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MINIMAL VERTEX DEGREE SUM OF A 3-PATH IN PLANE MAPS

O.V. Borodin*

Novosibirsk State University
Novosibirsk, 630090, Russia

Abstract

Let w_k be the minimum degree sum of a path on k vertices in a graph. We prove for normal plane maps that: (1) if $w_2 = 6$, then w_3 may be arbitrarily big, (2) if $w_2 > 6$, then either $w_3 \leq 18$ or there is a ≤ 15-vertex adjacent to two 3-vertices, and (3) if $w_2 > 7$, then $w_3 \leq 17$.

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Let $d(v)$ be the degree of a vertex v in a 3-polytope, i.e. in a 3-connected planar graph. Franklin [5] proves that in each simplicial 3-polytope (whose all faces are triangles) of minimal degree 5 there is a path uvw such that $d(u) \leq 6$, $d(v) = 5$, and $d(w) \leq 6$. Both 6's here are best possible. Kotzig [8] proves that each 3-polytope has an edge uv such that $d(u) + d(v) \leq 13$; the bound is best possible. For a graph G having at least one path consisting of k vertices, called hereafter a *k-path*, we denote by $w_k(G)$, or sometimes w_k, the minimal vertex degree sum of a k-path in G. A plane map is *normal* if its every edge and face is incident with at least three edges. If a plane map is not normal, then as seen from $K_{2,n}$, not only w_3, but also w_2 can be arbitrarily big. It is proved in [1] that each normal plane graph of minimal degree 5 has a face uvw such that $d(u) + d(v) + d(w) \leq 17$, which bound is best possible. If a 3-polytope is simplicial and no 4-vertex is adjacent to that of degree ≤ 4, then, as proved in [2], there is a face uvw such that $d(u) + d(v) + d(w) \leq 29$, which bound is also sharp. Jendrol' [6] proves that each 3-polytope has a path uvw such that $\max\{d(u), d(v), d(w)\} \leq 15$ (the bound is precise). Jendrol' [7] further shows that such a path must

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belong to one of ten classes, in which $d(u) + d(v) + d(w)$ varies from 23 to 15. As reported by Enomoto and Ota in [3], Ando, Iwasaki and Kaneko [4] prove $w_3 \leq 21$ for each 3-polytope, which is best possible due to Jendrol's construction [7].

It is natural to describe the classes of normal plane maps in which w_3 is bounded above. Consider the following construction with $w_2 = 6$ and w_3 unbounded: join two vertices by n edges and place two adjacent 3-vertices inside each 2-face. It turns out that not all $3,3$-edges are responsible for the unboundedness of w_3, but only those lying on 3-faces. More specifically, the purpose of this note is to prove the following

Theorem 1. *Each normal plane map without triangles incident with two 3-vertices has*

(i) *either $w_3 \leq 18$ or a vertex of degree ≤ 15 adjacent to two 3-vertices, and*

(ii) *either $w_3 \leq 17$ or $w_2 = 7$.*

Corollary 2. *Each normal plane map with $w_2 > 6$ has $w_3 \leq 21$.*

In particular, Theorem 1 immediately implies that Franklin's bound $w_3 \leq 17$ is valid for all normal plane maps of minimal degree ≥ 4.

Corollary 3. *Each normal plane map without 3-vertices has $w_3 \leq 17$.*

The upper bound in the following statement is also immediate:

Corollary 4. *In each 3-polytope without 3-vertices there is a path uvw such that* $\max\{d(u), d(v), d(w)\} \leq 9$.

To attain the bound in Corollary 4, take the dual of the well-known (3,5,3,5)-Archimedean solid, and join every two 5-vertices lying in a common face by a path consisting of two 4-vertices. (The former 3-vertices now have degree 9, while the former 5-ones become 10-vertices.)

***Proof of Theorem* 1.** Suppose that M' is a counterexample to (i) or (ii) of Theorem 1. In particular, M' has $w_3 > 17$. Let M be a counterexample on the same vertex set with the greatest number of edges.

(A) *M is a triangulation.*

Suppose there is a > 3-face $f = abc\ldots$. Further suppose b is a vertex with the minimal degree among all vertices incident with f. Then $M + ac$ is also a counterexample to the same statement (i) or (ii) as M. First observe that if M has no 4-vertex adjacent to a 3-vertex or a ≤ 15-vertex adjacent to two 3-vertices, then so does $M + ac$. Secondly, suppose $w_3(M + ac) < w_3(M)$. Then in $M + ac$ there is a path zac or acz, say zac, such that $z \neq b$ and $d_M(z)+d_M(a)+1+d_M(c)+1 < w_3(M)$. But since $d_M(b) \leq d_M(c)$, we have $w_3(M) \leq d_M(z)+d_M(a)+d_M(b) < d_M(z)+d_M(a)+1+d_M(c)+1 < w_3(M)$, which is a contradiction.

The next property follows immediately from (A):

(B) *No* 3*-vertex of* M *is adjacent to a* 3*-vertex.*

Throughout the paper, we denote the vertices adjacent to a vertex v in a cyclic order by $v_1, \ldots, v_{d(v)}$.

