

## SHAPE OPTIMIZATION FOR DYNAMIC CONTACT PROBLEMS

ANDRZEJ MYŚLIŃSKI

*System Research Institute*

*01-447 Warsaw, ul. Newelska 6, Poland*

**e-mail:** `myslinsk@ibspan.waw.pl`

### Abstract

The paper deals with shape optimization of dynamic contact problem with Coulomb friction for viscoelastic bodies. The mass nonpenetrability condition is formulated in velocities. The friction coefficient is assumed to be bounded. Using material derivative method as well as the results concerning the regularity of solution to dynamic variational inequality the directional derivative of the cost functional is calculated and the necessary optimality condition is formulated.

**Keywords:** dynamic unilateral problem, shape optimization, sensitivity analysis, necessary optimality condition.

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## 1 Introduction

This paper deals with formulation of a necessary optimality condition for a shape optimization problem of a viscoelastic body in unilateral dynamic contact with a rigid foundation. It is assumed that the contact with given friction, described by Coulomb law [2], occurs at a portion of the boundary of the body. The contact condition is described in velocities. This first order approximation seems to be physically realistic for the case of small distance between the body and the obstacle and for small time intervals. The friction coefficient is assumed to be bounded. The equilibrium state of this contact problem is described by a hyperbolic variational inequality of the second order [2, 3, 5, 7, 15].

The shape optimization problem for the elastic body in contact consists in finding, in a contact region, such shape of the boundary of the domain occupied by the body that the normal contact stress is minimized. It is assumed that the volume of the body is constant.

Shape optimization of static contact problems was considered, among others, in [3, 8, 9, 10, 11, 14]. In [3, 8] the existence of optimal solutions and convergence of finite-dimensional approximation was shown. In [9, 10, 11, 14] necessary optimality conditions were formulated using the material derivative approach (see [14]). Numerical results are reported in [3, 11].

In this paper, we shall study this shape optimization problem for a viscoelastic body in unilateral dynamical contact. The essential difficulty to deal with the shape optimization problem for dynamic contact problem is regularity of solutions to the state system. Assuming small friction coefficient and suitable regularity of data it can be shown [6, 7] that the solution to dynamic contact problem is regular enough to differentiate it with respect to parameter. Using material derivative method [14] as well as the results of regularity of solutions to the dynamic variational inequality [6, 7] we calculate the directional derivative of the cost functional and we formulate the necessary optimality condition for this problem.

We shall use the following notation:  $\Omega \in R^2$  will denote the bounded domain with Lipschitz continuous boundary  $\Gamma$ . The time variable will be denoted by  $t$  and the time interval  $I = (0, \mathcal{T}), \mathcal{T} > 0$ . By  $H^k(\Omega)$ ,  $k \in (0, \infty)$  we will denote the Sobolev space of functions having derivatives in all directions of the order  $k$  belonging to  $L^2(\Omega)$  [1]. For an interval  $I$  and a Banach space  $B$   $L^p(I; B)$ ,  $p \in (1, \infty)$  denotes the usual Bochner space [2].  $u_t = du/dt$  and  $u_{tt} = d^2u/dt^2$  will denote first and second order derivatives, respectively, with respect to  $t$  of function  $u$ .  $u_{tN}$  and  $u_{tT}$  will denote normal and tangential components, respectively, of function  $u_t$ .  $Q = I \times \Omega$ ,  $\gamma_i = I \times \Gamma_i$ ,  $i = 1, 2, 3$  where  $\Gamma_i$  are pieces of the boundary  $\Gamma$ .

## 2 Contact problem formulation

Consider deformations of an elastic body occupying domain  $\Omega \in R^2$ . The boundary  $\Gamma$  of domain  $\Omega$  is Lipschitz continuous. The body is subjected to body forces  $f = (f_1, f_2)$ . Moreover, surface tractions  $p = (p_1, p_2)$  are applied to a portion  $\Gamma_1$  of the boundary  $\Gamma$ . We assume that the body is clamped along the portion  $\Gamma_0$  of the boundary  $\Gamma$  and that the contact conditions are

prescribed on the portion  $\Gamma_2$  of the boundary  $\Gamma$ . Moreover,  $\Gamma_i \cap \Gamma_j = \emptyset$ ,  $i \neq j$ ,  $i, j = 0, 1, 2$ ,  $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ .

