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LEFT ANNIHILATOR OF IDENTITIES WITH GENERALIZED DERIVATIONS IN PRIME AND SEMIPRIME RINGS

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Abstract

Let R be a noncommutative prime ring of char $(R) \neq 2$, F a generalized derivation of R associated to the derivation d of R and I a nonzero ideal of R. Let $S \subseteq R$. The left annihilator of S in R is denoted by $l_R(S)$ and defined by $l_R(S) = \{x \in R \,|\, xS = 0\}$. In the present paper, we study the left annihilator of the sets $\{F(x) \circ_n F(y) - x \circ_n y \,|\, x, y \in I\}$ and $\{F(x) \circ_n F(y) - d(x \circ_n y) \,|\, x, y \in I\}$.

Keywords: prime ring, derivation, generalized derivation, extended centroid, Utumi quotient ring.

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1. Introduction

Throughout this paper, R always denotes an associative ring with center $Z(R)$. For any $a, b \in R$, a ring R is said to be prime, if $aRb = (0)$ implies either $a = 0$ or $b = 0$ and is semiprime if for any $a \in R$, $aRa = (0)$ implies $a = 0$. A mapping f is said to be an additive mapping on R if $f(x + y) = f(x) + f(y)$ holds for all $x, y \in R$. An additive mapping $d : R \to R$ defined by $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$ is called a derivation on R. The map $d(x) = [a, x]$ for all $x \in R$ and for some fixed $a \in R$, is called an inner derivation of R. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ stands for the anti-commutator $xy + yx$. For given $x, y \in R$, we set $x \circ_0 y = x$, $x \circ_1 y = xy + yx$, and inductively $x \circ_n y = (x \circ_{n-1} y) \circ y$ for $n > 1$. Let $S \subseteq R$. Then $r_R(S)$ denotes the right annihilator of S in R, that is, $r_R(S) = \{x \in R | Sx = 0\}$ and $l_R(S)$ denotes the left annihilator of S in R that is, $l_R(S) = \{x \in R | xS = 0\}$. If

$r_R(S) = l_R(S)$, then $r_R(S)$ is called an annihilator ideal of R and is written as $ann_R(S)$.

Ashraf and Rehman [3] proved that if R is a prime ring of char $(R) \neq 2$, I is a nonzero ideal of R and $d \neq 0$ such that $d(x) \circ d(y) = x \circ y$ for all $x, y \in I$, then R is commutative.

In [1], Ali and Huang studied the case for semiprime ring. They proved that if R is a 2-torsion free semiprime ring and I a nonzero ideal of R and $d(I) \neq (0)$ such that $d(x) \circ d(y) = x \circ y$ for all $x, y \in I$, then R has a nonzero central ideal.

There is ongoing interest to investigate the situations replacing derivations with generalized derivations. An additive mapping $F : R \to R$ is said to be a generalized derivation of R, if there exists a derivation d of R such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. In particular, when $d = 0$, then F becomes a left multiplier map on R. Thus a generalized derivation covers both the concept of derivation and left multiplier map.

In [12], Huang proved that if R is a prime ring with char $(R) \neq 2$, L is a square closed Lie ideal of R and F a generalized derivation associated with derivation d of R such that $d(x) \circ F(y) = x \circ y$ for all $x, y \in L$, then either $d = 0$ or $L \subseteq Z(R)$. In [4], Bell and Rehman studied the situation for prime ring R that $F(x) \circ F(y) = x \circ y$ for all $x, y \in R$.

