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TRI-QUASI IDEALS OF Γ-SEMIRINGS

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Abstract

In this paper, as a further generalization of ideals, we introduce the notion of tri-quasi ideal as a generalization of ideal, left ideal, right ideal, bi-ideal, quasi ideal, interior ideal, bi-interior ideal,weak interior ideal, bi-quasi ideal, tri-ideal,quasi-interior ideal and bi-quasi-interior ideal of Γ-semiring. Some charecterizations of Γ-semiring,regular Γ-semiring and simple Γ-semiring using tri-quasi ideals are given and study the properties of tri-quasi ideals of Γ-semiring.

Keywords: bi-quasi-interior ideal, bi-interior ideal, bi-quasi ideal, bi-ideal, quasi ideal, interior ideal, regular Γ-semiring, tri-quasi simple Γ-semiring.

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1. Introduction

The algebraic structures play a prominent role in mathematics with wide range of applications. Generalization of ideals of algebraic structures and ordered algebraic structure plays a very remarkable role and also necessary for further advance studies and applications of various algebraic structures. Many mathematicians proved important results and charecterized the algebraic structures by using the concept and the properties of generalization of ideals. During 1950–1980, the concepts of bi-ideals, quasi ideals and interior ideals were studied by many mathematicians.Then the author [8–13] introduced and studied weak interior ideals,bi-interior ideals, bi quasi ideals,quasi interior ideals and bi quasi interior ideals as a generalization of bi-ideal, quasi ideal and interior ideal of algebraic structures and charecterized regular algebraic structures as well as simple algebraic structures using these ideals.

In 1995, Rao [6, 7] introduced the notion of Γ-semiring as a generalization of Γ- ring, ternary semiring and semiring. As a generalization of ring, the notion of a Γ-ring was introduced by Nobusawa [14] in 1964. In 1981, Sen [15] introduced the notion of a Γ-semigroup as a generalization of semigroup. The notion of a semiring was introduced by Vandiver [17] in 1934, but semirings had appeared in earlier studies on the theory of ideals of rings. We know that the notion of a one sided ideal of any algebraic structure is a generalization of an ideal. The quasi ideals are generalization of left ideal and right ideal whereas the bi-ideals are generalization of quasi ideals.

In 1952, the concept of bi-ideals was introduced by Good and Hughes [1] for semigroups. The notion of bi-ideals in rings and semigroups were introduced by Lajos and Szasz [4]. The concept of interior ideals was introduced by Lajos [5] for semigroups. Steinfeld [16] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki [3], Henriksen [2] studied ideals for semirings.In this paper, as a further generalization of ideals, we introduce the notion of tri-quasi ideal as a generalization of bi-ideal, quasi ideal, interior ideal,bi-interior ideal, tri-ideal, bi-quasi-interior ideal and bi-quasi ideal of Γ-semiring and study some of the properties of tri-quasi ideals of Γ-semirings.

2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1 [6]**.** Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. Then we call M a Γ-semiring, if there exists a mapping $M \times \Gamma \times M \to M$ (images of (x, α, y) will be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) such that it satisfies the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$

(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$

(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Definition 2.2 [6]**.** A Γ-semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x$ and $0\alpha x = x\alpha 0 = 0$, for all $x \in M, \alpha \in \Gamma$.

Definition 2.3 [6]**.** A Γ-semiring M is said to be commutative Γ-semiring if $x\alpha y = y\alpha x$, for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 2.4 [7]**.** Let M be a Γ-semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 2.5 [7]**.** An element $a \in M$ is said to be invertible if there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 1$.

Definition 2.6 [6]**.** An element a in a Γ-semiring M is said to be idempotent if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$.

Definition 2.7 [6]**.** Let M be a $\Gamma-$semiring. An element $a \in M$, is said to be regular element of M if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

Definition 2.8 [6]**.** Let M be a $\Gamma-$semiring. Every element of M, is a regular element of M, then M is said to be regular Γ-semiring M.

Definition 2.9 [6]**.** Every element of M is an idempotent of M, then M is said to be idempotent Γ-semiring M.

Definition 2.10 [8]**.** A Γ-semiring M is called a division Γ-semiring if for each non-zero element of M has multiplicative inverse.

