dv)$ $= L'_x(t, x^*(t),\ u^*(t))$ *a.e. on* T $\varphi(b) = 0$ (*"adjoint equation"*);

(b) $(\varphi(t), \int_V f(t, x^*(t), v)(\lambda - \lambda^*(t))(dv)) \leq 0$ *a.e on T for all $\lambda \in \Sigma(t)$.*

(c) $L(t, x^*(t), u) - L(t, x^*(t), u^*(t)) + (B(t)^*\varphi(t), u - u^*(t)) \geq 0$ *a.e. on T for all $\lambda \in \Sigma(t)$.*

Proof. Since by (u^*, λ^*) is saddle point of $J(\cdot, \cdot)$, we have

(15) $$J(u^*, \lambda) \leq J(u^*, \lambda^*) \leq J(u, \lambda^*) \text{ for all } [u, \lambda] \in \hat{U}_{ad} \times S_\Sigma.$$

Let $v \in \hat{U}_{ad}$. Then by virtue of Lemma 1, given $\delta > 0$ we can find $C_\delta \subseteq T$ such that $|C_\delta| = \delta|T|$ and $\sup |\delta \int_0^t h(s)ds - \int_{C_\delta \cap [0,t]} h(s)ds| = o(\delta)$, where

$$h(s) = \begin{pmatrix} B(s)(v(s) - u^*(s)) \\ L(s, x^*(s), v(s)) - L(s, x^*(s), u(s)) \end{pmatrix}.$$

We introduce the following spike variations of the optimal control u^*:

$$u_\delta(t) = \begin{cases} u^*(t) & \text{if } \quad t \in T \setminus C_\delta \\ v(t) & \text{if } \quad t \in C_\delta. \end{cases}$$

Evidently $\hat{d}(u_\delta, u^*) = \delta$. Let $y_\delta = x(u_\delta, \lambda^*)$. We have:

$$\begin{aligned} &((\dot{y} - \dot{x}^*, y_\delta - x^*))_t + ((\hat{A}(y)_\delta - \hat{A}(x^*), y_\delta - x^*))_t \\ &= \int_0^t \int_V (f(t, y_\delta(t), v) - f(t, x^*(t), v), y_\delta(t) - x^*(t))\lambda^*(t)(dv)dt \\ &+ (\hat{B}(u_\delta - u^*), y_\delta - x^*)_{pqt}. \end{aligned}$$

Using the integration by parts formula, the monotonicity of $\hat{A}$ and hypothesis $H(f)_3$ (ii) we obtain

$$\frac{1}{2}|y_\delta(t) - x^*(t)|^2 \leq \int_0^1 (k(s)|y_\delta(s) - x^*(s)| + |B(s)(u_\delta(s) - u^*(s)|)|y_\delta(s) - x^*(s)|ds.$$

Invoking Lemma A.5 p. 157 of Brezis [6], we have

$$\begin{aligned} |y_\delta(t) - x^*(t)| &\leq \int_0^1 (k(s)|y_\delta(s) - x^*(s)| + |B(s)(u_\delta(s) - u^*(s)|)ds \\ &\leq \int_0^1 (k(s)|y_\delta(s) - x^*(s)| + ||B(s)||_\infty 2|U|\hat{d}(u_\delta, u^*). \end{aligned}$$

Here $|U| = \sup\{||u||_Y : u \in U\}$. Using Gronwall's inequality, we finally have

$$|y_\delta(t) - x^*(t)| \leq \beta_1 \hat{d}(u_\delta, u^*) \text{ for some } \beta_1 > 0 \text{ and all } t \in T$$

$$\Rightarrow y_\delta \to x^* \text{ in } C(T, H) \text{ as } \delta \downarrow 0.$$

On the other hand from the uniform monotonicity of $\hat{A}$ (cf. hypothesis $H(A)_2$ (ii)), we have

$$\beta||y_\delta - x^*||_p^p \le ||y_\delta - x^*||_\infty^2 ||k||_1 + ||y_\delta - x^*||_\infty ||b||_\infty 2|U| \to 0 \text{ as } \delta \downarrow 0;$$

$$\Rightarrow y_\delta \to x^* \text{ in } L^p(T,X) \text{ as } \delta \downarrow 0.$$

Now set $w_\delta = \frac{1}{\delta}(y_\delta - x^*) \in W_{pq}(T)$. From hypotheses $H(A)_2$ (ii) and $H(f)_3$ (iii) we have that

$$\begin{aligned}&\dot{w}_\delta(t) + A_x'(t, x^*(t))w_\delta(t)\\ &= \int_V f_x'(t, x^*(t), v)w_\delta(t)\lambda^*(t)(dv) + B(t)\frac{1}{\delta}(u_\delta(t) - u^*(t)) + \frac{o(t,\delta)}{\delta}\end{aligned}$$

with $o(\cdot,\delta) \in L^q(T, X^*)$. From the choice of the set $C_\delta \subseteq T$ we have