Ashraf *et al.* [2] proved that the prime ring R must be commutative if R admits a generalized derivation F associated with a nonzero derivation d satisfying $d(x) \circ F(y) = x \circ y$ for all $x, y \in I$, where I is a nonzero ideal of R. Then Dhara *et al.* [9] studied the same situation in 2-torsion free semiprime ring and obtained that R has a nonzero central ideal. Recently, in [18], Raza and Rehman studied the cases $F(x) \circ_m F(y) = (x \circ y)^n$ for all $x, y \in I$ and $F(x) \circ_m d(y) = d(x \circ y)^n$ for all $x, y \in I$ in prime and semiprime rings, where I is a nonzero ideal of R, F is a generalized derivation of R with associated derivation d and m, n are fixed positive integers. In the present paper, we investigate the left annihilator condition of the identities, that is $a\{F(x) \circ_n F(y) - (x \circ_n y)\} = 0$ for all $x, y \in I$ and $a\{F(x) \circ_n F(y) - d(x \circ_n y)\} = 0$ for all $x, y \in I$.

Let R be a prime ring with center $Z(R)$ and U is the Utumi quotient ring of R. It is well known that any derivation of R can be uniquely extended to a derivation of U, and so any derivation of R can be defined on the whole of U. Moreover U is a prime ring as well as R and the extended centroid C of R coincides with the center of U. Note that C is a field. We refer to [16] for more details.

We mention a very important result which will be used quite frequently as follows.

Theorem 1.1 (Kharchenko [14])**.** *Let R be a prime ring, d a nonzero derivation on R and I a nonzero ideal of R. If I satisfies the differential identity*

$$f(r_1, r_2, \ldots, r_n, d(r_1), d(r_2), \ldots, d(r_n)) = 0$$

for any $r_1, r_2, \ldots, r_n \in I$*, then either*

(i) *I satisfies the generalized polynomial identity*

$$f(r_1, r_2, \ldots, r_n, x_1, x_2, \ldots, x_n) = 0$$

or

(ii) *d is inner i.e., for some $q \in U$, $d(x) = [q, x]$ for all $x \in R$ and I satisfies the generalized polynomial identity*

$$f(r_1, r_2, \ldots, r_n, [q, r_1], [q, r_2], \ldots, [q, r_n]) = 0.$$

2. Main results

We begin with the theorem.

Theorem 2.1. *Let R be a noncommutative prime ring of char $(R) \neq 2$ with its Utumi ring of quotients U, $C = Z(U)$ the extended centroid of R, F a generalized derivation of R and I a nonzero ideal of R. Suppose that there exists $0 \neq a \in R$ such that $a\big(F(x)^n - x^n\big) = 0$ for all $x \in I$, where $n \geq 1$ is a fixed integer. Then one of the following holds.*

(1) *$n = 1$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$ with $a(b-1) = 0$;*

(2) *$n \geq 2$ and there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$ with $\lambda^n = 1$.*

Proof. In light of [15, Theorem 3], there exist $b \in U$ and derivation d of U such that $F(x) = bx + d(x)$. Since I, R and U satisfy the same generalized polynomial identities (see [8]) as well as the same differential identities (see [16]), without loss of generality, we have $a\big(F(x)^n - x^n\big) = 0$ for all $x \in U$, where d is the derivation of U, that is,

$$a\Big((bx + d(x))^n - x^n\Big) = 0$$

for all $x \in U$.

If $F = 0$, then our hypothesis reduces to $ax^n = 0$ for all $x \in U$. Replacing x with xa yields $(xa)^{n+1} = 0$ for all $x \in U$. Since R is a prime ring, by Levitzki's Lemma [11, Lemma 1.1], $a = 0$, a contradiction.

Now we assume $F \neq 0$. By Kharchenko's theorem, we divide the proof in two cases.

Case 1. Let d be an outer derivation of U. Then by Kharchenko's theorem [14], we have by our assumption that

$$a\Big((bx + u)^n - x^n\Big) = 0$$

for all $x, u \in U$. In particular, for $x = 0$, we have $au^n = 0$ for all $u \in U$. Again, this implies $a = 0$, a contradiction.

