

PARTITION PROBLEMS AND KERNELS OF GRAPHS

IZAK BROERE

Department of Mathematics, Rand Afrikaans University
P.O. Box 524, Auckland Park, 2006 South Africa
email: ib@rau3.rau.ac.za

PÉTER HAJNAL*

Bolyai Intézet, University of Szeged
Aradi Vértanúk tere 1., Szeged, Hungary 6720
email: hajnal@inf.u-szeged.hu

PETER MIHÓK

Mathematical Institute, Slovak Academy of Sciences
Grešákova 6, 040 01 Košice
email: mihok@kosice.upjs.sk

1. INTRODUCTION

The graphs we consider are finite, simple and undirected. The number of vertices in a longest path in a graph G is denoted by $\tau(G)$. For positive integers k_1 and k_2 a graph G is (τ, k_1, k_2) -partitionable if there exists a partition $\{V_1, V_2\}$ of $V(G)$ such that $\tau(G[V_1]) \leq k_1$ and $\tau(G[V_2]) \leq k_2$. If this can be done for every pair of positive integers (k_1, k_2) satisfying $k_1 + k_2 = \tau(G)$, we say that G is τ -partitionable.

Let H_v denote the fact that the graph H is rooted at v . The set $S \subseteq V(G)$ is an H_v -kernel if

- (i) there is no subgraph of $G[S]$ isomorphic to H and
- (ii) for every $x \in V(G) - S$ there is a subgraph of $G[S \cup \{x\}]$ isomorphic to H_v with its root v at x .

*Partially supported by OTKA T016349 and F021271.

Similarly, a graph is H_v -saturated if it has a subset $S \subseteq V(G)$ such that

- (i) H is not a subgraph of $G[S]$ and
- (ii) for every $x \in V(G) - S$ which is adjacent to some vertex of S the graph H is a subgraph of $G[S \cup \{x\}]$ with its root v at x .

A graph G is called *decomposable* if it is the join of two graphs.

2. THE PROBLEMS

We start with a problem which is formulated as a conjecture in [3] and [1] (see also in [2]).

Conjecture 1. *Every graph is τ -partitionable.*

In [1] it is shown amongst others that every decomposable graph is τ -partitionable.

For a given (rooted) graph H_v , the question whether every graph G has an H_v -kernel is discussed in [2], [4] and [5]. It is shown amongst others that

- (a) Every graph has an H_v -kernel if and only if every graph is H_v -saturated.
- (b) Every graph has a P_v -kernel where P_v is a path of order at most six and v is an endvertex of P .
- (c) Every graph has an S_v -kernel where S_v is a star and v is the center of the star or v is an endvertex of the star.

Clearly, if H_v is a vertex transitive graph, then every graph has an H_v -kernel (any maximal set of vertices inducing an H_v -free graph is an H_v -kernel). The fact that there are graphs H_v and G for which G has no H_v -kernel is illustrated in [2] and [4]. The general problem therefore is

Problem. *Describe the rooted graphs H_v for which every graph G has an H_v -kernel.*

Let the path P_v of order n be rooted at an endvertex. If every graph G has a P_v -kernel for every n then Conjecture 1 is true: If $\tau(G) = k_1 + k_2$, let V_1 be a Q_v -kernel where Q_v is a path (rooted at an endvertex) of order $k_1 + 1$ and let $V_2 = V(G) - S$. From (b) we immediately obtain that every graph is (τ, k_1, k_2) -partitionable if $\min\{k_1, k_2\} \leq 5$.

We are inclined to think that the following conjecture is also true for every path P_v rooted at an endvertex v .

Conjecture 2. *Every graph has a P_v -kernel.*

References

- [1] I. Broere, M. Dorfling J. Dunbar and M. Frick, *A path(ological) partition problem* (submitted).
- [2] P. Hajnal, Graph partitions (in Hungarian) (Thesis, supervised by L. Lovász, J.A. University, Szeged, 1984).
- [3] J.M. Laborde, C. Payan and N.H. Xuong, *Independent sets and longest directed paths in digraphs*, in: Graphs and other combinatorial topics (Prague, 1982), (Teubner-Texte Math., 59, 1983), 173–177.
- [4] P. Mihók, Problem 4, p. 86, in: Graphs, Hypergraphs and Matroids (M. Boro-wiecki and Z. Skupień, eds., Zielona Góra 1985).
- [5] J. Vronka, Vertex sets of graphs with prescribed properties (in Slovak) (Thesis, supervised by P. Mihók, P.J. Šafárik University, Košice, 1986).

Received 17 September 1997