PARTITION PROBLEMS AND KERNELS OF GRAPHS

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1. Introduction

The graphs we consider are finite, simple and undirected. The number of vertices in a longest path in a graph \( G \) is denoted by \( \tau(G) \). For positive integers \( k_1 \) and \( k_2 \) a graph \( G \) is \((\tau, k_1, k_2)\)-partitionable if there exists a partition \( \{V_1, V_2\} \) of \( V(G) \) such that \( \tau(G[V_1]) \leq k_1 \) and \( \tau(G[V_2]) \leq k_2 \). If this can be done for every pair of positive integers \((k_1, k_2)\) satisfying \( k_1 + k_2 = \tau(G) \), we say that \( G \) is \( \tau \)-partitionable.

Let \( H_v \) denote the fact that the graph \( H \) is rooted at \( v \). The set \( S \subseteq V(G) \) is an \( H_v \)-kernel if

(i) there is no subgraph of \( G[S] \) isomorphic to \( H \) and
(ii) for every \( x \in V(G) - S \) there is a subgraph of \( G[S \cup \{x\}] \) isomorphic to \( H_v \) with its root \( v \) at \( x \).

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Similarly, a graph is $H_v$-saturated if it has a subset $S \subseteq V(G)$ such that
(i) $H$ is not a subgraph of $G[S]$ and
(ii) for every $x \in V(G) - S$ which is adjacent to some vertex of $S$ the graph $H$ is a subgraph of $G[S \cup \{x\}]$ with its root $v$ at $x$.
A graph $G$ is called decomposable if it is the join of two graphs.

2. The problems

We start with a problem which is formulated as a conjecture in [3] and [1] (see also in [2]).

Conjecture 1. Every graph is $\tau$-partitionable.

In [1] it is shown amongst others that every decomposable graph is $\tau$-partitionable.

For a given (rooted) graph $H_v$, the question whether every graph $G$ has an $H_v$-kernel is discussed in [2], [4] and [5]. It is shown amongst others that
(a) Every graph has an $H_v$-kernel if and only if every graph is $H_v$-saturated.
(b) Every graph has a $P_v$-kernel where $P_v$ is a path of order at most six and $v$ is an endvertex of $P$.
(c) Every graph has an $S_v$-kernel where $S_v$ is a star and $v$ is the center of the star or $v$ is an endvertex of the star.

Clearly, if $H_v$ is a vertex transitive graph, then every graph has an $H_v$-kernel (any maximal set of vertices inducing an $H_v$-free graph is an $H_v$-kernel). The fact that there are graphs $H_v$ and $G$ for which $G$ has no $H_v$-kernel is illustrated in [2] and [4]. The general problem therefore is

Problem. Describe the rooted graphs $H_v$ for which every graph $G$ has an $H_v$-kernel.

Let the path $P_v$ of order $n$ be rooted at an endvertex. If every graph $G$ has a $P_v$-kernel for every $n$ then Conjecture 1 is true: If $\tau(G) = k_1 + k_2$, let $V_1$ be a $Q_v$-kernel where $Q_v$ is a path (rooted at an endvertex) of order $k_1 + 1$ and let $V_2 = V(G) - S$. From (b) we immediately obtain that every graph is $(\tau, k_1, k_2)$-partitionable if $\min\{k_1, k_2\} \leq 5$.

We are inclined to think that the following conjecture is also true for every path $P_v$ rooted at an endvertex $v$.

Conjecture 2. Every graph has a $P_v$-kernel.
References


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