Discussiones Mathematicae
Differential Inclusions, Control and Optimization 20 (2000) 41–50

AN OPTIMAL SHAPE DESIGN PROBLEM FOR A HYPERBOLIC HEMIVARIATIONAL INEQUALITY

Leszek Gasiński

Jagiellonian University, Institute of Computer Science
ul. Nawojki 11, 30–072 Cracow, Poland

e-mail: gasinski@ii.uj.edu.pl

Abstract

In this paper we consider hemivariational inequalities of hyperbolic type. The existence result for hemivariational inequality is given and the existence theorem for the optimal shape design problem is shown.

Keywords and phrases: optimal shape design, mapping method, hemivariational inequalities, Clarke subdifferential.

1991 Mathematics Subject Classification: 49J24.

1 Introduction

Hemivariational inequalities were introduced in the 80's by P.D. Panagiotopoulos as a natural description of physical problems governed by nonmonotone and possibly multivalued laws (see Panagiotopoulos [14, 15], Moreau, Panagiotopoulos and Strang [11]). The mathematical models for such problems deal with potentials given by nonconvex, possibly nondifferentiable functions. In [16], Panagiotopoulos introduced the notion of a nonconvex superpotential, being a generalization of the convex superpotential introduced by Moreau [10]. This generalization led to a new type of variational inequalities, called hemivariational inequalities, which cover boundary value problems for PDEs with nonmonotone, nonconvex and possibly multivalued laws.

The aim of this paper is to present an existence result for an optimal shape design problem for a system described by a hemivariational inequality of hyperbolic type. Such a problem may be formulated as a control problem, in which a hyperbolic hemivariational inequality appears as a state equation and the role of controls is played by sets from a family of admissible shapes. The cost functional to be minimized is of general (not necessary integral) form.

The proof of the existence of an optimal shape is based on the direct method of the calculus of variations. We use the mapping method introduced by Micheletti [9] (see also Murat and Simon [12] or Sokołowski and Zolesio [18]), which provides both a class of admissible shapes and a topology in this class of domains. The admissible shapes are obtained as the images of a fixed open bounded subset of $\mathbb{R}^N$ through regular bijections in $\mathbb{R}^N$. The boundary of these open sets should be regular (as the used method is valid in such case), but it does not have to be connected (see Section 2 for details).

The plan of the paper is as follows. In Section 2, we recall the notation and properties of the Clarke subdifferential and the mapping method. In Section 3, we formulate a hyperbolic hemivariational inequality as well as an optimal shape design problem described by this inequality. In Section 4, we proof an existence result for an optimal shape design problem.

2 Preliminaries

First of all we recall the notion of the Clarke subdifferential as well as some its properties.

Let Y be a Banach space and Y' its topological dual. By $\langle \cdot, \cdot \rangle_{Y' \times Y}$ we denote the duality brackets between Y' and Y. For a locally Lipschitz function $f : Y \longmapsto \mathbb{R}$, every $x \in Y$ and $h \in Y$, we define the Clarke directional derivative of f at x in the direction h by

$$f^0(x;h) \stackrel{df}{=} \limsup_{\substack{y \to x \text{ in } Y \\ t \searrow 0 \text{ in } \mathbb{R}}} \frac{f(y+th) - f(y)}{t}.$$

It is easy to check that the function $Y \ni h \longmapsto f^0(x;h) \in \mathbb{R}$ is sublinear and continuous (in fact $|f^0(x;h)| \leq k_x \|h\|_Y$ and hence $f^0(x;\cdot)$ is Lipschitz). So by the Hahn-Banach theorem $f^0(x;\cdot)$ is the support function of a nonempty, convex and w^*-compact set $\partial f(x)$ defined by

$$\partial f(x) \stackrel{df}{=} \{x^* \in Y' : \ f^0(x;h) \geq \langle x^*, h \rangle_{Y' \times Y} \text{ for all } h \in Y\},$$