Case 2. Let d be inner derivation of U, that is, $d(x) = [c, x]$ for all $x \in R$ and for some $c \in U$. Since $d \neq 0$, $c \notin C$. Thus $a\big((bx + [c, x])^n - x^n\big) = 0$ is a nontrivial generalized polynomial identity (GPI) for U. Denote by E either the algebraic closure of C or C according as C is either infinite or finite, respectively. Then, by a standard argument (see for instance, [16, Proposition]), $a\big((bx + [c, x])^n - x^n\big) = 0$ is also a GPI for $U \otimes_C E$. Since $U \otimes_C E$ is centrally closed prime E-algebra [10, Theorem 2.5 and Theorem 3.5], by replacing R, C with $U \otimes_C E$ and E, respectively, we may assume R is centrally closed and C is either finite or algebraically closed. By Martindale's theorem [17], R is then a primitive ring having nonzero socle H with C as the associated division ring. Hence by Jacobson's theorem [13, p.75] R is isomorphic to a dense ring of linear transformations of some vector space V over C, and H consists of the linear transformations in R of finite rank.

If V is finite dimensional over C, then by the density of R on V, $R \cong M_k(C)$ where $k = \dim_C V$.

Since R is noncommutative, $\dim_C V \geq 2$.

We show that v and cv are linearly C-dependent for any $v \in V$. Suppose that v and cv are linearly independent for some $v \in V$. By the density there exists $x \in R$ such that

$$xv = 0, \quad xcv = v.$$

Then

$$0 = a\Big((bx + [c, x])^n - x^n\Big)v = av.$$

If for some $u \in V$, $\{u, v\}$ is linearly C-dependent, then $au = 0$. Since $a \neq 0$, there exists $w \in V$ such that $aw \neq 0$ and so $\{w, v\}$ is linearly C-independent. Moreover, $a(w + v) = aw \neq 0$ and $a(w - v) = aw \neq 0$. By the above argument, it follows that w and cw are linearly C-dependent, as are $\{w + v, c(w + v)\}$ and $\{w - v, c(w - v)\}$. Therefore there exist $\alpha_w, \alpha_{w+v}, \alpha_{w-v} \in C$ such that

$$cw = \alpha_w w, \quad c(w + v) = \alpha_{w+v}(w + v), \quad c(w - v) = \alpha_{w-v}(w - v).$$

This gives

$$\alpha_w w + cv = \alpha_{w+v} w + \alpha_{w+v} v \tag{1}$$

and

$$\alpha_w w - cv = \alpha_{w-v} w - \alpha_{w-v} v. \tag{2}$$

Now (1) and (2) together yields

$$(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0 \tag{3}$$

and

$$2cv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v. \tag{4}$$

By (3), and since $\{w, v\}$ are C-independent and char $(R) \neq 2$, we have $\alpha_w = \alpha_{w+v} = \alpha_{w-v}$. Thus by (4), we have $cv = \alpha_w v$. This leads to a contradiction with the fact that $\{v, cv\}$ is linear C-independent.

In light of this, we may assume that for any $v \in V$ there exists a suitable $\alpha_v \in C$ such that $cv = \alpha_v v$. By standard argument, there is $\alpha \in C$ such that $cv = \alpha v$ for all $v \in V$. Now let $r \in R$, $v \in V$. Since $cv = \alpha v$,

$$[c, r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0. \tag{5}$$

Thus $[c, r]v = 0$ for all $v \in V$, i.e., $[c, r]V = 0$. Since $[c, r]$ acts faithfully as a linear transformation on the vector space V, $[c, r] = 0$ for all $r \in R$. Therefore, $c \in Z(R)$. Then

$$a\Big((bx)^n - x^n\Big) = 0$$

for all $x \in R$. If $n = 1$, then $a(b - 1)R = (0)$ implying $a(b - 1) = 0$.