We denote by  $u = (u_1, u_2)$ ,  $u = u(t, x)$ ,  $x \in \Omega$ ,  $t \in [0, T]$ ,  $T > 0$  the displacement of the body and by  $\sigma = \{\sigma_{ij}(u(t, x))\}$ ,  $i, j = 1, 2$ , the stress field in the body. We shall consider elastic bodies obeying Hook's law [2, 3, 5, 15]:

$$(1) \quad \sigma_{ij}(u) = c_{ijkl}^0(x)e_{kl}(u) + c_{ijkl}^1(x)e_{kl}(u_t) \quad x \in \Omega, \quad e_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}),$$

$i, j, k, l = 1, 2$ ,  $u_{k,l} = \partial u_k / \partial x_l$ . We use here the summation convention over repeated indices [2].  $c_{ijkl}^0(x)$  and  $c_{ijkl}^1(x)$ ,  $i, j, k, l = 1, 2$  are components of Hook's tensor. It is assumed that elasticity coefficients  $c_{ijkl}^0$  and  $c_{ijkl}^1$  satisfy usual symmetry, boundedness and ellipticity conditions [2, 3, 5]. In an equilibrium state a stress field  $\sigma$  satisfies the system [2, 3, 6, 7]:

$$(2) \quad u_{tti} - \sigma_{ij}(x)_{,j} = f_i(x), \quad (t, x) \in (0, T) \times \Omega \quad i, j = 1, 2,$$

where  $\sigma_{ij}(x)_{,j} = \partial \sigma_{ij}(x) / \partial x_j$ ,  $i, j = 1, 2$ . The following boundary conditions are given:

$$(3) \quad u_i(x) = 0 \quad \text{on } (0, T) \times \Gamma_0 \quad i = 1, 2,$$

$$\sigma_{ij}(x)n_j = p_i \quad \text{on } (0, T) \times \Gamma_1 \quad i, j = 2,$$

$$(4) \quad u_{tN} \leq 0, \quad \sigma_N \leq 0, \quad u_{tN}\sigma_N = 0, \quad \text{on } (0, T) \times \Gamma_2,$$

$$(5) \quad \begin{aligned} u_{tT} = 0 &\Rightarrow |\sigma_T| \leq \mathcal{F} |\sigma_N|, \\ u_{tT} \neq 0 &\Rightarrow \sigma_T = -\mathcal{F} |\sigma_N| \frac{u_{tT}}{|u_{tT}|}. \end{aligned}$$

Here we denote:  $u_N = u_i n_i$ ,  $\sigma_N = \sigma_{ij} n_i n_j$ ,  $(u_T)_i = u_i - u_N n_i$ ,  $(\sigma_T)_i = \sigma_{ij} n_j - \sigma_N n_i$ ,  $i, j = 1, 2$ ,  $n = (n_1, n_2)$  is the unit outward versor to the boundary  $\Gamma$ . The following initial conditions are given:

$$(6) \quad u_i(0, x) = u_0 \quad u_{ti}(0, x) = u_1, \quad i = 1, 2, \quad x \in \Omega.$$

We shall consider problem (2) – (6) in the variational form. Let us assume,

$$(7) \quad \begin{aligned} f &\in H^{1/4}(I; (H^1(\Omega; R^2))^*) \cap L^2(Q; R^2), \\ p &\in L^2(I; (H^{1/2}(\Gamma_1; R^2))^*), \\ u_0 &\in H^{3/2}(\Omega; R^2) \quad u_1 \in H^{3/2}(\Omega; R^2), \quad u_1|_{\Gamma_2} = 0, \\ \mathcal{F} &\in L^\infty(\Gamma_2; R^2), \quad \mathcal{F}(\cdot, x) \text{ is continuous for a.e. } x \in \Gamma_2 \end{aligned}$$

be given. The space  $L^2(Q; R^2)$  and the Sobolev spaces  $H^{1/4}(I; (H^1(\Omega; R^2))^*)$  as well as  $(H^{1/2}(\Gamma_1); R^2)$  are defined in [1, 2]. Let us introduce:

$$(8) \quad F = \{z \in L^2(I; H^1(\Omega; R^2)) : z_i = 0 \text{ on } (0, T) \times \Gamma_0, \quad i = 1, 2\},$$

$$(9) \quad K = \{z \in F : z_{tN} \leq 0 \text{ on } (0, T) \times \Gamma_2\}.$$