(see Clarke [4], Proposition 2.1.2, p. 27). The set $\partial f(x)$ is called the Clarke subdifferential of f at x. For every $x \in Y$ there exists $k_x > 0$ such that for every $x^* \in \partial f(x)$ we have $\|x^*\|_{Y'} \leq k_x$. Also, if $f, g : Y \longmapsto \mathbb{R}$ are locally Lipschitz functions, then $\partial(f + g)(x) \subseteq \partial f(x) + \partial g(x)$ and $\partial(\alpha f)(x) = \alpha \partial f(x)$ for all $\alpha \in \mathbb{R}$. Moreover, if $f : Y \longmapsto \mathbb{R}$ is convex (so locally Lipschitz as well), then the Clarke subdifferential defined above and subdifferential in the sense of convex analysis coincide and if f is strictly differentiable at x, then $\partial f(x) = \{f'(x)\}$.

For a given $\beta \in L^\infty_{loc}(\mathbb{R})$ by $\widehat{\beta}: \mathbb{R} \longmapsto 2^{\mathbb{R}}$ we denote a multifunction obtained from β by "filling in the gaps" at its discontinuity points, i.e.

$$\widehat{\beta}(\xi) = [\underline{\beta}(\xi), \overline{\beta}(\xi)],$$

where

$$\underline{\beta}(\xi) = \lim_{\delta \to 0^+} \operatorname*{ess\,inf}_{|\zeta - \xi| \leq \delta} \beta(\zeta), \quad \overline{\beta}(\xi) = \lim_{\delta \to 0^+} \operatorname*{ess\,sup}_{|\zeta - \xi| \leq \delta} \beta(\zeta)$$

and $[\cdot, \cdot]$ denotes the interval. It is well known (cf. Chang [3]) that a locally Lipschitz function $j: \mathbb{R} \longmapsto \mathbb{R}$ can be determined up to an additive constant by the relation

$$j(\xi) = \int_0^\xi \beta(\zeta)\, d\zeta$$

and that $\partial j(\xi) \subset \widehat{\beta}(\xi)$. Moreover, if for every $\xi \in \mathbb{R}$ the limits $\beta(\xi \pm 0)$ exist, then $\partial j(\xi) = \widehat{\beta}(\xi)$.

Next let us recall the notion and basic properties of the mapping method (cf. Micheletti [9], Murat and Simon [12], Sokołowski and Zolesio [18]), which will play the crucial role in the formulating of our optimal shape design problem. Roughly speaking, this method consists in finding the optimal shapes in a class of admissible domains obtained as images of a fixed set. An appropriate topology in the class will allow us to obtain an existence result for the optimal shape design problem.

Let C be a bounded open subset of $\mathbb{R}^N$ with a boundary ∂C of class $W^{i,\infty}$, $i \geq 1$ and such that $\mathrm{int}\overline{C} = C$. Then, following Murat and Simon [12], we introduce, for $k \geq 1$, the following spaces

$$W^{k,\infty}(\mathbb{R}^N; \mathbb{R}^N) \overset{df}{=} \left\{ \varphi \,|\, D^\alpha \varphi \in L^\infty(\mathbb{R}^N; \mathbb{R}^N) \text{ for all } \alpha, 0 \leq |\alpha| \leq k \right\},$$

where derivatives $D^\alpha \varphi$ are understood in the distributional sense. By $\mathcal{O}^{k,\infty}$ we will denote the space of bounded open subsets of $\mathbb{R}^N$, which are isomorphic with C, i.e.

$$\mathcal{O}^{k,\infty} \overset{df}{=} \{\Omega \mid \Omega = T(C),\ T \in \mathcal{F}^{k,\infty}\},$$

where $\mathcal{F}^{k,\infty}$ is the space of regular bijections in $\mathbb{R}^N$, defined by

$$\mathcal{F}^{k,\infty} \stackrel{df}{=} \{T\colon \mathbb{R}^N \longmapsto \mathbb{R}^N \mid T \text{ is bijective and } T, T^{-1} \in \mathcal{V}^{k,\infty}\},$$

where

$$\mathcal{V}^{k,\infty} \stackrel{df}{=} \{T\colon \mathbb{R}^N \longmapsto \mathbb{R}^N \mid T - I \in W^{k,\infty}(\mathbb{R}^N;\mathbb{R}^N)\}.$$