So, let $n > 1$. Suppose that v and bv are linearly independent for some $v \in V$. By the density there exists $x \in R$ such that

$$xv = v, \qquad xbv = 0,$$

and hence

$$0 = a\Big((bx)^n - x^n\Big)v = -av.$$

Since $a \neq 0$, by the same argument as earlier, it yields $b \in C$. Then $a\big(b^n x^n - x^n\big) = 0$, i.e., $a(b^n - 1)x^n = 0$ for all $x \in R$. By the same argument as before, we have $a(b^n - 1) = 0$. Since $a \neq 0$, it yields $b^n = 1$. This completes the proof. ∎

Corollary 2.2. *Let R be a noncommutative prime ring of char $R \neq 2$ with its Utumi ring of quotients U, $C = Z(U)$ the extended centroid of R, F a generalized derivation of R and I a nonzero ideal of R. Suppose that there exists $0 \neq a \in R$ such that $a\big(F(x) \circ_n F(y) - x \circ_n y\big) = 0$ for all $x, y \in I$, where $n \geq 0$ is a fixed integer. Then one of the following holds.*

(1) *$n = 0$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$ with $a(b - 1) = 0$;*

(2) *$n \geq 1$ and there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$ with $\lambda^{n+1} = 1$.*

Proof. In particular, for $x = y$, we have $a\big(F(x)^{n+1} - x^{n+1}\big) = 0$ for all $x \in I$. Then by Theorem 2.1, we conclude one of the following.

(1) $n = 0$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$ with $a(b-1) = 0$;

(2) $n \geq 1$ and there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$ with $\lambda^{n+1} = 1$. ■

In particular, for $F = d$, we have the following corollary.

Corollary 2.3. *Let R be a prime ring of char $(R) \neq 2$ and $0 \neq a \in R$. Suppose that d is a nonzero derivation of R and $n \geq 0$ a fixed integer such that $a\big(d(x) \circ_n d(y) - x \circ_n y\big) = 0$ for all $x, y \in I$, then R is commutative.*

Theorem 2.4. *Let R be a noncommutative prime ring of char $(R) \neq 2$ with its Utumi ring of quotients U, $C = Z(U)$ the extended centroid of R, F a generalized derivation of R with associated derivation d of R and I a nonzero ideal of R. Suppose that there exists $0 \neq a \in R$ such that $a\big(F(x)^n - d(x^n)\big) = 0$ for all $x \in I$, where $n \geq 1$ is a fixed integer. Then there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$ with $ab = 0$.*

Proof. In light of [15, Theorem 3], we may assume that there exist $b \in U$ and derivation d of U such that $F(x) = bx + d(x)$. Since I, R and U satisfy the same generalized polynomial identities (see [8]) as well as the same differential identities (see [16]), without loss of generality, we have $a\big(F(x)^n - d(x^n)\big) = 0$ for all $x \in U$, where d is derivation on U, that is

$$a\left\{(bx + d(x))^n - \sum_{i=0}^{n} x^i d(x) x^{n-i-1}\right\} = 0$$

for all $x \in U$.

In light of Kharchenko's theorem, we divide the proof in two cases.

Case 1. If d is not U-inner, then by Kharchenko's theorem [14] we have

$$a\left\{(bx + y)^n - \sum_{i=0}^{n-1} x^i y x^{n-i-1}\right\} = 0$$

for all $x, y \in U$. If $n > 1$, then in particular for $x = 0$, we have $ay^n = 0$ for all $y \in U$. This yields $a = 0$, a contradiction.

On the other hand, if $n = 1$, then $abx = 0$ for all $x \in U$, implying $ab = 0$.