The problem (1) – (6) is equivalent to the following variational problem [6, 7]: find  $u \in L^\infty(I; H^1(\Omega; R^2)) \cap H^{1/2}(I; L^2(\Omega; R^2)) \cap K$  such that  $u_t \in L^\infty(I; L^2(\Omega; R^2)) \cap H^{1/2}(I; L^2(\Omega; R^2)) \cap K$  and  $u_{tt} \in L^\infty(I; H^{-1}(\Omega; R^2)) \cap (H^{1/2}(I; L^2(\Omega; R^2)))^*$  satisfying the following inequality [6, 7]:

$$(10) \quad \begin{aligned} & \int_Q u_{tti} dx d\tau + \int_Q \sigma_{ij}(u) e_{ij}(v_i - u_{ti}) dx d\tau \\ & + \int_{\gamma_2} \mathcal{F} |\sigma_N(u)| (|v_T| - |u_{tT}|) dx d\tau \geq \int_Q f_i(v_i - u_{ti}) dx d\tau \\ & + \int_{\gamma_1} p_i(v_i - u_{ti}) dx d\tau \quad \forall v \in H^{1/2}(I; H^1(\Omega; R^2)) \cap K. \end{aligned}$$

Note that from (7) as well as from Imbedding Theorem of Sobolev spaces [1] it follows that  $u_0$  and  $u_1$  in (6) are continuous on the boundary of cylinder  $Q$ . The existence of solutions to system (1) – (6) was shown in [6, 7]:

**Theorem 21.** *Assume: (i) the data are smooth enough, i.e. (7) is satisfied (ii)  $\Gamma_2$  is of class  $C^{1,1}$  (ii) the friction coefficient is small enough. Then there exists a unique weak solution to the problem (1) – (6).*

**Proof.** The proof is based on penalization of the inequality (10), friction regularization and employment of localization and shifting technique due to Lions and Magenes. For details of the proof see [7].  $\blacksquare$

For the sake of brevity we shall consider the contact problem with prescribed friction, i.e. we shall assume

$$(11) \quad \mathcal{F} |\sigma_N| = \sigma_T \leq 1.$$

The condition (5) is replaced by the following one

$$(12) \quad u_{tT} \sigma_T + |u_{tT}| = 0, \quad |\sigma_T| \leq 1 \text{ on } I \times \Gamma_2.$$

Let us introduce the space

$$(13) \quad \Lambda = \{\lambda \in L^2(I; L^\infty(\Gamma_2)) : |\lambda| \leq 1 \text{ on } I \times \Gamma_2\}.$$

Taking into account (12) the system (10) takes the form: find  $u \in K$  and  $\lambda \in \Lambda$  such that:

$$(14) \quad \begin{aligned} & \int_Q u_{tti} dx d\tau + \int_Q \sigma_{ij}(u) e_{ij}(v_i - u_{ti}) dx d\tau \\ & - \int_{\gamma_2} \lambda_T (v_T - u_{tT}) dx d\tau \geq \int_Q f_i (v_i - u_{ti}) dx d\tau \\ & + \int_{\gamma_1} p_i (v_i - u_{ti}) dx d\tau \quad \forall v \in H^{1/2}(I; H^1(\Omega; R^2)) \cap K \end{aligned}$$

$$(15) \quad \int_{\gamma_2} \sigma_T u_{tT} ds d\tau \leq \int_{\gamma_2} \lambda_T u_{tT} ds d\tau \quad \forall \lambda_T \in \Lambda.$$

### 3 Formulation of the shape optimization problem

We are going to consider a family  $\{\Omega_s\}$  of the domains  $\Omega_s$  depending on parameter  $s$ . For each  $\Omega_s$  we formulate a variational problem corresponding to (14) – (15). In this way we obtain a family of the variational problems depending on  $s$  and for this family we shall study a shape optimization problem, i.e. we minimize with respect to  $s$  a cost functional associated with the solutions to (14) – (15).