$$\begin{aligned}&\sup_{t\in T} \delta\Big|\int_0^t B(s)(v(s) - u^*(s))ds - \int_{C_\delta \cap [0,t]} B(s)(v(s) - u^*(s)ds\Big| = o(\delta)\\ &\Rightarrow \sup_{t\in T}\Big|\delta\int_0^t B(s)(v(s) - u^*(s))ds - \int_0^t B(s)(\frac{u_\delta(s) - u^*(s)}{\delta})ds\Big| = \frac{o(\delta)}{\delta}\\ &\Rightarrow B(\cdot)\frac{u_\delta - u^*(\cdot)}{\delta} \xrightarrow{||\cdot||_w} B(\cdot)(v(\cdot) - u^*(\cdot)) \text{ as } \delta \downarrow 0,\end{aligned}$$

where by $||\cdot||_w$ we denote the "weak norm" on $L^1(T,H)$; i.e, if $h \in L^1(T,H)$

$$||h||_w = \sup_{t\in T}\Big|\int_0^t h(s)ds\Big|.$$

But from Papageorgiou [28] (see Lemma, p. 327), we know that

$$B(\cdot)\frac{u_\delta(\cdot) - u^*(\cdot)}{\delta} \xrightarrow{w} B(\cdot)(v(\cdot) - u^*(\cdot)) \text{ in } L^q(T,H) \text{ as } \delta \downarrow 0.$$

Using standard a priori estimation (see for example Papageorgiou-Shahzad [32]), we can check that $\{w_\delta\}_{\delta>0} \subseteq W_{pq}(T)$ is relatively weakly compact. So we have $w_\delta \xrightarrow{w} z$ in $W_{pq}(T)$ as $\delta \downarrow 0$ and

$$\left\{\begin{array}{l}\dot{z}(t) + A_x'(t, x^*(t))z(t) = \displaystyle\int_V f_x'(t, x^*(t), v)z(t)\lambda^*(t)(dv)\\ + B(t)(v(t) - u^*(t)) \text{ a.e. on } T\\ z(0) = 0\end{array}\right\} \tag{16}$$

We introduce the adjoint equation

$$\left\{\begin{array}{l} -\dot{\varphi}(t)+A_x'(t,x^*(t))^*\varphi(t)-\int_V f_x'(t,x^*(t),v)^*\varphi(t)\lambda^*(t)(dv) \\ =L_x'(t,x^*(t),u^*(t)) \text{ a.e. on } T \\ \varphi(b)=0 \end{array}\right\}$$

From Theorem 30.A, p. 771 of Zeidler [37], we know that this problem has a unique solution $\varphi \in W_{pq}(T)$. So we obtain part (a) of the theorem. We have

$$((-\dot{\varphi},z))+((\hat{A}_x'(x^*)^*\varphi,z)) \ -\int_0^b \left(\int_V f_x'(t,x^*(t),v)^*\varphi(t)\lambda^*(t)(dv),z(t)\right)dt \\ =(\hat{L}_x'(x^*,u^*),z)_{pq},$$

where $\hat{A}_x'(x^*)(\cdot)=A_x'(\cdot,x^*(\cdot))$ and $\hat{L}_x'(x^*,u^*)(\cdot)=L_x'(\cdot,x^*(\cdot),u^*(\cdot))$. From the integration by parts formula for functions in $W_{pq}(T)$, we have

$$((-\dot{\varphi},z))=((\varphi,\dot{z}))$$

Therefore we obtain

$$\begin{aligned} &((\varphi,\dot{z}))+((\varphi,\hat{A}_x'(x^*)z)) \\ (17)\quad &-\int_0^b(\varphi(t),\int_V f_x'(t,x^*(t),v)\lambda^*(t)(dv)z(t))dt=(\hat{L}_x'(x^*,u^*),z)_{pq} \\ &\Rightarrow ((\varphi,\hat{B}(v-u^*)))=(L_x'(x^*,u^*),z)_{pq} \text{ (see (16)).} \end{aligned}$$

From the second inequality in (15), we know that $J(u^*,\lambda^*)\le J(u,\lambda^*)$ for all $u\in S_U^q$. So we have:

$$\begin{aligned} &\int_0^b L(t,x^*(t),u^*(t))dt \le \int_0^b L(t,x(u,\lambda^*)(t),u(t)dt \text{ for all } u\in S_U^q \\ &\Rightarrow \int_0^b (L(t,y_\delta(t),u_\delta(t))-L(t,x^*(t),u^*(t))dt \ge 0 \text{ for all } \delta>0 \\ (18)\quad &\Rightarrow \int_{C_\delta}(L(t,y_\delta(t),v(t))-L(t,x^*(t),u^*(t))dt \\ &+\int_{T\setminus C_\delta}(L(t,y_\delta(t),u^*(t))-L(t,x^*(t),u^*(t))dt \ge 0. \end{aligned}$$

By hypothesis $H(L)_2$ (iii), $L(t,\cdot,u)$ is continuously Frechet differentiable.