Case 2. We assume the case when d is U-inner derivation, that is for some $c \in U$, $d(x) = [c, x]$ for all $x \in U$. Since $d \neq 0$, $c \notin C$. Hence $a\big((bx + [c, x])^n - [c, x^n]\big) = 0$ is a nontrivial generalized polynomial identity (GPI) for U. Denote by E either the algebraic closure of C or C according as C is either infinite or finite, respectively. Then, by a standard argument (see for instance, [16, Proposition]), $a\big((bx + [c, x])^n - [c, x^n]\big) = 0$ is also a GPI for $U \otimes_C E$. Since $U \otimes_C E$ is centrally closed prime E-algebra [10, Theorem 2.5 and Theorem 3.5], by replacing R, C with $U \otimes_C E$ and E, respectively, we may assume that R is centrally closed and C is either finite or algebraically closed. By Martindale's theorem [17], R is then a primitive ring having nonzero socle H with C as the associated division ring. Hence by Jacobson's theorem [13, p.75] R is isomorphic to a dense ring of linear transformations of some vector space V over C, and H consists of the linear transformations in R of finite rank. If V is finite dimensional over C, then by the density of R on V gives $R \cong M_k(C)$, where $k = \dim_C V$.

Since R is noncommutative, $\dim_C V \geq 2$.

We prove now that for any $v \in V$, v and cv are linearly C-dependent. Suppose on the contrary that v and cv are linearly independent for some $v \in V$. By the density, there exists $x \in R$ such that

$$xv = 0, xcv = v.$$

Then

$$0 = a\Big((bx + [c, x])^n - [c, x^n]\Big)v = av.$$

If for any $u \in V$, $\{u, v\}$ is linearly C-dependent, then $au = 0$. Since $a \neq 0$, there exists $w \in V$ such that $aw \neq 0$ and so $\{w, v\}$ are linearly C-independent. Also $a(w + v) = aw \neq 0$ and $a(w - v) = aw \neq 0$. By the above argument, it follows that w and cw are linearly C-dependent, as are $\{w + v, c(w + v)\}$ and $\{w - v, c(w - v)\}$. Therefore there exist $\alpha_w, \alpha_{w+v}, \alpha_{w-v} \in C$ such that

$$cw = \alpha_w w, \quad c(w + v) = \alpha_{w+v}(w + v), \quad c(w - v) = \alpha_{w-v}(w - v).$$

Thus we have

$$\alpha_w w + cv = \alpha_{w+v} w + \alpha_{w+v} v \tag{6}$$

and

$$\alpha_w w - cv = \alpha_{w-v} w - \alpha_{w-v} v. \tag{7}$$

Now (6) and (7) together yields

$$(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0 \tag{8}$$

and

$$2cv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v. \tag{9}$$

By (8), and since $\{w, v\}$ are C-independent, $2\alpha_w - \alpha_{w+v} - \alpha_{w-v} = 0$ and $\alpha_{w-v} - \alpha_{w+v} = 0$. These relations imply by using char $(R) \neq 2$, that $\alpha_w = \alpha_{w+v} = \alpha_{w-v}$. By (9) it follows $cv = \alpha_w v$. This leads to a contradiction with the fact that $\{v, cv\}$ is linear C-independent.

In light of this, we may assume that for any $v \in V$ there exists a suitable $\alpha_v \in C$ such that $cv = \alpha_v v$, and standard argument shows that there is $\alpha \in C$ such that $cv = \alpha v$ for all $v \in V$. Then by the same argument as in Theorem 2.1, $c \in Z(R)$. Then $a(bx)^n = 0$ for all $x \in R$. Replacing x with xa, we have $(xab)^{n+1} = 0$ for all $x \in R$. By Levitzki's Lemma [11, Lemma 1.1], $ab = 0$. ■

Corollary 2.5. *Let R be a noncommutative prime ring of char $(R) \neq 2$ with its Utumi ring of quotients U, $C = Z(U)$ the extended centroid of R, F a generalized derivation of R with associated derivation d and I a nonzero ideal of R. Suppose that there exists $0 \neq a \in R$ such that $a\big(F(x) \circ_n F(y) - d(x \circ_n y)\big) = 0$ for all $x, y \in I$, where $n \geq 0$ is a fixed integer. Then $d = 0$ and there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$ with $ab = 0$.*

Proof. In particular, for $x = y$, we have $a\big(F(x)^{n+1} - d(x^{n+1})\big) = 0$ for all $x \in I$. Then by Theorem 2.4, we conclude that there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$ with $ab = 0$. ■

In particular, for $F = d$, we have the following corollary.