We shall consider the domain  $\Omega_s$  as an image of a reference domain  $\Omega$  under a smooth mapping  $\mathbf{T}_s$ . To describe the transformation  $\mathbf{T}_s$  we shall use the speed method [14]. Let us denote by  $V(s, x)$  regular enough vector field depending on parameter  $s \in [0, \vartheta)$ ,  $\vartheta > 0$ :

$$V : [0, \vartheta) \times R^2 \rightarrow R^2,$$

$$(16) \quad V(s, \cdot) \in C^2(R^2, R^2) \quad \forall s \in [0, \vartheta), \quad V(\cdot, x) \in C([0, \vartheta), R^2) \quad \forall x \in R^2.$$

Let  $\mathbf{T}_s(V)$  denote the family of mappings:  $\mathbf{T}_s(V) : R^2 \ni X \rightarrow x(t, X) \in R^2$  where the vector function  $x(\cdot, X) = x(\cdot)$  satisfies the systems of ordinary differential equations:

$$(17) \quad \frac{d}{d\tau} x(\tau, X) = V(\tau, x(\tau, X)), \quad \tau \in [0, \vartheta), \quad x(0, X) = X \in R.$$

We denote by  $D\mathbf{T}_s$  the Jacobian of the mapping  $\mathbf{T}_s(V)$  at a point  $X \in R^2$ . We denote by  $D\mathbf{T}_s^{-1}$  and  ${}^*D\mathbf{T}_s^{-1}$  the inverse and the transpose inverse of the Jacobian  $D\mathbf{T}_s$  respectively.  $J_s = \det D\mathbf{T}_s$  will denote the determinant

of the Jacobian  $D\mathbf{T}_s$ . The family of domains  $\{\Omega_s\}$  depending on parameter  $s \in [0, \vartheta)$ ,  $\vartheta > 0$ , is defined as follows:  $\Omega_0 = \Omega$

$$(18) \quad \begin{aligned} \Omega_s &= \mathbf{T}_s(\Omega)(V) = \{x \in R^2 : \exists X \in R^2 \\ &\text{such that } x = x(s, X), \text{ where the function } x(\cdot, X) \text{ satisfies} \\ &\text{equation (17) for } 0 \leq \tau \leq s\}. \end{aligned}$$

Let us consider problem (14) – (15) in the domain  $\Omega_s$ . Let  $F_s$ ,  $K_s$ ,  $\Lambda_s$  be defined, respectively, by (8), (9), (13) with  $\Omega_s$  instead of  $\Omega$ . We shall write  $u_s = u(\Omega_s)$ ,  $\sigma_s = \sigma(\Omega_s)$ . The problem (14) – (15) in the domain  $\Omega_s$  takes the form: find  $u_s \in K_s$  and  $\lambda_s \in \Lambda_s$  such that

$$(19) \quad \begin{aligned} &\int_{Q_s} u_{ttsi} v_i dx d\tau + \int_{Q_s} \sigma_{ij}(u_s) e_{ij}(v_i - u_{ttsi}) dx d\tau \\ &- \int_{\gamma_{s2}} \lambda_{sT}(v_T - u_{tsT}) dx d\tau \geq \int_{Q_s} f_i(v_i - u_{ttsi}) dx d\tau \\ &+ \int_{\gamma_{s1}} p_i(v_i - u_{ttsi}) dx d\tau \quad \forall v \in H^{1/2}(I; H^1(\Omega_s; R^2)) \cap K \end{aligned}$$

$$(20) \quad \int_{\gamma_{s2}} \sigma_{sT} u_{tsT} ds d\tau \leq \int_{\gamma_{s2}} \lambda_{sT} u_{tsT} ds d\tau \quad \forall \lambda_{sT} \in \Lambda_s.$$

We are ready to formulate the optimization problem. By  $\hat{\Omega} \subset R^2$  we denote a domain such that  $\Omega_s \subset \hat{\Omega}$  for all  $s \in [0, \vartheta)$ ,  $\vartheta > 0$ . Let  $\phi \in M$  be a given function. The set  $M$  is determined by:

$$(21) \quad \begin{aligned} M &= \{\phi \in L^\infty(I; H_0^2(\hat{\Omega}; R^2)) : \\ &\phi \leq 0 \text{ on } I \times \hat{\Omega}, \|\phi\|_{L^\infty(I; H_0^2(\hat{\Omega}; R^2))} \leq 1\}. \end{aligned}$$

