

## PARTITION PROBLEMS AND KERNELS OF GRAPHS

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### 1. INTRODUCTION

The graphs we consider are finite, simple and undirected. The number of vertices in a longest path in a graph  $G$  is denoted by  $\tau(G)$ . For positive integers  $k_1$  and  $k_2$  a graph  $G$  is  $(\tau, k_1, k_2)$ -partitionable if there exists a partition  $\{V_1, V_2\}$  of  $V(G)$  such that  $\tau(G[V_1]) \leq k_1$  and  $\tau(G[V_2]) \leq k_2$ . If this can be done for every pair of positive integers  $(k_1, k_2)$  satisfying  $k_1 + k_2 = \tau(G)$ , we say that  $G$  is  $\tau$ -partitionable.

Let  $H_v$  denote the fact that the graph  $H$  is rooted at  $v$ . The set  $S \subseteq V(G)$  is an  $H_v$ -kernel if

- (i) there is no subgraph of  $G[S]$  isomorphic to  $H$  and
- (ii) for every  $x \in V(G) - S$  there is a subgraph of  $G[S \cup \{x\}]$  isomorphic to  $H_v$  with its root  $v$  at  $x$ .

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\*Partially supported by OTKA T016349 and F021271.

Similarly, a graph is  $H_v$ -saturated if it has a subset  $S \subseteq V(G)$  such that

- (i)  $H$  is not a subgraph of  $G[S]$  and
- (ii) for every  $x \in V(G) - S$  which is adjacent to some vertex of  $S$  the graph  $H$  is a subgraph of  $G[S \cup \{x\}]$  with its root  $v$  at  $x$ .

A graph  $G$  is called *decomposable* if it is the join of two graphs.

## 2. THE PROBLEMS

We start with a problem which is formulated as a conjecture in [3] and [1] (see also in [2]).

**Conjecture 1.** *Every graph is  $\tau$ -partitionable.*

In [1] it is shown amongst others that every decomposable graph is  $\tau$ -partitionable.

For a given (rooted) graph  $H_v$ , the question whether every graph  $G$  has an  $H_v$ -kernel is discussed in [2], [4] and [5]. It is shown amongst others that

- (a) Every graph has an  $H_v$ -kernel if and only if every graph is  $H_v$ -saturated.
- (b) Every graph has a  $P_v$ -kernel where  $P_v$  is a path of order at most six and  $v$  is an endvertex of  $P$ .
- (c) Every graph has an  $S_v$ -kernel where  $S_v$  is a star and  $v$  is the center of the star or  $v$  is an endvertex of the star.

Clearly, if  $H_v$  is a vertex transitive graph, then every graph has an  $H_v$ -kernel (any maximal set of vertices inducing an  $H_v$ -free graph is an  $H_v$ -kernel). The fact that there are graphs  $H_v$  and  $G$  for which  $G$  has no  $H_v$ -kernel is illustrated in [2] and [4]. The general problem therefore is

**Problem.** *Describe the rooted graphs  $H_v$  for which every graph  $G$  has an  $H_v$ -kernel.*

Let the path  $P_v$  of order  $n$  be rooted at an endvertex. If every graph  $G$  has a  $P_v$ -kernel for every  $n$  then Conjecture 1 is true: If  $\tau(G) = k_1 + k_2$ , let  $V_1$  be a  $Q_v$ -kernel where  $Q_v$  is a path (rooted at an endvertex) of order  $k_1 + 1$  and let  $V_2 = V(G) - S$ . From (b) we immediately obtain that every graph is  $(\tau, k_1, k_2)$ -partitionable if  $\min\{k_1, k_2\} \leq 5$ .

We are inclined to think that the following conjecture is also true for every path  $P_v$  rooted at an endvertex  $v$ .

**Conjecture 2.** *Every graph has a  $P_v$ -kernel.*

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Received 17 September 1997