Corollary 2.6. *Let R be a prime ring of char $(R) \neq 2$ and $0 \neq a \in R$. Suppose that d is a nonzero derivation of R and $n \geq 0$ a fixed integer such that $a\big(d(x) \circ_n d(y) - d(x \circ_n y)\big) = 0$ for all $x, y \in I$. Then R is commutative.*

Example 2.7. Let $\mathbb{Z}$ be the set of all integers. Consider

$$R = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}.$$

Notice that R is not prime ring. We define maps $F, d : R \to R$ by $F \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $d \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then it is easy to verify that F is a generalized derivation associated with a derivation d on R. We choose $0 \neq a = \begin{pmatrix} 0 & a_1 & 0 \\ 0 & 0 & b_1 \\ 0 & 0 & 0 \end{pmatrix} \in R$ such that $a(F(x) \circ_n F(y) - x \circ_n y) = 0$ and $a(F(x) \circ_n F(y) - d(x \circ_n y)) = 0$ for all $x, y \in R$ and for any integer $n \geq 1$. But $d \neq 0$ and so F can not be written as $F(x) = bx$ for all $x \in R$, for some $b \in R$. Thus the primeness hypothesis in Corollary 2.2 and Corollary 2.5 is not superfluous.

3. The results on semiprime rings

In this section we extend Corollary 2.3 and Corollary 2.6 to the semiprime ring. Let R be a semiprime ring and U be its left Utumi ring of quotients. Then $C = Z(U)$ is the extended centroid of R [7, p. 38]. We know the fact.

Fact 1. *Any derivation of a semiprime ring R can be uniquely extended to a derivation of its left Utumi quotient ring U and so any derivation of R can be defined on the whole of U* [16, Lemma 2].

Let $M(C)$ be the set of all maximal ideals of C.

By the standard theory of orthogonal completions for semiprime rings, we have the following lemma.

Lemma 3.1 ([6], Lemma 1 and Theorem 1)**.** *Let R be a 2-torsion free semiprime ring and P a maximal ideal of C. Then PU is a prime ideal of U invariant under all derivations of U. Moreover, $\bigcap\{PU : P \in M(C)$ with U/PU 2-torsion free$\}$ $= 0$.*

Theorem 3.2. *Let R be a noncommutative 2-torsion free semiprime ring, U the left Utumi quotient ring of R and $0 \neq a \in R$. Let d be a nonzero derivation of R such that $a(d(x) \circ_n d(y) - x \circ_n y) = 0$ for all $x, y \in R$. Then R contains a nonzero central ideal.*

Proof. By Fact 1 and since U and R satisfy the same differential identities (see [16]), we have $a(d(x) \circ_n d(y) - x \circ_n y) = 0$ for all $x, y \in U$. Let $P \in M(C)$ be such that U/PU is 2-torsion free. It is clear that U is 2-torsion free semiprime ring. Then PU is a prime ideal of U invariant under d by Lemma 3.1. Denote $\overline{U} = U/PU$ and $\overline{d}$ the derivation induced by d on $\overline{U}$, that is $\overline{d}(\overline{x}) = \overline{d(x)}$ for all $x \in U$. For any $\overline{x}, \overline{y} \in \overline{U}$, we get $\overline{a}(\overline{d}(\overline{x}) \circ_n \overline{d}(\overline{y}) - \overline{x} \circ_n \overline{y}) = 0$. Moreover $\overline{U}$ is a prime ring so by Corollary 2.3, we get either $\overline{d} = 0$ or $[\overline{U}, \overline{U}] = 0$. In any case we have $d(U)[U, U] \subseteq PU$ for all $P \in M(C)$. In view of Lemma 3.1, $\bigcap\{PU : P \in M(C)$ *with* U/PU 2-*torsion free*$\} = 0$. Then $d(U)[U, U] = 0$. In particular we get $d(R)[R, R] = 0$. These imply that $0 = d(R)[R^2, R] = d(R)R[R, R] + d(R)[R, R]R = d(R)R[R, R]$. In particular $d(R)R[R, d(R)] = 0$.