In other words $\mathcal{F}^{k,\infty}$ represents the set of essentially bounded perturbations (with essentially bounded derivatives) of identity in $\mathbb{R}^N$. It can be seen that if C has a $W^{i,\infty}$ boundary, then every set $\Omega \in \mathcal{O}^{k,\infty}$ also has the boundary of class $W^{i,\infty}$. Endowing the space $W^{k,\infty}(\mathbb{R}^N;\mathbb{R}^N)$ with the norm

$$\|\varphi\|_{k,\infty} \stackrel{df}{=} \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} \left(\sum_{0\le|\alpha|\le k} |D^\alpha \varphi|^2_{\mathbb{R}^N} \right)^{\frac{1}{2}},$$

we define on $\mathcal{O}^{k,\infty} \times \mathcal{O}^{k,\infty}$ a function

$$\delta_{k,\infty}(\Omega_1,\Omega_2) \stackrel{df}{=} \inf_{\substack{T \in \mathcal{F}^{k,\infty},\\ T(\Omega_1)=\Omega_2}} \Big(\|T-I\|_{k,\infty} + \|T^{-1}-I\|_{k,\infty} \Big).$$

Function $\delta_{k,\infty}$ is a pseudo-distance on $\mathcal{O}^{k,\infty}$ since it does not satisfy the triangle inequality (see Murat and Simon [12], Section 2.4) but it can be easily modified into a distance function. Namely, there exists a positive constant μ_k such that function $d_{k,\infty} = \sqrt{\min(\delta_{k,\infty},\mu_k)}$ is a metric on $\mathcal{O}^{k,\infty}$. Moreover the space $\Big(\mathcal{O}^{k,\infty}, d_{k,\infty}\Big)$ is a complete metric space. If $k \ge 2$, then the injection from $\mathcal{O}^{k,\infty}$ into $\mathcal{O}^{k-1,\infty}$ is compact. More precisely, if $\mathcal{B}$ is a bounded (in $\delta_{k,\infty}$) and closed subset of $\mathcal{O}^{k,\infty}$, then for any sequence $\{\Omega_n\}_{n\ge1} \subset \mathcal{B}$, there exist a subsequence $\{\Omega_{n\nu}\}_{\nu\ge1}$ of $\{\Omega_n\}_{n\ge1}$ and a set $\Omega \in \mathcal{B}$ such that $\Omega_{n\nu} \longrightarrow \Omega$ in $\mathcal{O}^{k-1,\infty}$ (see Murat and Simon [12], Proposition 2.3, Theorem 2.2 and Theorem 2.4).

It is also known that $\Omega_n \longrightarrow \Omega$ in $\mathcal{O}^{k,\infty}$ iff there exist T_n and T in $\mathcal{F}^{k,\infty}$ such that $T_n(C) = \Omega_n$, $T(C) = \Omega$ and $T_n - T \longrightarrow 0$, $T_n^{-1} - T^{-1} \longrightarrow 0$ in $W^{k,\infty}(\mathbb{R}^N;\mathbb{R}^N)$. Some other facts on the mapping method, are summarized in the following lemmas.

Lemma 1. *Let $k \ge 1$. Then*

(a) *If $T \in \mathcal{F}^{1,\infty}$, $\Omega = T(C)$, then $u \in L^2(\Omega)$ iff $u \circ T \in L^2(C)$; $u \in H^1(\Omega)$ iff $u \circ T \in H^1(C)$. Moreover, if $u_n \longrightarrow u$ in $H^1(\Omega)$ (or in $H^1(C)$) and $T \in \mathcal{F}^{k,\infty}$, then $u_n \circ T \longrightarrow u \circ T$ in $H^1(C)$ (or $u_n \circ T^{-1} \longrightarrow u \circ T^{-1}$ in $H^1(\Omega)$).*

(b) *Let* $u \in H^l(\mathbb{R}^N)$ *with* $l = 0$ *or* 1. *Then the mapping* $T \longmapsto u \circ T$ *is continuous from* $\mathcal{V}^{k,\infty}$ *to* $H^l(\mathbb{R}^N)$ *at every point* $T \in \mathcal{F}^{k,\infty}$.