Definition 2.11 [9, 10, 11, 12, 13]**.** A non-empty subset A of a Γ-semiring M is called.

(i) A bi-interior ideal of M if A is a Γ-subsemiring of M and $M\Gamma A\Gamma M \cap A\Gamma M\Gamma A \subseteq A$.

(ii) A left (right) bi-quasi ideal of M if A is a subsemigroup of $(M, +)$ and $M\Gamma A \cap A\Gamma M\Gamma A \subseteq A$ $(A\Gamma M \cap A\Gamma M\Gamma A \subseteq A)$.

(iii) A left (right) weak-interior ideal of M if A is a Γ-subsemiring of M and $M\Gamma A\Gamma A \subseteq A($ $A\Gamma A\Gamma M \subseteq A)$.

(iv) A left (right) quasi-interior ideal) of M if A is a Γ-subsemiring of M and $M\Gamma A\Gamma M\Gamma A \subseteq A$ $(A\Gamma M\Gamma A\Gamma M \subseteq A)$.

(v) A left (right) tri- ideal of M if A is a Γ-subsemiring of M and $A\Gamma M\Gamma A\Gamma A \subseteq A$ $(A\Gamma A\Gamma M\Gamma A \subseteq A)$.

(vi) A bi-quasi-interior ideal of M if A is a Γ-subsemiring of M and $A\Gamma M\Gamma A\Gamma M \Gamma A \subseteq A$.

3. Tri-quasi ideals of Γ-semirings

In this section, we introduce the notion of tri-quasi ideal as a generalization of bi-ideal, quasi-ideal and interior ideal of Γ-semiring and study the properties of tri-quasi ideal of Γ-semiring.

Definition 3.1. A non-empty subset B of a Γ-semiring M is said to be tri-quasi ideal of M if B is a Γ-subsemiring of M and $B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$.

Every tri-quasi ideal of a Γ-semiring M need not be bi-ideal, quasi-ideal, interior ideal, bi-interior ideal and bi-quasi ideals of M.

Example 3.2. If $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Q \right\}$ and $\Gamma = M$ then M is a Γ-semiring with respect to usual addition of matrices and ternary operation is defined as usual matrix multiplication and $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$. Then A is neither a bi-ideal nor an interior ideal of the Γ-semiring M.

Example 3.3. Let $M = \{0, 1, 2, 3, 4\}$ and $\Gamma = M$ be sets. Define the binary operation as $(x, y) \rightarrow x + y$, tennary operation is defined as $(x, \alpha, y) \rightarrow x + \alpha + y$, and $x+y = x+y$, if $x+y \in M$ and $x+y = 4$, if $x+y \notin M$, for all $x, y \in M, \alpha \in \Gamma$, where $+$ is the usual addition.Then M is a Γ-semiring. A subset $I = \{0, 2, 4\}$ of M is a tri-quasi-interior ideal of M but not bi-ideal, quasi-ideal, interior ideal, bi-interior ideal of the Γ-semiring M.

In the following theorem, we mention some important properties and we omit the proofs since proofs are straight forward.

Theorem 3.4. *Let M be a Γ-semring. Then the following are hold.*

(1) *Every left ideal is a tri-quasi ideal of M.*
(2) *Every right ideal is a tri-quasi ideal of M.*
(3) *Every quasi ideal is a tri-quasi ideal of M.*
(4) *Every ideal is a tri-quasi ideal of M.*
(5) *Intersection of a right ideal and a left ideal of M is a tri-quasi ideal of M.*
(6) *If L is a left ideal and R is a right ideal of M then $B = R\Gamma L$ is a tri-quasi ideal of M.*
(7) *Every bi-ideal of M is a tri-quasi ideal of M.*
(8) *Every interior ideal of M is a tri-quasi ideal of M.*
(9) *Let B be bi-ideal of M and I be interior ideal of M. Then $B \cap I$ is a tri-quasi ideal of M.*
(10) *If B is a bi-interior ideal of M, then B is a tri-quasi ideal of M.*
(11) *If B is a left bi-quasi ideal of M, then B is a tri-quasi ideal of M.*
(12) *If B is a right bi-quasi ideal of M, then B is a tri-quasi ideal of M.*
(13) *If B is a bi-quasi ideal of M, then B is a tri-quasi ideal of M.*
(14) *Let A and C be Γ-subsemirings of M and $B = A\Gamma C$. If A is the left ideal then B is a tri-quasi-interior ideal of M.*