So we have

$$\begin{aligned}&\int_{T\setminus C_\delta}(L(t,y_\delta(t),u^*(t))-L(t,x^*(t),u^*(t)))dt\\&=\int_{T\setminus C_\delta}(L_x^{'}(t,x^*(t),u^*(t)),y_\delta(t)-x^*(t))dt+o(\delta)\\&=\int_T(L_x^{'}(t,x^*(t),u^*(t)),y_\delta(t)-x^*(t))dt\\&-\int_{C_\delta}(L_x^{'}(t,x^*(t),u^*(t)),y_\delta(t)-x^*(t))dt+o(\delta).\end{aligned}$$

Note that

$$\begin{aligned}&\Big|\int_{C_\delta}(L_x^{'}(t,x^*(t),u^*(t)),y_\delta(t)-x^*(t))dt\Big|\\&\leq||y-x^*||_\infty\int_{C_\delta}||L_x^{'}(t,x^*(t),u^*(t))||_{\mathcal{L}}dt=o_1(\delta)\end{aligned}$$

since $y_\delta\to x^*$ in $C(T,H)$ and $\int_{C_\delta}||L_x^{'}(t,x^*(t),u^*(t))||_{\mathcal{L}}dt\to 0$ as $\delta\downarrow 0$ (see hypothesis $H(L)_2$ (iii) and recall that $|C_\delta|=\delta|T|\downarrow 0$ as $\delta\downarrow 0$). So we have

$$\begin{aligned}&\int_{T\setminus C_\delta}(L(t,y_\delta(t),u^*(t))-L(t,x^*(t),u^*(t)))dt\\&\leq\int_T(L_x^{'}(t,x^*(t),u^*(t)),y_\delta(t)-x^*(t))dt+o_2(\delta).\end{aligned}\tag{19}$$

Also we have

$$\begin{aligned}&\int_{C_\delta}(L(t,y_\delta(t),v(t))-L(t,x^*(t),u^*(t)))dt\\&=\int_{C_\delta}(L(t,y_\delta(t),v(t))-L(t,x^*(t),v(t)))dt\\&+\int_{C_\delta}(L(t,x^*(t),v(t))-L(t,x^*(t),u^*(t))dt\\&=\int_{C_\delta}(L_x^{'}(t,x^*(t),v(t)),y_\delta(t)-x(t))dt+o_3(\delta)\\&+\int_{C_\delta}(L(t,x^*(t),v(t))-L(t,x^*(t),u^*(t)))dt.\end{aligned}$$

By virtue of the choice of C_δ we have

$$\begin{aligned}&\int_{C_\delta}(L(t,x^*(t),v(t))-L(t,x^*(t),u^*(t)))dt\\&\leq\int_0^b(L(t,x^*(t),v(t))-L(t,x^*(t),u^*(t)))dt+o_4(\delta).\end{aligned}$$

So we can write that

$$
(20)\qquad \begin{aligned}
&\int_{C_\delta}(L(t,y_\delta,v(t))-L(t,x^*(t),u^*(t)))dt \le \\
&\le \int_{C_\delta}(L_x^{'}(t,x^*(t),v(t)),y_\delta(t)-x^*(t))dt \\
&+\delta\int_0^b(L(t,x^*(t),v(t))-L(t,x^*(t),u^*(t))dt+0_4(\delta).
\end{aligned}
$$

Using (19) and (20) in (18), we obtain

$$
\begin{aligned}
0 \le &\int_T(L_x^{'}(t,x^*(t),u^*(t)),y_\delta(t)-x^*(t))dt \\
&+\delta\int_{C_\delta}(L_x^1(t,x^*(t),v(t)),y_\delta(t)-x^*(t))dt \\
&+\delta\int_0^b(L(t,x^*(t),v(t))-L(t,x^*(t),u^*(t)))dt+o_5(\delta).
\end{aligned}
$$

Divide by $\delta>0$ and let $\delta\downarrow 0$. Recall that $w_\delta=\frac{1}{\delta}(y^*-x^*)\xrightarrow{w} z$ in $W_{pq}(T)$ as $\delta\downarrow 0$ and that

$$
\begin{aligned}
&\int_{C_\delta}(L_x^{'}(t,x^*(t),v(t)),y_\delta(t)-x^*(t))dt \\
&\le ||y_\delta-x^*||_\infty \textstyle\int_{C_\delta}||L_x^{'}(t,x^*(t),v(t))||_{\mathcal{L}}dt\to 0 \text{ as } \delta\downarrow 0.
\end{aligned}
$$

So in the limit we have

$$
\begin{aligned}
0 \le &\int_0^b(L_x^{'}(t,x^*(t),u^*(t)),z(t))dt \\
+&\int_0^b(L(t,x^*(t),v(t))-L(t,x^*(t),u^*(t))dt \ \text{ for all } \ v\in S_U^q.
\end{aligned}
$$

Using (17), it follows that

$$
\begin{aligned}
0 \le &\int_0^b(\varphi(t),B(t)(v(t)-u^*(t))dt \\
+&\int_0^b(L(t,x^*(t),v(t))-L(t,x^*(t),u^*(t))dt \ \text{ for all } \ v\in S_U^q.
\end{aligned}
$$

Let $\{u_k\}_{k\ge 1}$ be dense in U and let

$$
\begin{aligned}
&\sigma_k(s)=(\varphi(s),B(s)(v_k-u^*(s))) \\
&+L(s,x^*(s),v_k)-L(s,x^*(s),u^*(s)) \ \text{ for all } \ k\ge 1.
\end{aligned}
$$