(c) *The following mappings are continuous*

$$T \longmapsto J_T^{-1} \ \text{ from } \ \mathcal{V}^{k,\infty} \ \text{ to } \ W^{k-1,\infty}(\mathbb{R}^N;\mathbb{R}^{N^2}),$$
$$T \longmapsto \det J_T \ \text{ from } \ \mathcal{V}^{k,\infty} \ \text{ to } \ W^{k-1,\infty}(\mathbb{R}^N;\mathbb{R})$$

at every point $T \in \mathcal{F}^{k,\infty}$ *(*J_T *denotes here the standard Jacobian matrix of* T*).*

Lemma 2. *Let* $\{\Omega_n\}_{n\geq 1}$ *be a sequence of sets from* $\mathcal{O}^{k,\infty}$, *let* $T_n \in \mathcal{F}^{k,\infty}$ *be such that* $T_n(C) = \Omega_n$ *and* $u_n \in W(0,I;H^1(\Omega_n))$. *If* $\{\|u_n\|_{W(0,I;H^1(\Omega_n))}\}_{n\geq 1}$ *is a bounded bounded and sequences* $\{J_{T_n}\}_{n\geq 1}$, $\{J_{T_n}^{-1}\}_{n\geq 1}$ *are bounded in* $W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$, *then sequence* $\{\|\widehat{u}_n\|_{W(0,I;H^1(C))}\}_{n\geq 1}$ *is also bounded, where* $\widehat{u}_n(t,X) \stackrel{df}{=} u_n(t,T(X))$ *for a.e.* $(t,X) \in (0,I) \times C$.

Lemma 3. *If* $f, f_n \in L^2(\mathbb{R}^{N+1})$ *and* $f_n(t,x) \to f(t,x)$ *strongly in* $L^2(\mathbb{R}^{N+1})$, *and* $T_n - T \to 0$, $T_n^{-1} - T^{-1} \to 0$ *in* $W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$, *then* $f_n(t,T_n(X)) \to f(t,T(X))$ *strongly in* $L^2(\mathbb{R}^{N+1})$.

For the proofs of the above lemmas we refer to Murat and Simon [12], Lemmas 4.1, 4.4(i), 4.3 and 4.2 and to Liu and Rubio [8], Section 2.

It is interesting to observe some relationships between the convergence in $\mathcal{O}^{k,\infty}$ and other types of convergence of sets.

Let D be an open subset of $\mathbb{R}^N$. If $\Omega_n \longrightarrow \Omega_0$ in $\mathcal{O}^{k,\infty}$, then $1_{\Omega_n} \longrightarrow 1_{\Omega_0}$ in $L^2(\mathbb{R}^N)$, where by 1_D we denote the characteristic function of an open subset $D \subseteq \mathbb{R}^N$.

Let us denote by H^c, the Hausdorff complementary topology (see e.g. Pironnneau [17], Section 3.2.1). Then, if $\Omega_n \longrightarrow \Omega_0$ in $\mathcal{O}^{k,\infty}$ and $\operatorname{int}\overline{C} = C$, then $\Omega_n \stackrel{H^c}{\longrightarrow} \Omega_0$. H^c-convergence has an important property of "covering" of the compacts, namely, if $\Omega_n \stackrel{H^c}{\longrightarrow} \Omega_0$, then $\forall G \subset\subset \Omega_0\ \exists n_G \in \mathbb{N}\ \forall n \geq n_G :$ $G \subseteq \Omega_n$.