Thus $[d(R), R]R[d(R), R] = 0$. Since R is semiprime, we obtain that $[d(R), R] = 0$. Then by [5, Theorem 3], R contains a nonzero central ideal. ■

Similarly, we have

Theorem 3.3. *Let R be noncommutative 2-torsion free semiprime ring, U the left Utumi quotient ring of R and $0 \neq a \in R$. Let d be a nonzero derivation of R such that $a(d(x) \circ_n d(y) - d(x \circ_n y)) = 0$ for all $x, y \in R$. Then R contains a nonzero central ideal.*

References

[1] S. Ali and H. Shuliang, *On derivations in semiprime rings*, Algebra Represent Theor. **15** (2012) 1023–1033.
doi:10.1007/s10468-011-9271-9

[2] M. Ashraf, A. Ali and R. Rani, *On generalized derivations of prime rings*, Southeast Asian Bull. Math. **29** (2005) 669–675.

[3] M. Ashraf and N. Rehman, *On commutativity of rings with derivations*, Results Math. **42** (2002) 3–8.
doi:10.1007/BF03323547

[4] H.E. Bell and N. Rehman, *Generalized derivations with commutativity and anti-commutativity conditions*, Math. J. Okayama Univ. **49** (2007) 139–147.

[5] H.E. Bell and W.S. Martindale III, *Centralizing mappings of semiprime rings*, Canad. Math. Bull. **30** (1987) 92–101.
doi:10.4153/CMB-1987-014-x

[6] K.I. Beidar, *Rings of quotients of semiprime rings*, Vestnik Moskow. Univ. Ser. I. Mat. Mekh. (Engl. Transl: Moskow Univ. Math. Bull.) **33** (1978) 36–42.

[7] C.L. Chuang, *Hypercentral derivations*, J. Algebra **166** (1994) 34–71.
doi:10.1006/jabr.1994.1140

[8] C.L. Chuang, *GPIs having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc. **103** (1988) 723–728.
doi:10.1090/S0002-9939-1988-0947646-4

[9] B. Dhara, S. Ali and A. Pattanayak, *Identities with generalized derivations in semiprime rings*, Demonstratio Math. **XLVI** (2013) 453–460.
doi:10.1515/dema-2013-0471

[10] T.S. Erickson, W.S. Martindale III and J.M. Osborn, *Prime nonassociative algebras*, Pacific J. Math. **60** (1975) 49–63.
doi:10.2140/pjm.1975.60.49

[11] I.N. Herstein, Topics in Ring Theorey (The University of Chicago Press, Chicago, 1969).

[12] S. Huang, *Generalized derivations of prime rings*, Int. J. Math. & Math. Sci. Volume 2007, Article ID 85612, 6 pages.
doi:10.1155/2007/85612

[13] N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Pub. **37**, Amer. Math. Soc. (Providence, RI, 1964).

[14] V.K. Kharchenko, *Differential identity of prime rings*, Algebra and Logic. **17** (1978) 155–168.
doi:10.1007/BF01669313

[15] T.K. Lee, *Generalized derivations of left faithful rings*, Comm. Algebra **27** (1999) 4057–4073.
doi:10.1080/00927879908826682

[16] T.K. Lee, *Semiprime rings with differential identities*, Bull. Inst. Math. Acad. Sinica **20** (1992) 27–38.

[17] W.S. Martindale III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra **12** (1969) 576–584.
doi:10.1016/0021-8693(69)90029-5

[18] M.A. Raza and N.U. Rehman, *On generalized derivation in rings and Banach Algebras*, Kragujevac J. Math. **41** (2017) 105–120.
doi:10.5937/KgJMath1701105R

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