Theorem 3.5. *The intersection of $\{B_\lambda \mid \lambda \in A\}$ tri-quasi ideals of a Γ-semiring M is a tri-quasi-interior ideal of M.*

Proof. Let $B = \bigcap_{\lambda \in A} B_\lambda$. Then B is a Γ-subsemiring of M. Since B_λ is a tri-quasi ideal of M, we have

$$B_\lambda \Gamma B_\lambda \Gamma M \Gamma B_\lambda \Gamma B_\lambda \subseteq B_\lambda, \quad \text{for all } \lambda \in A$$
$$\Rightarrow B\Gamma B \Gamma M \Gamma B \Gamma B \subseteq B.$$

Hence B is a tri-quasi ideal of M. ■

Theorem 3.6. *Let M be a Γ-semiring. B is a tri-quasi ideal of M and $B\Gamma B = B$ if and only if there exist a left ideal L and a right ideal R such that $R\Gamma L \subseteq B \subseteq R \cap L$.*

Proof. Suppose B is a tri-quasi ideal of the Γ-semiring$M$. Then $B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$. Let $R = B\Gamma M$ and $L = M\Gamma B$. Then L and R are left and right ideals of M, respectively.Therefore $R\Gamma L \subseteq B \subseteq R \cap L$. Conversely suppose that there exist L and R are left and right ideals of M respectively such that $R\Gamma L \subseteq B \subseteq R \cap L$. Then $B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq (R \cap L)\Gamma(R \cap L)\Gamma M\Gamma(R \cap L)\Gamma(R \cap L)$
$\subseteq (R)\Gamma R\Gamma M\Gamma L\Gamma(L)$
$\subseteq R\Gamma L \subseteq B.$
Hence B is a tri-quasi ideal of the Γ-semiring M. ■

Theorem 3.7. *Let M be a Γ-semiring.Then B is a tri-quasi ideal of a Γ-semiring M if and only if B is a left ideal of some right ideal of a Γ-semiring M.*

Proof. Let B be a tri-quasi ideal of the Γ-semiring M. Then $B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$. Therefore $B\Gamma B$ is a left ideal of right ideal $B\Gamma B\Gamma M$ of a Γ-semiring$M$.

Conversely suppose that B is a left ideal of some right ideal R of the Γ-semiring$M$. Then $R\Gamma B \subseteq B, R\Gamma M \subseteq R$. Hence $B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B\Gamma M\Gamma B \subseteq R\Gamma M\Gamma B \subseteq R\Gamma B \subseteq B$. Therefore B is a tri-quasi ideal of the Γ-semiring M. ■

Corollary 3.8. *B is a tri-quasi ideal of a Γ-semiring M if and only if B is a right ideal of some left ideal of a Γ-semiring M.*

Theorem 3.9. *Let M be a Γ-semiring. If $M = M\Gamma < a >$, for all $a \in M$ where $< a >$ is the smallest tri-quasi ideal generated by a. Then B is a tri-quasi ideal of M if and only if B is a quasi ideal of M.*

Proof. Let B be a tri-quasi ideal of a Γ-semiring M and $a \in B$. Then

$$B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$$
$$\Rightarrow M\Gamma < a > \subseteq M\Gamma B$$
$$\Rightarrow M \subseteq M\Gamma B \subseteq M$$
$$\Rightarrow M\Gamma B = M$$
$$\Rightarrow B\Gamma M = B\Gamma M\Gamma B \subseteq B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$$
$$\Rightarrow M\Gamma B \cap B\Gamma M \subseteq M\Gamma M \cap B\Gamma M \subseteq B.$$

Therefore B is a quasi ideal of M. Converse is obvious. ■

4. Tri-quasi simple Γ-semiring and Regular Γ-semiring

In this section, we introduce the notion of a tri-quasi simple Γ-semiring and characterize the tri-quasi simple Γ-semiring using tri-quasi ideals of Γ-semiring and study the properties of minimal tri-quasi ideals of Γ-semiring. We also characterize regular Γ-semiring using tri-quasi ideals of Γ-semiring.