In the sequel we will use the following spaces:

$$\begin{aligned}
H &= H(\Omega) = L^2(\Omega),\\
V &= V(\Omega) = H^1(\Omega) = \{v:\ v \in L^2(\Omega),\ D^\alpha v \in L^2(\Omega) \text{ for } 0 \leq |\alpha| \leq 1\},\\
\mathcal{H} &= \mathcal{H}(\Omega) = L^2(0,I;H(\Omega)),\\
\mathcal{V} &= \mathcal{V}(\Omega) = L^2(0,I;V(\Omega)),\\
\mathcal{W} &= \mathcal{W}(\Omega) = \mathcal{W}(\Omega) = \{v:\ v \in \mathcal{V}(\Omega),\ v' \in \mathcal{V}'(\Omega))\}.
\end{aligned}$$

3 Formulation of the problem

We consider the following hyperbolic hemivariational inequality

$$(HVI) \qquad \begin{cases} u \in C(0,I;V), \text{ such that } u' \in \mathcal{W} \\ \langle u''(t), v\rangle_{V'\times V} + a(u'(t),v) + b(u(t),v) + (\chi(t),v)_H \\ \qquad\qquad = \langle f(t), v\rangle_{V'\times V}, \quad \forall v \in V, \quad \text{for a.e. } t \in (0,I), \\ u(0) = \psi_0, \quad u'(0) = \psi_1 \quad \text{in } \Omega, \\ \chi(t,x) \in \partial j(u(t,x)) \quad \text{for a.e.} \quad (t,x) \in (0,I)\times\Omega \\ \chi \in \mathcal{H}, \end{cases}$$

where $a, b : V \times V \longmapsto \mathbb{R}$ are two functionals, $j : \mathbb{R} \longmapsto \mathbb{R}$ is a function and $f \in \mathcal{H}(\mathbb{R}^N)$. If by $S(\Omega)$ we denote the set of solutions for (HVI), then optimal shape design probem consists in solving the following control problem:

$$(OSDP) \qquad \begin{cases} \text{Find } \Omega^* \in \mathcal{B} \text{ and } u^* \in S(\Omega^*) \text{ such that} \\ J(\Omega^*, u^*) = \min_{\Omega\in\mathcal{B}} \min_{u\in S(\Omega)} J(\Omega, u) \end{cases}$$

in which controls are the sets Ω changing in the family $\mathcal{B} \subseteq \mathcal{O}^{k,\infty}$ and J is a cost functional depending on sets Ω and on solutions u of (HVI) on sets Ω.

First of all we need to guarantee the existence of solutions for (HVI). In this purpose we state the following hypotheses on operators a, b, function j, right hand side f and functions ψ_0 and ψ_1 from the initial conditions:

$\underline{H(a)}$ The form $a : V \times V \longmapsto \mathbb{R}$ is defined by

$$a(u,v) = \int_\Omega [(A\nabla u, \nabla v) + a_0 uv] \ dx,$$

where

(i) the matrix $A \in [C(\mathbb{R}^N)]^{N^2} \cap [L^\infty(\mathbb{R}^N)]^{N^2}$ is coercive,

(ii) $a_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and there exists $\tilde{a} > 0$, such that $a_0(x) \geq \tilde{a}$ a.e. in $\mathbb{R}^N$.

$\underline{H(b)}$ The form $b : V \times V \longmapsto \mathbb{R}$ is defined by

$$b(u,v) = \int_\Omega [(B\nabla u, \nabla v) + b_0 uv] \ dx,$$

where

(i) the matrix $B \in [C(\mathbb{R}^N)]^{N^2} \cap [L^\infty(\mathbb{R}^N)]^{N^2}$ is symmetic and non-negative,

(ii) $b_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and there exists $\tilde{b} > 0$, such that $b_0(x) \geq \tilde{b}$ a.e. in $\mathbb{R}^N$.

$\underline{H(j)}$ $j : \mathbb{R} \longmapsto \mathbb{R}$ is a locally Lipschitz function such that

(i) $j(\xi) = \int_0^\xi \beta(s)\, ds$, where $\beta \in L^\infty_{loc}(\mathbb{R})$;

(ii) limits $\beta(\xi \pm 0)$ exist for each $\xi \in \mathbb{R}$,

(iii) there exists $c_0 > 0$ such that $|\beta(\xi)| \leq c_0(1 + |\xi|)$ for all $\xi \in \mathbb{R}$.