Then $\sigma_k \in L^1(T)$ and let D_k the set of Lebesque points of $\sigma_k(\cdot)$. From Lebesgue's theorem (see for example Oxtoby [23]), we know that $|D_k| = |T|$ for all $k \geq 1$. Fix $k \geq 1$ and $t \epsilon D_k$ and then for any given > 0 define

$$v(s) = \begin{cases} u^*(s) & \text{if } |s-t| > \varepsilon \\ v_k & \text{if } |s-t| \leq \varepsilon. \end{cases}$$

Evidently $v \in S_U^q$ and we have

$$0 \leq \int_{t-\varepsilon}^{t+\varepsilon} \sigma_k(s) ds.$$

Divide by $\varepsilon > 0$ and let $\varepsilon \downarrow 0$. We obtain

$$\begin{aligned} 0 \leq \sigma_k(t) &= (\varphi(t), B(t)(v_k - u^*(t))) \\ &+ (L(t, x^*(t), v_k) - L(t, x^*(t), u^*(t))) \ \text{ for all } \ t \in D_k, \ k \geq 1. \end{aligned}$$

Since $\{v_k\}_{k \geq 1}$ is dense in U, we conclude that

$$0 \leq (\varphi(t), B(t)(u_k - u^*(t))) + L(t, x^*(t), u_k) - L(t, x^*(t), u^*(t)) \text{ for all } u \in U.$$

This proves conclusion (c) of the theorem.

Next let $\lambda \in S_\Sigma$ and set $\lambda_\varepsilon = \lambda^* + (\lambda - \lambda^*)$ and $x_\varepsilon = x(u^*, \lambda)$. We have

$$\begin{aligned} &\dot{x}_\varepsilon(t) - \dot{x}(t) + A(t, x_\varepsilon(t)) - A(t, x^*(t)) = \int_V f(t, x_\varepsilon(t), v) \lambda_\varepsilon(t)(dv) \\ &- \int_V f(t, x^*(t), v) \lambda^*(t)(dv) \ \text{ a.e. on } \ T. \end{aligned}$$

Arguing as in the first part of the proof, we can have that

$$x_\varepsilon \to x^* \text{ in } C(T, H) \text{ and } x_\varepsilon \to x^* \text{ in } L^p(T, H) \text{ as } \varepsilon \downarrow 0.$$

Exploiting the Frechet differentiability of $A(t, \cdot)$ and $f(t, \cdot, v)$, we obtain

$$A(t, x_\varepsilon(t)) - A(t, x^*(t)) = A_x'(t, x^*(t))(x_\varepsilon(t) - x^*(t)) + o_1(t, \varepsilon)$$

a.e on T with $o_1(\cdot, \varepsilon) \in L^q(T, X^*)$ and

$$\begin{aligned} &\int_V f(t, x_\varepsilon(t), v) \lambda_\varepsilon(t)(dv) - \int_V f(t, x^*(t), v) \lambda^*(t)(dv) \\ &= \int_V f_x'(t, x^*(t), v)(x_\varepsilon(t) - x^*(t)) \lambda_\varepsilon(t)(dv) \\ &+ \int_V f(t, x^*(t), v)(\lambda_\varepsilon(t) - \lambda^*(t))(dv) + o_2(t, \varepsilon) \text{ a.e. on } T \end{aligned}$$

with $o_2(\cdot,\varepsilon)\in L^q(T,H)$. If we set $y_\varepsilon=\frac{1}{\varepsilon}(x_\varepsilon-x^*)$, we have

$$\dot{y}_\varepsilon(t)+A(t,x^*(t))=\int_V f_x^{'}(t,x^*(t),v)y_\varepsilon(t)\lambda_\varepsilon(t)(dv)$$
$$+\int_V f(t,x^*(t),v)(\lambda(t)-\lambda^*(t))(dv)+o(t,\varepsilon)\text{ a.e. on }T$$

with $o(\cdot,\varepsilon)\in L^q(T,H)$. As before we have that $y_\delta\to y$ in $W_{pq}(T)$ as $\delta\downarrow 0$, with $y\in W_{pq}(T)$ being the unique solution to this Cauchy problem:

$$\begin{aligned}\dot{y}(t)+A_x^{'}(t,x^*(t))y(t) &= \int_V f_x^{'}(t,x^*(t),v)y(t)\lambda^*(t)(dv)\\ &\quad+\int_V f(t,x^*(t),v)(\lambda(t)-\lambda^*(t))(dv)\text{ a.e. on }T\\ y(0) &= 0.\end{aligned}$$

Using the adjoint state $\varphi\in W_{pq}(T)$, as in the first part of the proof, we obtain

$$\int_0^b(\varphi(t),\int_V f(t,x^*(t),v)(\lambda(t)-\lambda^*(t)(dv))dt=(\hat{L}_x^{'}(x^*,u^*),y)_{pq}. \tag{21}$$

Using the first inequality in (15), we obtain

$$\int_0^b(\varphi(t),\int_V f(t,x^*(t),v)(\lambda(t)-\lambda^*(t)(dv))dt\le 0\text{ for all }\lambda\in S_\Sigma.$$

Let $\lambda_k:T\to M_+^1(V)$ be Lebesgue measurable maps such that $\Sigma(t)=\overline{\{\lambda_k(t)\}}_{k\ge1}$ for all $t\in T$ (see Hu-Papageorgiou [19], Theorem 2.4, p. 156). Set

$$\theta_k(t)=(\varphi(t),\int_V f(t,x^*(t),v)(\lambda_k(t)-\lambda^*(t))(dv))\ k\ge 1.$$