Definition 4.1. A Γ-semiring M is a left (right) simple Γ-semiring if M has no proper left (right) ideals of M.

Definition 4.2. A Γ-semiring M is said to be simple Γ-semring if M has no proper ideals of M.

Example 4.3. Let N be the set of all natural numbers, $M = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in N \right\}$. If $M = \Gamma$with respect to usual matrix addition and matrix multiplication then M is a right simple Γ-semiring.

Example 4.4. Let $M = \{0, 1, 2, 3, 4\}$ and $\Gamma = M$ be the sets. Define the binary operation as $(x, y) \to x + y$, tennary operation is defined as $(x, \alpha, y) \to x + \alpha + y$, where $+$ is the usual addition of integers and $x + y = x + y$, if $x + y \in M$ and $x + y = 4$, if $x + y \notin M$, for all $x, y \in M$, where $+$ is the usual addition. Then M is a simple Γ-semiring.

Definition 4.5. A Γ-semiring M is said to be tri-quasi simple Γ-semiring M if M has no tri-quasi ideals other than M itself.

Theorem 4.6. *If M is a division Γ-semiring then M is a tri-quasi simple Γ-semiring.*

Proof. Let B be a proper tri-quasi ideal of the division Γ-semring M and $0 \neq a \in B$. Since M is a division Γ-semiring, there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = 1$. Then there exist $\beta \in \Gamma, x \in M$ such that $a\alpha b\beta x = x = x\beta a\alpha b$. Then $x \in B\Gamma M$. Therefore $M \subseteq B\Gamma M$. We have $B\Gamma M \subseteq M$. Hence $M = B\Gamma M$. Similarly we can prove $M\Gamma B = M$.

$$\begin{aligned} M &= M\Gamma B \\ &= B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B \\ M &\subseteq B \\ \text{Therefore } M &= B. \end{aligned}$$

Hence division Γ-semiring M has no proper tri-quasi-interior ideals. ■

Theorem 4.7. *Let M be a left and a right simple Γ-semiring. Then M is a tri-quasi simple Γ-semiring.*

Proof. Let M be a simple Γ-semring and B be a tri-quasi ideal of M. Then $B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$ and $M\Gamma B$ and $B\Gamma M$ are left and right ideals of M. Since M is a left and right simple Γ-semiring, we have $M\Gamma B = M$. $B\Gamma M = M$. Hence

$$B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$$
$$\Rightarrow B\Gamma M\Gamma B \subseteq B. \Rightarrow M \subseteq B.$$

Hence the theorem. ■

Theorem 4.8. *Let M be a Γ-semiring. M is a tri-quasi simple Γ-semiring if and only if $< a >= M$, for all $a \in M$ and where $< a >$ is the smallest tri-quasi ideal generated by a.*

Proof. Let M be a Γ-semiring. Suppose M is a tri-quasi simple Γ-semiring, $a \in M$ and $B = M\Gamma a$. Then B is a left ideal of M. Therefore, by Theorem[3.4], B is a tri-quasi ideal of M. Therefore $B = M$. Hence $M\Gamma a = M$, for all $a \in M$.

$$M\Gamma a \subseteq < a > \subseteq M$$
$$\Rightarrow M \subseteq < a > \subseteq M.$$
$$\text{Therefore } M =< a > .$$

Suppose $< a >$ is the smallest tri-quasi ideal of M generated by a and $< a >= M$ and A is the tri-quasi ideal and $a \in A$. Then

$$< a > \subseteq A \subseteq M$$
$$\Rightarrow M \subseteq A \subseteq M.$$

Therefore $A = M$. Hence M is a tri-quasi simple Γ-semiring. ■

Theorem 4.9. *Let M be a Γ-semiring. Then M is a tri-quasi simple Γ-semriring if and only if $a\Gamma a\Gamma M\Gamma a\Gamma a = M$, for all $a \in M$.*