Euler's formula $|V| - |E| + |F| = 2$ for M may be written as

$$\sum_{v \in V} (d(v) - 6) = -12. \tag{1}$$

Every $\leq$ 5-vertex contributes a negative *charge* $\mu(v) = d(v) - 6$ to (1), while the charges of $\geq$ 6-vertices are non-negative. Using the properties of M as a counterexample to (i) or (ii), we define a local redistribution of μ's, preserving their sum, such that the *new contribution* $\mu'(v)$ is non-negative for all $v \in V$. This will contradict the fact that the sum of the new contributions is, by (1), equal to -12.

Rule 1. Suppose $d(v) = 3$. If each v_i has $d(v_i) \geq 7$, then each of them gives 1 to v. Suppose $4 \leq d(v_1) \leq 6$. Then each of v_2, v_3 gives $\frac{3}{2}$ to v.

Now $\mu'(v)$ is completely determined for $d(v) = 3$, and clearly $\mu'(v) = \mu(v) + 3 = 0$.

Rule 2. Suppose $d(v) = 4$. If v is not adjacent to $<$ 7-vertices, it receives 1/2 from each v_i. Suppose $5 \leq d(v_1) \leq 6$. Then v receives 2/3 from each of v_2, v_3, v_4. If $d(v_1) = 4$, then v_3 gives 4/5 to v, and each of v_2, v_4 gives 3/5. Finally, if $d(v_1) = 3$, then v_3 gives 1, while each of v_2, v_4 gives 1/2.

Clearly, each 4-vertex v has $\mu'(v) = \mu(v) + 2 = 0$.

Rule 3. Suppose $d(v) = 5$. If each of v_i has $d(v_i) \geq 6$, then four of v_i actually have $d(v_i) \geq 7$, and each $\geq$ 7-neighbour gives 1/4 to v. Otherwise, if say $3 \leq d(v_1) \leq 5$, then v receives 1/2 from each of v_3, v_4.

Clearly, $\mu'(v) \geq \mu(v) + 1 \geq 0$ if $d(v) = 5$.

Thus, if v is *minor*, i.e. has $d(v) \leq 5$, then $\mu'(v) \geq 0$. It remains to prove $\mu'(v) \geq 0$ for $d(v) \geq 7$, because 6-vertices do not participate in discharging.

If $7 \leq d(v) \leq 8$, then either v is adjacent to at most one minor vertex, or v is an 8-vertex with all minor neighbours having degree 5. By Rules 1-3, v cannot give >1 to any of its neighbours because $3 + 6 + 8 < w_3$, therefore $\mu'(v) \geq 7 - 6 - 1 = 0$ in the first subcase above. In the second, observe that by Rule 3, v can give a positive charge to a 5-vertex y only if there are triangles vxy and vzy, where $d(x) \geq 6$, $d(z) \geq 6$. We thus have $\mu'(v) \geq 7 - 6 - 3 \times 1/4 > 0$ if $d(v) = 7$, and $\mu'(v) \geq 8 - 6 - 4 \times 1/4 = 0$ if $d(v) = 8$.

Suppose $d(v) = 9$. If v has a 3-neighbour, then no other minor neighbour is possible because $9 + 3 + 5 < w_3$, therefore $\mu'(v) \geq 9 - 6 - 3/2 > 0$. Otherwise, v can do at most four transfers $(9 + 4 + 4 < w_3)$, each of which is not greater than 2/3, which implies $\mu'(v) \geq 9 - 6 - 4 \times 2/3 > 0$.

Suppose $d(v) = 10$. If v has a 3-neighbour, then 4-neighbours are impossible. Since v then can do at most five transfers in total, we have $\mu'(v) \geq 10 - 6 - 3/2 - 4 \times 1/2 > 0$. Assume v has no 3-neighbours. By Rule 2, v gives 4/5 to single 4-neighbours, and 3/5 to each of 4-twins. To estimate the total expenditure of v, undertake the following averaging of transfers from v to its neighbours: If v_i receives 4/5, then neither v_{i-1}, nor v_{i+1}, where indices are taken modulo $d(v)$, gets anything from v directly. We may imagine that v actually gives 1/5 to each of v_{i-1}, v_{i+1}, and only 2/5 to v_i. Similarly, if v_i gets 3/5 from v by Rule 3, we may imagine that v_i actually gets only 2/5, while the remaining 1/5 goes to that of v_{i-1}, v_{i+1} which is not a 4-vertex. Observe that those vertices which did not receive anything from v directly, now receive at most $2 \times 1/5 = 2/5$. Thus, v gives on average $\leq 2/5$ to each neighbour, that is $\mu'(v) \geq 10 - 6 - 10 \times 2/5 = 0$.