Let E_k be the set of Lebesgue points of θ_k. Recall that $|E_k|=|T|,k\ge 1$. Fix $k\ge 1$ and let $t\in E_k$. Define

$$\lambda(s)=\begin{cases}\lambda^*(s) & \text{if } |s-t|>r\\ \lambda_k(s) & \text{if } |s-t|\le r,\ r>0.\end{cases}$$

Evidently $\lambda \in S_\Sigma$ and we have

$$\int_0^b (\varphi(s), \int_V f(s, x^*(s), v)(\lambda(s) - \lambda^*(s))(dv)ds = \int_{t-r}^{t+r} \theta_k(s)ds \leq 0$$
$$\Rightarrow \frac{1}{r}\int_{t-r}^{t+r} \theta_k(s)ds \leq$$
$$\Rightarrow \theta_k(t) \leq 0 \ \text{ for all } \ t \in E_k, k \geq 1.$$

If we set $E = \cap E_k$, we have $|E| = |T|$ and for $t \in E$ we obtain

$$(\varphi(t), \int_V f(t, x^*(t), v)(\lambda(t) - \lambda^*(t))(dv)) \leq 0 \text{ for all } k \geq 1$$
$$\Rightarrow \quad (\varphi(t), \int_V f(t, x^*(t), v)(\lambda - \lambda^*(t))(dv)) \leq 0 \text{ a.e. on } T \text{ for all } \lambda \in \Sigma(t).$$

So we have proved conclusion (b) of the theorem. This concludes the proof of the theorem. ■

6. Examples

In this section, we present parabolic distributed parameter control systems where our abstract results can be applied.

So let $T = [0, b]$ and let $Z \subseteq \mathbb{R}^n$ be a bounded domain with C^1-boundary. We consider the following optimal control problem:

$$(22) \quad \left\{ \begin{array}{l} \int_0^b \int_Z L(t, z, x(t, z), u(t, z))dzdt + \hat{\varphi}(||x(b, \cdot)||_2) \to \inf = \beta \\ \text{s.t. } \frac{\partial x}{\partial t} - \sum_{k=1}^N D_k a_k(t, z, x, Dx) + a_0(t, z, x, Dx) \\ = f(t, z, x(t, z))u(t, z) \text{ a.e. on } T \times Z \\ x|_{T\times\Gamma} = 0, x(0, z) = x_0(z) \ \text{ a.e. on } \ Z, \\ |u(t, z)| \leq r(t, z) \ \text{ a.e. on } \ T \times Z \end{array} \right\}$$

The hypotheses on the data of (22) are the following:

$\mathbf{H(a)}:$ $a_k : T \times Z \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $k \in \{1, 2, ..., N\}$ are functions such that

(i) for all $(x, \eta) \in \mathbb{R} \times \mathbb{R}^N$, $(t, z) \to a_k(t, z, x, \eta)$ is measurable;

(ii) for almost all $(t, z) \in T \times Z$, $(x, \eta) \to a_k(t, z, x, \eta)$ is continuous;

(iii) for almost all $(t,z) \in T \times Z$ and all $x \in \mathbb{R}$, $\eta \in \mathbb{R}^N$, we have that

$$|a_k(t,z,x,\eta)| \leq \beta_1(t,z) + c_1(|x|^{p-1} + ||\eta||^{p-1})$$

with $\beta_1 \in L^q(T \times Z),\ c_1 > 0,\ 2 \leq p < \infty,\ \frac{1}{p} + \frac{1}{q} = 1;$

(iv) for almost all $(t,z) \in T \times Z$, all $x \in \mathbb{R}$ and all $\eta, \eta' \in \mathbb{R}^N,\ \eta \neq \eta'$, we have

$$\sum_{k=1}^{N} (a_k(t,z,x,\eta) - a_k(t,z,x,\eta'))(\eta - \eta^{'}) > 0;$$

(v) for almost all $(t,z) \in T \times Z$, all $x \in \mathbb{R}$ and all $\eta \in \mathbb{R}^N$ we have

$$\sum_{k=1}^{N} a_k(t,z,x,\eta) \geq c_2||\eta||^p - \theta(t)$$

with $\in L^1(T),\ c_2 > 0.$

$\mathbf{H(a_0)}:$ $a_0 : T \times Z \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R},\ \eta \in \mathbb{R}^n,\ (t,z) \to a_0(t,z,x,\eta)$ is measurable;

(ii) for almost all $(t,z) \epsilon T \times Z,\ (x,\eta) \to a_0(t,z,x,\eta)$ is continuous;

(iii) for almost all $(t,z) \in T \times Z$, all $x \in \mathbb{R}$ and all $\eta \in \mathbb{R}^N$ we have

$$|a_0(t,z,x,\eta)| \leq \beta_2(t,z) + c_2(|x|^{p-1} + ||\eta||^{p-1})$$

with $\beta_2 \in L^q(T \times Z),\ c_2 > 0.$

$\mathbf{H(L)_3}:$ $L : T \times Z \times \mathbb{R} \times \mathbb{R} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is an integrand such that

(i) $(t,z,x,u) \to L(t,z,x,u)$ is measurable;

(ii) for all $t \in T$ and almost all $z \in Z$, $(x,u) \to L(t,z,x,u)$ is lower semicontinuous;

(iii) for almost all $(t,z) \in T \times Z$ and all $x, u \in \mathbb{R}$ we have

$$\psi(t,z) - c_1(z)(|x| + |u|) \leq L(t,z,x,u)$$

with $\psi \in L^1(T \times Z)$ and $c_1 \in L^\infty(Z)$.