$\underline{H(f,\psi)}$ $f \in \mathcal{H}(\mathbb{R}^N)$, $\psi_0 \in V(\mathbb{R}^N)$, $\psi_1 \in H(\mathbb{R}^N)$.

Now we can formulate the existence theorem for (HVI):

Theorem 4. *If hypotheses $H(a)$, $H(b)$, $H(j)$ and $H(f,\psi)$ hold, then (HVI) admits a slution for any $\Omega \in \mathcal{O}^{k,\infty}$, i.e. $S(\Omega) \neq \emptyset$.*

The proof of Theorem 4 can be obtained, using the methods of Bian [1] or applying the existence theorem for more general formulation of (HVI) by Gasiński [7]. The latter exploits the surjectivity result for pseudomonotone operators.

In the next section we will need an apriori estimate on the solutions of (HVI), which in fact is employed also in the proof of Theorem 4.

Lemma 5. *Let assumptions $H(a)$, $H(b)$ $H(j)$ and $H(f,\psi)$ hold. If $u \in S(\Omega)$, then the following estimate holds:*

$$
\begin{aligned}
& \|u\|_{C(0,I;V)} + \|u'\|_{\mathcal{W}} \\
\leq\ & c\,(1 + |\Omega|)\left(1 + \|\psi_0\|_V^2 + \|\psi_1\|_H^2 + \|f\|_{\mathcal{V}'}\right)
\end{aligned}
$$

with constant $c = c(I, \tilde{a}, a_0, A, \tilde{b}, b_0, B, c_0) > 0$ not depending on Ω.

4 Existence result

In this section we will proof the existence theorem for $(OSDP)$. Our assumptions on family $\mathcal{B}$ of admissible shapes and on functional J are the following:

$\underline{H(C,\mathcal{B})}$ C is a bounded open set in $\mathbb{R}^N$ with boundary of class $W^{i,\infty}$, $i \geq 1$ such that $\operatorname{int} \bar{C} = C$ and $\mathcal{B}$ is a bounded closed subset of $\mathcal{O}^{k,\infty}$, with $k \geq 3$ and $1 \leq i \leq k$.

$\underline{H(J)}$ $J : D(J) \stackrel{df}{=} \bigcup_{\Omega\in\mathcal{B}} (\{\Omega\} \times S(\Omega)) \longmapsto \mathbb{R}$ is a functional which is lower semicontinuous with respect to the following convergence in $D(J)$:

$(\Omega_n, u_n) \longrightarrow (\Omega_0, u_0)$ *in* $D(J)$ *iff* $\Omega_n \longrightarrow \Omega_0$ *in* $\mathcal{O}^{k-1,\infty}$ *and* $\underline{u_n} \longrightarrow \underline{u_0}$ *in* $\mathcal{H}(\mathbb{R}^N)$*, where by* $\underline{u}$ *we denote the extension by zero of the function* $u \in \mathcal{V}(\Omega)$*, namely*

$$\underline{u}(t,x) \stackrel{df}{=} \begin{cases} u(t,x), & \text{if } x \in \Omega \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

The assumption of lower semicontinuity of functional J with respect to the above defined convergence is slightly weaker than the lower semicontinuity with respect to the local convergence (compare Gasiński [6], Definition 3, p. 313). In our assumptions we do not need to specify the form of the cost functional J. Nevertheless, in practice, it is usually of integral form, namely

$$J(\Omega, u) = \int_0^I \int_\Omega l(t,x,u)\, dx\, dt.$$

In the proof of our existence theorem the crucial role will play the fact that the map $\mathcal{B} \ni \Omega \longmapsto S(\Omega) \subseteq \mathcal{W}(\Omega)$ has a graph closed in the sense of the following lemma.