Next suppose $d(v) = 11$. Still, at most one 3-neighbour is possible. Besides, no 4-neighbour of v is adjacent to a 3-vertex by (i) and (ii), so that no 4-neighbour of v can receive 1 from v. If v_i receives 3/2 from v, then by Rule 1 we may assume that $d(v_{i+1}) \leq 6$. Observe that v_{i+2} does not receive anything from v, and therefore v may split its donation of 3/2 to v_i among v_{i-1}, v_i, v_{i+1} and v_{i+2} as follows: 3/2=1/5+7/10+2/5+1/5. The argument used for $d(v) = 10$ says that after averaging such a v_i receives < 1 from v, while each of the other ten neighbours receives $\leq 2/5$, i.e. $\mu'(v) \geq 11 - 6 - 1 - 10 \times 2/5 = 0$. If there does exist a 3-neighbour receiving 1, or there is no 3-neighbours at all, then the argument used for $d(v) = 10$ is still valid, because if v_i receives t from v where $2/5 \leq t \leq 4/5$, then v_{i-1} and v_{i+1} receive nothing from v (directly). Hereafter suppose $d(v) \geq 12$.

Case 1. M contradicts (ii).
It follows that if $d(v_i) = 3$, then $d(v_{i+1}) > 4$, whenever $1 \le i \le d(v)$. By Rule 3, if $d(v_i) = 3$, then neither v_{i+1}, nor v_{i-1} receives anything. We employ another averaging of the donations of v to its neighbours. If v_2 receives 3/2, then assume $d(v_3) \le 6$ and split this 3/2 amongst $v_1, \ldots, v_4$ as follows: 1/4+1/2+1/2+1/4, respectively. Otherwise, whenever v_2 receives $> 1/2$ from v, it actually receives ≤ 1. We then direct 1/4 to each of v_1, v_3, and $\le 1/2$ remains for v_2. As a result, each neighbour receives $\le 1/2$ from v, which implies $\mu'(v) \ge d(v) - 6 - d(v)/2 = (d(v) - 12)/2 \ge 0$. This completes the proof of (ii).

Case 2. M contradicts (i).
If $d(v) \le 15$, then v cannot be adjacent to two 3-vertices, and the 2/5-argument given for $d(v) = 11$ is valid. Therefore assume $d(v) \ge 16$. We now employ yet another averaging. Whenever $d(v_i) = 3$ and $d(v_{i+1}) = 4$, we redistribute what they receive from v among $v_{i-1}, \ldots, v_{i+2}$ as follows 3/2+1/2=1/3+2/3+2/3+1/3, respectively.

If $d(v_i) = 3$ and $5 \le d(v_{i+1}) \le 6$, i.e. v_i receives 3/2 by Rule 1, while v_{i+1} does nothing, then both v_{i-1} and v_{i+2} still receive nothing from v directly, and we split 3/2=1/3+1/3+1/2+1/3, respectively. If v_i receives ≤ 1, while each of v_{i-1}, v_{i+1} receives nought, then we instead send 1/3 to each of v_{i-1}, v_{i+1}, so that $\le 1/3$ remains for v_i. Observe that in the last two cases v_i saves for v at least 1/3 with respect to the normal level 2/3 of donations of v to its neighbours.

Clearly, after this averaging every neighbour of v indirectly receives from v at most 2/3. It follows, $\mu'(v) \ge d(v) - 6 - 2d(v)/3 = 2(d(v) - 18) \ge 0$, i.e. we are done with $d(v) \ge 18$.

It remains to prove $\mu'(v) \ge 0$ if $16 \le d(v) \le 17$. Observe that $d(v) - 6 - 2d(v)/3$ is -1/3 and -2/3 for $d(v) = 17$ and $d(v) = 16$, respectively. So, it suffices to find one or, respectively, two vertices receiving $\le 1/3$ after averaging to complete the proof in these two cases.

Consider a partition of cycle $C_v = v_1 \ldots v_{d(v)}$ into segments $R_{i,j} = v_i \ldots v_j$ where $i < j$, called *receivers*, such that neither v_i nor v_j receives by Rule 1-3 anything from v, whereas each v_q does whenever $i < q < j$. Clearly, $j - i \le 3$ because in M there are no three minor vertices in a row. If there is $R_{i,j}$ such that $j - i = 1$, then each of v_i, v_j obviously has $\le 1/3$ after averaging, which implies $\mu'(v) \ge 0$ as mentioned above. Assume each $R_{i,j}$ is either *singular* $(j - i = 2)$ or *double* $(j - i = 3)$. Due to the residues of 17 and 16 modulo 3, there should be at least one singular $R_{i,j}$ if $d(v) = 17$, and at least two if $d(v) = 16$. But in each singular $R_{i,j}$, vertex v_{i+1} receives $\leq 1/3$, i.e. saves $1/3$ for v. This completes the proof of $\mu'(v) \geq 0$ if $16 \leq d(v) \leq 17$.

Thus we have proved $\mu'(v) \geq 0$ for every $v \in V$, which contradicts (1):

$$0 \leq \sum_{v \in V} \mu'(v) = \sum_{v \in V} \mu'(v) = -12.$$

This completes the proof of Theorem 1. ∎

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