$\mathbf{H(f)_3}:$ $f : T \times Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}$, $(t,z) \to f(t,z,x)$ is measurable;
(ii) for almost all $(t,z) \in T \times Z$, $x \to f(t,z,x)$ is continuous;
(iii) for almost all (t,z) and all $x \in \mathbb{R}$, $|f(t,z,x)| \leq \beta_3(t,z)$ with $\beta_3 \in L^q(T, L^2(Z))$.

$\mathbf{H(r)}$: $r \in L^\infty(T, L^2(Z))$.

$\mathbf{H}(\hat{\varphi})$: $\hat{\varphi} : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous.

Proposition 7. *If hypotheses $H(a), H(a_0), H(L)_3, H(f)_3, H(r), H(\hat{\varphi})$ hold and $x_0 \in L^2(Z)$, then problem (22) admits a solution $[x^*, u^*] \in (C(T, L^2(Z)) \cap L^p(T, W_0^{1,p}(Z)) \times (L^2(T \times Z)$ and $\frac{\partial x^*}{\partial t} \in L^q(T, W^{-1,q}(Z))$.*

Proof. In this case the evolution triple is $X = W_0^{1,p}(Z)$, $H = L^2(Z)$, $X^* = W^{-1,q}(Z)$. The embeddings are compact. Let $A : T \times T \to X^*$ be defined by

$$< A(t,x), y > = \int_Z \sum_{k=1}^N a_k(t,z,x,Dx) D_k y dz + \int_Z a_0(t,z,x,Dx) y(z) dz.$$

Hypotheses $H(a)$ and $H(a_0)$ imply that $H(A)_1$ holds. In particular, the pseudomonotonicity of the operator $A(t,\cdot)$ follows from the result of Gossez-Mustonen [15]. Let $Y = L^2(Z) = H$ and set $U(t) = \{u \epsilon Y : ||u||_2 \leq \hat{r}(t)\}$ with $\hat{r}(t) = ||r(t,\cdot)||_2 \in L^\infty(T)$ (see hypothesis H(r)). So we see that $H(U)_1$ holds.

Next let $\hat{f} : T \times H \to L(Y,H) = L(H)$ be defined by

$$\hat{f}(t,x)u(\cdot) = f(t,\cdot,x(\cdot))u(\cdot).$$

By virtue of hypothesis $H(f)_3$, we see that hypothesis $H(f)_1$ holds.

Finally, let $\hat{L} : T \times H \times Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ and $\varphi : H \to \overline{\mathbb{R}}$ be defined by

$$\hat{L}(t,x,u) = \int_Z L(t,z,x(z),u(z)) dz \text{ and } \varphi(x) = \hat{\varphi}(||x||).$$

Using hypothesis $H(L)_3$ and $H(\hat{\varphi})$, we can easily check that hypotheses $H(L)$ and $H(\varphi)$ hold. So we can apply Theorem 4 and produce an optimal pair $(x^*, u^*) \in (C(T, L^2(Z) \cap L^p(T, W_0^{1,p}(Z)) \times L^2(T \times Z)$ for problem (22) and $\frac{\partial x^*}{\partial t} \in L^q(T, W^{-1,q}(Z))$. ∎

Next consider the following parabolic distributed parameter control system. Here V is a compact subset of the Euclidean space $\mathbb{R}^m, m \geq 1$.

$$(23)\quad \left\{ \begin{array}{l} \dfrac{\partial x}{\partial t} - diva(t,z,Dx) \\ = \displaystyle\int_V g(t,z,v)\lambda(t)(dv)x(t,z) + \beta_1(t,z)u(t,z) \text{ a.e on } T \times Z \\ x|_{Tx} = 0,\ (0,z) = x_0(z) \text{ a.e on } Z, \\ |u(t,z)| \leq r(z) \text{ a.e on } Z,\ \lambda \in S_\Sigma \end{array} \right\}$$

Also we are given the integral cost function

$$J(u,\lambda) = \int_0^b \int_Z \hat{L}(t,z,x(u,\lambda)(t,z),u(t,z))dzdt.$$

We consider the minimax control problem

$$\inf_u \sup_\lambda [J(u,\lambda) : u \in L^q(T \times Z), |u(t,z)| \leq r(z) \text{ a.e on } Z,\ \lambda \in S_\Sigma] = \beta. \quad (24)$$

We assume that (x^*, u^*) is a saddle point optimal pair for the minimax control problem 24. We make the following hypotheses:

$\mathbf{H(a)_1}$: $a : T \times Z \times \mathbb{R}^N \to \mathbb{R}^N$ is a function such that

(i) for all $\eta \in \mathbb{R}^N$, $(t,z) \to a(t,z,\eta)$ is measurable;