Lemma 6. Let hypotheses $H(C,\mathcal{B})$, $H(j)$, $H(a)$, $H(b)$, $H(f,\psi)$ hold. Let $\{\Omega_n\}_{n\geq 1} \subseteq \mathcal{B}$, $\Omega_0 \in \mathcal{B}$, $\{T_n\}_{n\geq 1} \subseteq \mathcal{F}^{k,\infty}$, $T_0 \in \mathcal{F}^{k,\infty}$ be such that $\Omega_n = T_n(C)$ for $n \geq 1$ and $\Omega_0 = T_0(C)$. Let $u_n \in S(\Omega_n)$, $\widehat{u}_n(t,X) \stackrel{df}{=} u_n(t,T_n(X))$, for $n \geq 1$ and $u^* \in \mathcal{W}(C)$. If $\Omega_n \longrightarrow \Omega_0$ in $\mathcal{O}^{k,\infty}$, $\widehat{u}_n \longrightarrow u^*$ weakly in $\mathcal{W}(C)$, then there exists $u_0 \in S(\Omega_0)$ such that $u^*(t,X) = u_0(t,T_0(X))$.

Now we can formulate and prove the existence theorem for $(OSDP)$:

Theorem 7. *If hypotheses* $H(C,\mathcal{B})$*,* $H(J)$*,* $H(j)$*,* $H(a)$*,* $H(b)$*,* $H(f,\psi)$ *hold, then problem* $(OSDP)$ *admits at least one solution.*

Proof. We apply the direct method of the calculus of variations. Let $\{(\Omega_n,u_n)\}_{n\geq 1} \subseteq D(J)$ be a minimizing sequence for $(OSDP)$. As the injection $\mathcal{O}^{k,\infty}$ into $\mathcal{O}^{k-1,\infty}$ is compact (see Section 2) so $\mathcal{B}$ is compact in $\mathcal{O}^{k-1,\infty}$ and we can choose a subsequence of Ω_n (still indexed by n) and a set $\Omega_0 \in \mathcal{B}$ such that $\Omega_n \longrightarrow \Omega_0$ in $\mathcal{O}^{k-1,\infty}$. This means that there exist $\{T_n\}_{n\geq 1} \subseteq \mathcal{F}^{k-1,\infty}$ and $T_0 \in \mathcal{F}^{k-1,\infty}$ such that $\Omega_n = T_n(C)$, $\Omega_0 = T_0(C)$ and $T_n - T_0 \longrightarrow 0$, $T_n^{-1} - T_0^{-1} \longrightarrow 0$ in $W^{k-1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$.

From the relationship between $\mathcal{O}^{k,\infty}$-convergence and the convergence of characteristic functions of open sets (see Section 2), we obtain that $1_{\Omega_n} \longrightarrow 1_{\Omega_0}$ in $H(\mathbb{R}^N)$ which gives, in particular, that the sequence $\{|\Omega_n|\}_{n\geq 1}$ is bounded, so also sequences $\{\|\psi_0\|_{V(\Omega_n)}\}_{n\geq 1}$, $\{\|\psi_1\|_{H(\Omega_n)}\}_{n\geq 1}$ and $\{\|f\|_{\mathcal{H}(\Omega_n)}\}_{n\geq 1}$ are bounded. Since $u_n \in S(\Omega_n)$, so from Lemma 5, we obtain that the sequence $\{\|u_n\|_{\mathcal{W}(\Omega_n)}\}_{n\geq 1}$ is bounded. Putting $\widehat{u}_n(t,X) \overset{df}{=} u_n(t,T_n(X))$ and using Lemma 2, we obtain that the sequence $\{\|\widehat{u}_n\|_{\mathcal{W}(C)}\}_{n\geq 1}$ is bounded. Thus, taking a next subsequence if necessary, we have $\widehat{u}_n \longrightarrow u^*$ weakly in $\mathcal{W}(C)$, with some $u^* \in \mathcal{W}(C)$. From the compactness of the embedding $\mathcal{W}(C) \subset \mathcal{H}(C)$, we get

$$\widehat{u}_n \longrightarrow u^* \quad \text{in } \mathcal{H}(C).$$

From Lemma 6, we have that $u^*(t,X) = u_0(t,T_0(X))$ with some $u_0 \in S(\Omega_0)$. So the pair (Ω_0, u_0) is admissible for $(OSDP)$.