Proof. Suppose M is left bi-quasi simple Γ-semiring and $a \in M$. Therefore $a\Gamma a\Gamma M\Gamma a\Gamma a = M$ is a tri-quasi ideal of M. Hence $a\Gamma M\Gamma a\Gamma M\Gamma a = M$, for all $a \in M$. Conversely suppose that $a\Gamma a\Gamma M\Gamma a\Gamma a = M$, for all $a \in M$. Let B be a tri-quasi ideal of the Γ-semiring M and $a \in B$.

$$M = a\Gamma a\Gamma M\Gamma a\Gamma a$$
$$M = a\Gamma a\Gamma M\Gamma a\Gamma a$$
$$\subseteq B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$$
$$\text{Therefore} M = B.$$

Hence M is a tri-quasi simple Γ-semiring. ■

Theorem 4.10. *If B is a minimal tri-quasi ideal of a Γ-semiring M then any two non-zero elements of B generated the same right ideal of M.*

Proof. Let B be a minimal tri-quasi ideal of M and $x \in B$. Then $(x)_R \cap B$ is a tri-quasi ideal of M. Therefore $(x)_R \cap B \subseteq B$. Since B is a minimal tri-quasi ideal of M, we have $(x)_R \cap B = B \Rightarrow B \subseteq (x)_R$. Suppose $y \in B$. Then $y \in (x)_R$, $(y)_R \subseteq (x)_R$. Therefore $(x)_R = (y)_R$. Hence the theorem. ∎

Corollary 4.11. *If B is a minimal tri-quasi ideal of a Γ-semiring M then any two non-zero elements of B generates the same left ideal of M.*

Theorem 4.12. *Let M be a Γ-semiring and B be a tri-quasi ideal of M. Then B is minimal tri-quasi ideal of M if and only if B is a tri-quasi simple Γ-subsemiring.*

Proof. Let B be a minimal tri-quasi ideal of the Γ-semiring M and C be a tri-quasi ideal of B. Then $C\Gamma C\Gamma B\Gamma C\Gamma C \subseteq C$. Therefore $C\Gamma C\Gamma B\Gamma C\Gamma C$ is a tri-quasi ideal of M. Since C is a tri-quasi ideal of B,

$$\begin{aligned} &C\Gamma C\Gamma B\Gamma C\Gamma C = B\\ \Rightarrow &B = C\Gamma C\Gamma B\Gamma C\Gamma C \subseteq C\\ \Rightarrow &B = C. \end{aligned}$$

Conversely suppose that B is a tri-quasi simple Γ-subsemiring of M. Let C be a tri-quasi ideal of M and $C \subseteq B$.

$$\begin{aligned} &C\Gamma C\Gamma B\Gamma C\Gamma C = C\\ \Rightarrow &C\Gamma C\Gamma B\Gamma C\Gamma C \subseteq C\Gamma C\Gamma B\Gamma C\Gamma C \subseteq B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B,\\ \Rightarrow &C \text{ is a tri-quasi ideal of } B\ ,\\ \Rightarrow &B = C, \text{since } B \text{ is a tri-quasi simple } \Gamma\text{-subsemiring.} \end{aligned}$$

Hence B is a minimal tri-quasi ideal of M. ∎

Theorem 4.13. *Let M be a Γ-semiring and $B = R\Gamma L$, where L and R are minimal left and right ideals of M repectively. Then B is a minimal tri-quasi ideal of M.*

Proof. Obviously $B = R\Gamma L$ is a tri-quasi ideal of M. Let A be a tri-quasi ideal of M such that $A \subseteq B$. Then $M\Gamma A\Gamma A$ is a left ideal of M.

$$\begin{aligned} \Rightarrow M\Gamma A\Gamma A &\subseteq M\Gamma B\Gamma B\\ &= M\Gamma R\Gamma L\Gamma R\Gamma L\\ &\subseteq L, \text{ since } L \text{ is a left ideal of } M. \end{aligned}$$

$$\begin{aligned} \text{Similarly, we can prove } & A\Gamma A\Gamma M \subseteq R \\ \text{Therefore } & M\Gamma A\Gamma A = L, \quad A\Gamma A\Gamma M = R \\ \text{Hence } B &= A\Gamma A\Gamma M\Gamma M\Gamma A\Gamma A \\ &\subseteq A\Gamma A\Gamma M\Gamma A\Gamma A. \\ &\subseteq A. \end{aligned}$$