(ii) for almost all $(t,z) \in T \times Z, \eta \to a(t,z,\eta)$ is a C^1-function and $a'_\eta(\cdot,\cdot,x^*(\cdot,\cdot)) \in L^q(T \times Z, \mathbb{R}^N)$;

(iii) for almost all $(t,z) \in T \times Z$ and all $\eta \in \mathbb{R}^N$ we have

$$||a(t,z,)|| \leq a_1(t,z) + c_1||\eta||^{p-1} \text{ with } a_1 \in L^q(T \times Z) \text{ and } c_1 > 0;$$

(iv) for almost all $t \in T$, almost all $z \in Z$ and all $\eta, \eta' \in \mathbb{R}^N$ we have

$$(a(t,z,\eta) - a(t,z,\eta'), \eta - \eta')_{\mathbb{R}^N} \geq c||\eta - \eta'||^p;$$

and $a(t,z,0) = 0$.

$\mathbf{H(g)}$: $g : T \times Z \times V \to \mathbb{R}$ is a function such that

(i) for all $v \in V, (t,z) \to g(t,z,v)$ is measurable;

(ii) for all $t \in T$ and almost all $z \in Z, v \to g(t,z,v)$ is continuous;

(iii) for almost all $(t,z) \in T \times Z$ and all $v \in V$ we have $|g(t,z,v)| \leq a_2(t,z)$ with $a_2 \in L^q(T \times Z)$.

$\mathbf{H_1}:$ $r \in L^\infty(Z)$ and $\beta \in L^\infty(T \times Z)$;

$\mathbf{H(L)_4}:$ $\hat{L} : T \times Z \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an integral such that

(i) for all $(x,u) \in \mathbb{R} \times \mathbb{R},\ (t,z) \to \hat{L}(t,z,x,u)$ is measurable ;

(ii) for almost all $(t,z) \in T \times Z,\ (x,u) \to \hat{L}(t,z,x,u)$ is continuous;

(iii) for all $t \in T$, almost all $z \in Z$ and all $u \in \mathbb{R}, x \to \hat{L}(t,z,x,u)$ is C^1 and

$$\hat{L}'_x(\cdot,\cdot,x^*(\cdot,\cdot),u^*(\cdot,\cdot)) \in L^1(T, L^2(Z)).$$

Again the evolution triple is $X = W_0^{1,p}(Z),\ H = L^2(Z),\ X^* = W^{-1,q}(Z)$. Set $f(t,x,v)(\cdot) = g(t,\cdot,x(\cdot),v) \in H$ for all $x \in H$, $Y = L^2(Z) = H, U(t) = \{u \in Y : ||u||_2 \leq ||r||_\infty\},\ B(t)u(\cdot) = \beta(t,\cdot)u(\cdot) \in H$ for all $u \in Y$ and $L(t,x,u) = \int_Z \hat{L}(t,z,x(z),u(z))dz$ for all $t \in T$ and all $x,u \in H = Y$. Then we can easily verify that hypotheses $H(A)_2, H(f)_3, H(B), H(L)_3$ hold. Assume also $H(\Sigma)$. So we can apply Theorem 5 and obtain:

Proposition 8. *If hypotheses $H(a)_1, H(g), H_1, H(\Sigma), H(L)_4$ hold and (u^*,λ^*) is a saddle point for the cost functional $J(u,\lambda)$ then there exists $\varphi \in C(T,H) \cap L^p(T, W_0^{1,p}(Z))$ with $\frac{\partial \varphi}{\partial t} \in L^q(T, W^{-1,q}(Z))$ such that:*

(a) $-\frac{\partial \varphi}{\partial t} - div(a'_x(t,z,Dx^*(z))\varphi(t,z) - \int_V g'_x(t,z,v)\lambda^*(t)(dv)x(t,z) = \hat{L}'_x(t,z,x^*(t,z),u^*(t,z))$ *a.e. on* $T \times Z$ $\varphi|_{T\times\Gamma} = 0,\ \varphi(b,z) = 0$ *a.e on* Z*;*

(b) $\int_Z \varphi(t,z) \int_V g(t,z,v)(\lambda - \lambda^*(t))(dv)x^*(t,z)dz \leq 0$ *a.e on* T *for all* $\lambda \in \Sigma(t)$;

(c) $\int_Z (\hat{L}(t,z,x^*(t,z),u(z)) - \hat{L}(t,z,x^*(t,z),u^*(t,z))dz$
$+ \int_Z \beta(t,z)\varphi(t,z)(u(z) - u^*(t,z))dz \geq 0$ *a.e on* T *for all* $u \in U$. ■

We can also deal with higher order systems. So consider

$$\left\{\begin{array}{l} \dfrac{\partial x}{\partial t} - \displaystyle\sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(t,z,\eta(x),\theta(x)) \\ = f(t,z,\eta(x))u(t,z) \text{ a.e. on } T \times Z \\ x|_{T\times\Gamma} = 0, x(0,z) = x_0(z) \text{ a.e. on } Z, \\ |u(t,z)| \leq r(t,z) \text{ a.e. on } Z \end{array}\right\} \tag{25}$$