Let us introduce, for given  $\phi \in M$ , the following cost functional:

$$(22) \quad J_\phi(u_s) = \int_{\gamma_{s2}} \sigma_{sN} \phi_{tNs} dz d\tau$$

where  $\phi_{tNs}$  and  $\sigma_{sN}$  are normal components of  $\phi_{ts}$  and  $\sigma_s$ , respectively, depending on parameter  $s$ . Note that the cost functional (22) approximates the normal contact stress [3, 8, 11]. We shall consider such family of domains  $\{\Omega_s\}$  that every  $\Omega_s$ ,  $s \in [0, \vartheta)$ ,  $\vartheta > 0$  has constant volume  $c > 0$ , i.e. every  $\Omega_s$  belongs to the constraint set  $U$  given by:

$$(23) \quad U = \left\{ \Omega_s : \int_{\Omega_s} dx = c \right\}.$$

We shall consider the following shape optimization problem:

$$(24) \quad \begin{aligned} & \text{For given } \phi \in M, \text{ find the boundary } \Gamma_{2s} \text{ of the domain } \Omega_s \\ & \text{occupied by the body, minimizing the cost functional (22)} \\ & \text{subject to } \Omega_s \in U. \end{aligned}$$

The set  $U$  given by (23) is assumed to be nonempty.  $(u_s, \lambda_s) \in K_s \times \Lambda_s$  satisfy (19) – (20). Note that the goal of the shape optimization problem (24) is to find such boundary  $\Gamma_2$  of the domain  $\Omega$  occupied by the body that the normal contact stress is minimized. Remark that the cost functional (22) can be written in the following form [3, 15]:

$$(25) \quad \begin{aligned} \int_{\gamma_{2s}} \sigma_{sN} \phi_{tN} ds d\tau &= \int_{Q_s} u_{tt_s} \phi_{ts} dx d\tau + \int_{Q_s} \sigma_{sij}(u_s) e_{kl}(\phi_{ts}) dx d\tau \\ &- \int_{Q_s} f \phi_{ts} dx d\tau - \int_{\gamma_{1s}} p_s \phi_{ts} ds d\tau - \int_{\gamma_{2s}} \sigma_{sT} \phi_{tT_s} ds d\tau. \end{aligned}$$

We shall assume there exists at least one solution to the optimization problem (24). It implies compactness assumption of the set (23) in suitable topology. For a detailed discussion concerning the conditions assuring the existence of optimal solutions see [3, 14].

## 4 Shape derivatives of contact problem solution

In order to calculate Euler derivative (43) of the cost functional (22) we have to determine shape derivatives  $(u', \lambda') \in F \times \Lambda$  of a solution  $(u_s, \lambda_s) \in K_s \times \Lambda_s$  to the system (19) – (20). Let us recall from [14]:

**Definition 41.** The shape derivative  $u' \in F$  of the function  $u_s \in F_s$  is determined by:

$$(26) \quad (\tilde{u}_s)|_{\Omega} = u + s u' + o(s),$$

where  $\|o(s)\|_F / s \rightarrow 0$  for  $s \rightarrow 0$ ,  $u = u_0 \in F$ ,  $\tilde{u}_s \in F(R^2)$  is an extension of the function  $u_s \in F_s$  into the space  $F(R^2)$ .  $F(R^2)$  is defined by (8) with  $R^2$  instead of  $\Omega$ .

In order to calculate shape derivatives  $(u', \lambda') \in F \times \Lambda$  of a solution  $(u_s, \lambda_s) \in K_s \times \Lambda_s$  to the system (19), (20), first we calculate material derivatives  $(\dot{u}, \dot{\lambda}) \in F \times \Lambda$  of the solution  $(u_s, \lambda_s) \in K_s \times \Lambda_s$  to the system (19), (20). Let us recall the notion of material derivative [14]:

**Definition 42.** The material derivative  $\dot{u} \in F$  of the function  $u_s \in K_s$  at a point  $X \in \Omega$  is determined by:

$$(27) \quad \lim_{s \rightarrow 0} \| [(u_s \circ \mathbf{T}_s) - \sigma]/s - \dot{u} \|_F = 0,$$

where  $u \in K$ ,  $u_s \circ \mathbf{T}_s \in K$  is an image of function  $u_s \in K_s$  in the space  $F$  under the mapping  $\mathbf{T}_s$ .