Therefore $A = B$. Hence B is a minimal tri-quasi ideal of M. ■

Theorem 4.14. *Let M be a regular idempotent Γ-semiring. Then B is a tri-quasi ideal of M if and only if $B\Gamma B\Gamma M\Gamma B\Gamma B = B$, for all tri-quasi ideals B of M.*

Proof. Suppose M is a regular Γ-semiring, B is a tri-quasi ideal of M and $x \in B$. Then $B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$ and there exist $y \in M$, $\alpha, \beta, \delta \in \Gamma$, such that $x = x\delta x\alpha y\beta x\delta x \in B\Gamma B\Gamma M\Gamma B\Gamma B$. Therefore $x \in B\Gamma B\Gamma M\Gamma B\Gamma B$. Hence $B\Gamma B\Gamma M\Gamma B\Gamma B = B$.

Conversely suppose that $B\Gamma B\Gamma M\Gamma B\Gamma B = B$, for all tri-quasi ideals B of M. Let $B = R \cap L$, where R is a right ideal and L is a left ideal of M. Then B is a tri-quasi ideal of M. Therefore $(R\cap L)\Gamma M\Gamma(R\cap L)\Gamma M\Gamma(R\cap L) = R\cap L$

$$\begin{aligned} R\cap L &= (R\cap L)\Gamma(R\cap L)M\Gamma M\Gamma(R\cap L)\Gamma(R\cap L) \\ &\subseteq R\Gamma M\Gamma L\Gamma M\Gamma L \\ &\subseteq R\Gamma L \\ &\subseteq R\cap L \text{ (since } R\Gamma L \subseteq L \text{ and } R\Gamma L \subseteq R). \end{aligned}$$

Therefore $R\cap L = R\Gamma L$. Hence M is a regular Γ-semiring. ■

Theorem 4.15. *Let M be a regular commutative Γ-semiring. Then every tri-quasi ideal of M is an ideal of M.*

Proof. Let B be a tri-quasi ideal of M and $C = B\Gamma B\Gamma M\Gamma B\Gamma B$.
Then $C = B\Gamma B\Gamma M\Gamma B\Gamma B = B$
$\Rightarrow B\Gamma M = C\Gamma M \subseteq C\Gamma M\Gamma C$, since M is regular
$\Rightarrow B\Gamma M \subseteq B\Gamma B\Gamma M\Gamma B\Gamma B\Gamma M\Gamma B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$. Hence the theorem. ■

Theorem 4.16. *M is regular Γ-semiring if and only if $A\Gamma B = A\cap B$ for any right ideal A and left ideal B of Γ-semiring M.*

Theorem 4.17. *Let B be Γ-subsemiring of a regular idempotent Γ-semiring M. B can be represented as $B = R\Gamma L$, where R is a right ideal and L is a left ideal of M if and only if B is a tri-quasi ideal of M.*

Proof. Suppose $B = R\Gamma L$, where R is right ideal of M and L is a left ideal of M.

$$\begin{aligned} B\Gamma B\Gamma M\Gamma B\Gamma B &= R\Gamma L\Gamma R\Gamma L\Gamma M\Gamma R\Gamma L\Gamma R\Gamma L \\ &\subseteq R\Gamma L = B. \end{aligned}$$

Hence B is a tri-quasi ideal of the Γ-semiring M. Conversely suppose that B is a tri-quasi ideal of the regular idempotent Γ-semiring M. Then $B\Gamma B\Gamma M\Gamma B\Gamma B = B$. Let $R = B\Gamma M$ and $L = M\Gamma B$. Then $R = B\Gamma M$ is a right ideal of M and $L = M\Gamma B$ is a left ideal of M.

$$\begin{aligned} & B\Gamma M \cap M\Gamma B \subseteq B\Gamma B\Gamma M\Gamma B\Gamma B = B \\ \Rightarrow & B\Gamma M \cap M\Gamma B \subseteq B \\ \Rightarrow & R \cap L \subseteq B. \\ \text{We have } & B \subseteq B\Gamma M = R \text{ and } B \subseteq M\Gamma B = L \\ \Rightarrow & B \subseteq R \cap L \\ \Rightarrow & B = R \cap L = R\Gamma L, \text{ since } M \text{ is a regular } \Gamma\text{-semiring.} \end{aligned}$$

Hence B can be represented as $R\Gamma L$, where R is the right ideal and L is the left ideal of M. Hence the theorem. ■

The following theorem is a necessary and sufficient condition for Γ-semiring M to be regular using tri-quasi ideal.