Here $\alpha = (\alpha_k)_{k=1}^N,k$ is a nonnegative integer, $|\alpha| = \sum_{k=1}^N \alpha_k$ (the length of the multi-index) $D^\alpha = D_1^{\alpha_1}...D_N^{\alpha_N}$ with $D_k = \frac{\partial}{\partial z_k}, \eta(x) = \{D_x^\alpha : |\alpha| \le m-1\}$ and $\theta(x) = \{D_x^\alpha : |\alpha| = m\}$. We make the following hypotheses:

$\mathbf{H(a)_2}:$ $A_\alpha : T \times Z \times \mathbb{R}^{N_{m-1}} \times \mathbb{R}^{\hat{N}_m} \to \mathbb{R}$
$(N_{m-1} = \frac{(N+m-1)!}{N!(m-1)!}$, $N_m = \frac{(N+m)!}{N!m!}$, $\hat{N}_m = N_m - N_{m-1})$ are such that

(i) for all $\eta \in \mathbb{R}^{N_{m-1}}$ and $\theta \in \mathbb{R}^{\hat{N}_m}$, $(t,z) \to A_\alpha(t,z,\eta,\theta)$ is measurable;

(ii) for almost all $(t,z) \in T \times Z$, $(\eta,\theta) \to A_\alpha(t,z,\eta,\theta)$ is continuous;

(iii) for almost all $(t,z) \in T \times Z$, all $\eta \in \mathbb{R}^{N_{m-1}}$ and all $\theta \in \mathbb{R}^{\hat{N}_m}$ we have

$$|A_\alpha(t,z,\eta,\theta)| \le \beta_1(t,z) + c_1(||\eta||^{p-1} + ||\theta||^{p-1})$$

with $\beta_1 \in L^q(T \times Z)$, $c_1 > 0$;

(iv) for almost all $(t,z) \in T \times Z$, all $\eta \in \mathbb{R}^{N_{m-1}}$ and all $\theta, \theta' \in \Re^{N_m}$, $\theta \ne \theta'$, we have

$$\sum_{|\alpha|=m} (A_\alpha(t,z,\eta,\theta) - A_\alpha(t,z,\eta,\theta'))(\theta_\alpha - \theta'_\alpha) > 0;$$

(v) for almost all $(t,z) \in T \times Z$ and all $\eta \in \mathbb{R}^{N_{m-1}}$, $\theta \in \mathbb{R}^{\hat{N}_m}$ we have

$$\sum_{|\alpha|\le m} A_\alpha(t,z,\eta,\theta)\theta_\alpha \ge c_2||\theta||^p - \gamma(t,z)$$

with $\gamma \in L^1(T \times Z)$, $c_2 > 0$.

$\mathbf{H(f)_4}:$ $f : T \times Z \times \mathbb{R}^{N_{m-1}} \to \mathbb{R}$ is a function such that

(i) for all $\eta \in \mathbb{R}^{N_{m-1}}$ $(t,z) \to f(t,z,\eta)$ is measurable;

(ii) for all $t \in T$ and almost all $z \in Z$, $\eta \to f(t,z,\eta)$ is continuous;

(iii) for almost all $(t,z) \in T \times Z$ and all $\eta \in \mathbb{R}^{N_{m-1}}$ we have

$$f(t,z,\eta)| \le \beta_2(t,z) + c_2||\eta||^{p-1} \text{ with } \beta_2 \in L^q(T \times Z), c_2 > 0$$

and

$$f(t,z,\eta_0,\eta')\eta_0 \le \gamma_1 \text{ for some } \gamma_1 > 0, \text{ with } \eta = (\eta_0,\eta'), \ \eta_0 \in \mathbb{R}.$$

In this case the evolution triple is $X = W_0^{m,p}(Z)$, $H = L^2(Z)$ and $X^* = W^{-m,q}(Z)$. Again all embeddings are compact (Sobolev embedding theorem). The operator $A : T \times X \to X^*$ is defined by

$$< A(t,x), y >= \int_Z \sum_{|\alpha|\leq m} A_\alpha(t,z,\eta(x),\theta(x)) D^\alpha y(z) dz, \; x, y \in W_0^{1,p}(Z).$$

The pseudomonotonicity of $A(t,\cdot)$ follows from Browder [7]. Also we introduce $\hat{f} : T \times X \to H$ defined by $\hat{f}(t,x)(\cdot) = f(t,\cdot,\eta(x)(\cdot))$. Using Proposition 6 we obtain the following result:

Proposition 9. *If hypotheses $H(a)_2, H(f)_4$ hold, $r \in L^\infty(T \times Z)$ and $x_0 \in L^2(Z)$ then the set of all admissible solutions of* (25) *is weakly compact in $W_{pq}(T)$ and compact in $C(T, L^2(Z))$.*

Remark. A similar system has been studied by Ahmed-Xiang [1]. The authors claim that it can be analyzed using their results. However, this claim is inaccurate. Hypothesis (A8) (iv) in that paper (monotonicity only on the higher order terms; compare with $H(a)_2$ (iv)), does not imply the monotonicity of $A(t,\cdot)$. In Ahmed-Xiang [1], it is assumed that $A(t,\cdot)$ is monotone. So the example of Ahmed-Xiang should be modified and assume that A_α depends only on $D^m x$ and not on $(x, Dx, D^m x)$.

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