Let $\underline{\widehat{u}_n}$ and $\underline{u^*}$ denote the functions in $\mathcal{H}(\mathbb{R}^N)$ obtained from $\widehat{u}_n$ and u^*, respectively, by extending them by zero outside C. So, we have

$$\underline{\widehat{u}_n} \longrightarrow \underline{u^*} \quad \text{in } \mathcal{H}(\mathbb{R}^N).$$

From Lemma 3, we also have

$$\underline{u_n} \longrightarrow \underline{u_0} \quad \text{in } \mathcal{H}(\mathbb{R}^N),$$

where

$$\underline{u_n}(t,x) = \begin{cases} u_n(t,x), & \text{if } x \in \Omega_n \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \Omega_n \end{cases}$$

and

$$\underline{u_0}(t,x) = \begin{cases} u_0(t,x), & \text{if } x \in \Omega_0 \\ 0, & \text{if} x \in \mathbb{R}^N \setminus \Omega_0. \end{cases}$$

Hence, due to the hypothesis $H(J)$, we conclude that (Ω_0, u_0) solves the problem $(OSDP)$ and the proof of the theorem is complete. ■

References

[1] W. Bian, *Existence Results for Second Order Nonlinear Evolution Inclusions*, Indian J. Pure Appl. Math. **29** (11) (1998), 1177–1193.

[2] F.E. Browder and P. Hess, *Nonlinear Mappings of Monotone Type in Banach Spaces*, J. of Funct. Anal. **11** (1972), 251–294.

[3] K.C. Chang, *Variational methods for nondifferentiable functionals and applications to partial differential equations*, J. Math. Anal. Appl. **80** (1981), 102–129.

[4] F.H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley & Sons, New York 1983.

[5] Z. Denkowski and S. Migórski, *Optimal Shape Design Problems for a Class of Systems Described by Hemivariational Inequality*, J. Global. Opt. **12** (1998), 37–59.

[6] L. Gasiński, *Optimal Shape Design Problems for a Class of Systems Described by Parabolic Hemivariational Inequality*, J. Global. Opt. **12** (1998), 299–317.

[7] L. Gasiński, *Hyperbolic Hemivariational Inequalities*, in preparation.

[8] W.B. Liu and J.E. Rubio, *Optimal Shape Design for Systems Governed by Variational Inequalities, Part* 1: *Existence Theory for the Elliptic Case, Part* 2: *Existence Theory for Evolution Case*, J. Optim. Th. Appl. **69** (1991), 351–371, 373–396.

[9] A.M. Micheletti, *Metrica per famiglie di domini limitati e proprietà generiche degli autovalori*, Annali della Scuola Normale Superiore di Pisa **28** (1972), 683–693.

[10] J.J. Moreau, *Le Notions de Sur-potential et les Liaisons Unilatérales en Élastostatique*, C.R. Acad. Sc. Paris 267A (1968), 954–957.

[11] J.J. Moreau, P.D. Panagiotopoulos and G. Strang, *Topics in Nonsmooth Mechanics*, Birkhäuser, Basel 1988.

[12] F. Murat and J. Simon, *Sur le Controle par un Domaine Geometrique*, Preprint no. 76015, University of Paris **6** (1976), 725–734.

[13] Z. Naniewicz and P.D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, Dekker, New York 1995.

[14] P.D. Panagiotopoulos, *Nonconvex Superpotentials in the Sense of F.H. Clarke and Applications*, Mech. Res. Comm. **8** (1981), 335–340.

[15] P.D. Panagiotopoulos, *Nonconvex problems of semipermeable media and related topics*, Z. Angew. Math. Mech. **65** (1985), 29–36.

[16] P.D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications. Convex and Nonconvex Energy Functions*, Birkhäuser, Basel 1985.

[17] O. Pironneau, *Optimal Shape Design for Elliptic Systems*, Springer-Verlag, New York 1984.

[18] J. Sokołowski and J.P. Zolesio, *Introduction to Shape Optimization. Shape Sensitivity Analysis*, Springer Verlag 1992.

Received 16 November 1999
Revised 15 March 2000