Taking into account Definition 42 we can calculate material derivatives of a solution to the system (19) – (20):

**Lemma 41.** *The material derivatives  $(\dot{u}, \dot{\lambda}) \in K_1 \times \Lambda$  of a solution  $(u_s, \lambda_s) \in K_s \times \Lambda_s$  to the system (19) – (20) are determined as a unique solution to the following system:*

$$(28) \quad \begin{aligned} & \int_Q \{ (\dot{u}_{tt}\eta + u_{tt}\dot{\eta} + u_{tt}\eta \operatorname{div} V(0)(DV(0)u)_{tt}\eta + u_{tt}(DV(0)\eta) \\ & - \dot{f}\eta - f\dot{\eta} + (\sigma_{ij}(u)e_{kl}(\eta) - f\eta) \operatorname{div} V(0) \} dx d\tau \\ & - \int_{\gamma_1} (\dot{p}\eta + p\dot{\eta} + p\eta D) dx d\tau - \int_{\gamma_2} \{ (\dot{\lambda}\eta_{tT} + \lambda\dot{\eta}_{tT} \\ & + \lambda \nabla \eta_T V(0)n + \lambda \eta_{tT} D \} dx d\tau \geq 0 \quad \forall \eta \in K_1, \end{aligned}$$

$$(29) \quad \begin{aligned} & \int_{\gamma_2} (\dot{\lambda} - \mu)u_{tT} + (\lambda - \dot{\mu})u_{tT} \\ & + (\lambda - \mu)\dot{u}_{tT} + \lambda u_{tT} D \} dx d\tau \quad \forall \mu \in L_1, \end{aligned}$$

where  $V(0) = V(0, X)$ ,  $DV(0)$  denotes the Jacobian matrix of the matrix  $V(0)$ . Moreover,

$$(30) \quad K_1 = \{ \xi \in F : \xi = u - DVu \text{ on } \gamma_0, \xi n \geq nDV(0)u \text{ on } A_1, \\ \xi n = nDV(0)u \text{ on } A_2 \},$$

$$(31) \quad \begin{aligned} A_0 &= \{ x \in \gamma_2 : u_{tN} = 0 \}, \quad A_1 = \{ x \in A_0 : \sigma_N = 0 \}, \\ A_2 &= \{ x \in B : \sigma_N < 0 \}, \end{aligned}$$

$$(32) \quad \begin{aligned} B_0 &= \{ x \in \gamma_2 : \lambda_T = 1, u_{tT} \neq 0 \}, \\ B_1 &= \{ x \in \gamma_2 : \lambda_T = -1, u_{tT} = 0 \}, \\ B_2 &= \{ x \in \gamma_2 : \lambda_T = 1, u_{tT} = 0 \}, \end{aligned}$$



$$(33) \quad L_1 = \{\xi \in \Lambda : \xi \geq 0 \text{ on } B_2, \xi \leq 0 \text{ on } B_1, \xi = 0 \text{ on } B_0\}$$

and  $D$  is given by

$$(34) \quad D = \operatorname{div} V(0) - (DV(0)n, n).$$

**Proof** is based on the approach proposed in [14]. First we transport the system (19) – (20) to the fixed domain  $\Omega$ . Let  $u^s = u_s \circ \mathbf{T}_s \in F$ ,  $u = u_0 \in F$ ,  $\lambda^s = \lambda_s \circ \mathbf{T}_s \in \Lambda$ ,  $\lambda = \lambda_0 \in \Lambda$ . Since in general  $u^s \notin K(\Omega)$  we introduce a new variable  $z^s = D\mathbf{T}_s^{-1}u^s \in K$ . Moreover,  $\dot{z} = \dot{u} - DV(0)u$  [7, 13]. Using this new variable  $z^s$  as well as the formulae for transformation of the function and its gradient into reference domain  $\Omega$  [13, 14] we write the system (19) – (20) in the reference domain  $\Omega$ . Using the estimates on time derivative of function  $u$  [7] the Lipschitz continuity of  $u$  and  $\lambda$  satisfying (19) – (20) with respect to  $s$  can be proved. Applying to this system the result concerning the differentiability of solutions to variational inequality [13, 14] we obtain that the material derivative  $(\dot{u}, \dot{\lambda}) \in K_1 \times \Lambda$  satisfies the system (28) – (29). Moreover, from the ellipticity condition of the elasticity coefficients by a standard argument [13] it follows that  $(\dot{u}, \dot{\lambda}) \in K_1 \times \Lambda$  is a unique solution to the system (28) – (29). ■

Recall [14] that if the shape derivative  $u' \in F$  of the function  $u_s \in F_s$  exists, then the following condition holds:

$$(35) \quad u' = \dot{u} - \nabla u V(0),$$

where  $\dot{u} \in F$  is material derivative of the function  $u_s \in F_s$ .