Theorem 4.18. *M is a regular Γ-semriring if and only if $B \cap I \cap L \subseteq B\Gamma I\Gamma L$, for any tri-quasi ideal B, ideal I and left ideal L of M.*

Proof. Suppose M be a regular Γ-semiring, B, I and L are tri-quasi ideal, ideal and left ideal of M respectively.

Let $a \in B \cap I \cap L$. Then $a \in a\Gamma M\Gamma a$, since M is regular.

$$\begin{aligned} a \in a\Gamma M\Gamma a &\subseteq a\Gamma M\Gamma a\Gamma M\Gamma a \\ &\subseteq B\Gamma I\Gamma L. \end{aligned}$$

Hence $B \cap I \cap L \subseteq B\Gamma I\Gamma L$.

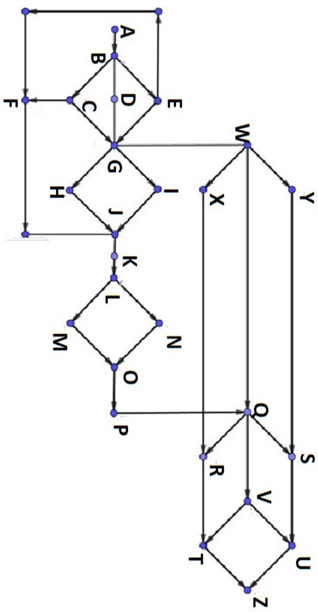
Conversely suppose that $B \cap I \cap L \subseteq B\Gamma I\Gamma L$, for any tri-quasi ideal B, ideal I and left ideal L of M. Let R be a right ideal and L be left ideal of M. Then by assumption, $R \cap L = R \cap M \cap L \subseteq R\Gamma M\Gamma L \subseteq R\Gamma L$. We have $R\Gamma L \subseteq R$, $R\Gamma L \subseteq L$. Therefore $R\Gamma L \subseteq R \cap L$. Hence $R \cap L = R\Gamma L$.

Thus M is a regular Γ-semiring. ■

5. CONCLUSION

As a further generalization of ideals, we introduced the notion of tri-quasi ideal of Γ-semiring as a generalization of ideal ,left ideal, right ideal, bi-ideal, quasi ideal and interior ideal of Γ-semiring and studied some of their properties. We introduced the notion of tri-quasi simple Γ-semiring and characterized the tri-quasi simple Γ-semiring, regular Γ-semiring using tri-quasi ideals of Γ-semiring. We proved every bi-quasi ideal and bi-interior ideal of Γ-semiring are tri-quasi ideals and studied some of the properties of tri-quasi ideals of Γ-semirings. In continuity of this paper, we study prime tri-quasi ideals, maximal and minimal tri-quasi ideals of ordered Γ-semirings.

The following figure helps us for visualising the relations between various ideals.



A	- Algebraic Structure	N	- Right bi quasi Ideal
B	- Ideal	O	- Bi quasi interior ideal
C	- Left Ideal	P	- Generalized bi ideal
D	- Interior Ideal	Q	- Tri ideal
E	- Right Ideal	R	- Left tri ideal
F	- Quasi Ideal	S	- Right tri ideal
G	- Quasi interior Ideal	T	- Generalized left tri ideal
H	- Left quasi interior Ideal	U	- Generalized right tri ideal
I	- Right quasi interior Ideal	V	- Generalized tri ideal
J	- Bi Ideal	W	- Weak interior Ideal
K	- Bi interior Ideal	X	- Left weak interior Ideal
L	- Bi quasi Ideal	Y	- Right weak interior Ideal
M	- Left bi quasi Ideal	Z	- Tri quasi ideal

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