From regularity result in [7] it follows that:

$$(36) \quad \nabla u V(0) \in F, \quad \nabla \lambda_T V(0) \in \Lambda,$$

where the spaces  $F$  and  $\Lambda$  are determined by (8) and (13), respectively.

Integrating by parts system (28), (29), taking into account (35), (36) we obtain a similar system to (28), (29) determining the shape derivative  $(u', \lambda'_T) \in F \times L$  of the solution  $(u_s, \lambda_{sT}) \in K_s \times L_s$  to the system (19) – (20):

$$(37) \quad \int_Q u'_{tt} \eta + u_t t \eta' + (DV(0) + {}^* DV(0)) u_{tt} \eta dx d\tau + \int_\gamma u_{tt} \eta V(0) n \\ + \int_Q \sigma_{ij}(u') e_{kl} \eta - \int_{\gamma_2} \lambda' \eta_{tT} + \lambda \eta'_{tT} \} dx d\tau \\ + I_1(u_t, \eta) + I_2(\lambda, u, \eta) \geq 0 \quad \forall \eta \in N_1,$$

$$(38) \quad \int_{\gamma_2} [u'_{tT}(\mu - \lambda) - u_{tT}\lambda'] dx d\tau + I_3(u, \mu - \lambda) \geq 0 \quad \forall \mu \in L_1,$$

$$(39) \quad N_1 = \{\eta \in F : \eta = \lambda - DuV(0), \lambda \in K_1\},$$

$$(40) \quad I_1(\varphi, \phi) = \int_{\gamma} \{\sigma_{ij}(\varphi) e_k l \phi - f \phi - ((\nabla p n) \phi + (p \nabla \phi) n + p \phi H) V(0) n\} dx d\tau,$$

$$(41) \quad I_2(\mu, \varphi, \phi) = \int_{\gamma_2} \{(\nabla \mu) n \nabla \phi + \mu (\nabla (\nabla \varphi n)) \varphi + \mu \nabla \varphi_{tT} H + \mu \nabla \varphi n\} V(0) n dx d\tau,$$

$$(42) \quad I_3(\varphi, \mu - \lambda) = \int_{\gamma_2} (\varphi n)(\mu - \lambda) + \varphi (\nabla \mu n) - \varphi (\nabla \lambda n) + \varphi (\mu - \lambda) H] V(0) n dx d\tau,$$

where  $H$  denotes a mean curvature of the boundary  $\Gamma$  [14].

## 5 Necessary optimality condition

Our goal is to calculate the directional derivative of the cost functional (22) with respect to the parameter  $s$ . We will use this derivative to formulate the necessary optimality condition for the optimization problem (24). First, let us recall from [14] the notion of Euler derivative of the cost functional depending on domain  $\Omega$ :

**Definition 51.** Euler derivative  $dJ(\Omega; V)$  of the cost functional  $J$  at a point  $\Omega$  in the direction of the vector field  $V$  is given by:

$$(43) \quad dJ(\Omega; V) = \limsup_{s \rightarrow 0} [J(\Omega_s) - J(\Omega)]/s.$$

The form of the directional derivative  $dJ_\phi(u; V)$  of the cost functional (22) is given in:

**Lemma 51.** *The directional derivative  $dJ_\phi(u; V)$  of the cost functional (22), for  $\phi \in M$  given, at a point  $u \in K$  in the direction of vector field  $V$  is determined by:*

$$\begin{aligned}
(44) \quad dJ_\phi(u; V) &= \int_Q [u'_{tt}\eta + u_{tt}\eta' + (DV(0) + {}^* DV(0))u_{tt}\eta] dx d\tau \\
&+ \int_\gamma u_{tt}\eta V(0)n + \int_Q (\sigma'_{ij}e_{kl}(\phi)) dx \\
&+ \int_\Gamma (\sigma_{ij}e_{kl}(\phi) - f\phi)V(0)nds - \int_{\Gamma_1} (\nabla p\phi V(0) \\
&+ p \nabla \phi V(0) + p\phi D) ds - \int_{\Gamma_2} \sigma'_T \phi_T ds + I_1(u, \phi) - I_2(\lambda, u, \phi),
\end{aligned}$$

where  $\sigma'$  is a shape derivative of the function  $\sigma_s$  with respect to  $s$ . This derivative is defined by (26).  $\nabla p$  is the gradient of function  $p$  with respect to  $x$ . Moreover,  $V(0) = V(0, X)$ ,  $\phi_T$  and  $\sigma_T$  are tangent components of functions  $\phi$  and  $\sigma$ , respectively, and  $D$  is given by (34).  $DV(0)$  denotes the Jacobian matrix of the matrix  $V(0)$  and  $\text{div}$  denotes divergence operator.

**Proof.** Taking into account (22), (25) as well as formulae for transformation of the gradient of the function defined on domain  $\Omega_s$  into the reference domain  $\Omega$  [14] and using the mapping (16) – (17) we can express the cost functional (22) defined on domain  $\Omega_s$  in the form of the functional  $J_\phi(u^s)$  defined on domain  $\Omega$ , determined by:

$$\begin{aligned}
(45) \quad J_\phi(u^s) &= \int_Q (D\mathbf{T}_s u^s)_{tt} D\mathbf{T}_s \phi_t^s \det D\mathbf{T}_s dx d\tau \\
&+ \int_Q [\sigma_{ij} D\mathbf{T}_s e_{kl}(D\mathbf{T}_s \phi_t^s) - f^s D\mathbf{T}_s \phi] \det D\mathbf{T}_s dx \\
&- \int_{\gamma_1} p^s D\mathbf{T}_s \phi \|\det D\mathbf{T}_s^* D\mathbf{T}_s^{-1} n\| ds \\
&- \int_{\gamma_2} \lambda_{sT} D\mathbf{T}_s \phi_T \|\det D\mathbf{T}_s^* D\mathbf{T}_s^{-1} n\| ds,
\end{aligned}$$

where  $u^s = u_s \circ \mathbf{T}_s \in F$ ,  $u = u_0 \in F$  and  $\lambda = \lambda_0 \in \Lambda$ . By (43) we have:

$$(46) \quad dJ_\phi(u; V) = \limsup_{t \rightarrow 0} [J_\phi(u^s) - J_\phi(u)]/s$$

It follows by standard arguments [3] that the pair  $(\sigma_s, u_s) \in Q_s \times K_s$ ,  $s \in [0, \vartheta]$ ,  $\vartheta > 0$ , satisfying the system (19) – (20) is Lipschitz continuous with respect to the parameters. Passing to the limit with  $s \rightarrow 0$  in (46) as well as taking into account the formula for derivatives of  $D\mathbf{T}_s^{-1}$  and  $\det D\mathbf{T}_s$  with respect to the parameter  $s$  [14] and (26) we obtain (44). ■

In order to eliminate the shape derivative  $(u', \lambda')$  from (44) we introduce an adjoint state  $(r, q) \in K_2 \times L_2$  defined as follows:

$$(47) \quad \int_Q r_{tt} \zeta dx d\tau + \int_Q \sigma_{ij}(\zeta) e_{kl}(\phi + r) dx d\tau + \int_{\gamma_2} \zeta_{tT} (q - \lambda) \zeta dx d\tau = 0 \quad \forall \zeta \in K_2$$

with

$$r(\mathcal{T}, x) = 0, \quad r_t(\mathcal{T}, x) = 0,$$

$$(48) \quad \int_{\gamma_2} (r_{tT} + \phi_{tT} - u_{tT}) \delta dx d\tau = 0, \quad \forall \delta \in L_2,$$

$$(49) \quad K_2 = \{\zeta \in K_1 : \zeta n = 0 \text{ on } A_0\},$$

$$(50) \quad L_2 = \{\delta \in \Lambda : \delta = 0 \text{ on } A_0 \cap B_0\}.$$

Since  $\phi \in M$  is a given element, then by means of the same arguments as used to show the existence of solution  $(u, \lambda) \in K \times L$  to the system (19) – (20) we can show the existence of the solution  $(r, q) \in K_2 \times L_2$  to the system (47), (48). From (44), (38), (38), (47), (48) we obtain:

$$(51) \quad dJ_\phi(u; V) = I_1(u, \phi + r) + I_2(\lambda, u, \phi + r) + I_3(u, q - \lambda).$$

The necessary optimality condition has a standard form:

**Theorem 51.** *There exists a Lagrange multiplier  $\mu \in R$  such that for all vector fields  $V$  determined by (16), (17) the following condition holds:*

$$(52) \quad dJ_\phi(u; V) + \mu \int_\Gamma V(0) n ds \geq 0,$$

where  $dJ_\phi(u; V)$  is given by (51).

**Proof** is given in [3, 4, 5, 14, 15]. ■

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