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ON THE GENUS OF THE IDEMPOTENT GRAPH OF A FINITE COMMUTATIVE RING

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Abstract

Let R be a finite commutative ring with identity. The *idempotent graph* of R is the simple undirected graph $I(R)$ with vertex set, the set of all non-trivial idempotents of R and two distinct vertices x and y are adjacent if and only if $xy = 0$. In this paper, we have determined all isomorphism classes of finite commutative rings with identity whose $I(R)$ has genus one or two. Also we have determined all isomorphism classes of finite commutative rings with identity whose $I(R)$ has crosscap one. Also we study the the book embedding of toroidal idempotent graphs and classify finite commutative rings whose $I(R)$ is a ring graph.

Keywords: idempotent graph, planar, genus, crosscap.

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1. Introduction

The study on linking commutative ring theory with graph theory has been started with the concept of the zero-divisor graph of a commutative ring which was first launched by I. Beck [3]. Recall that the *idempotent graph* of a commutative ring R, is a simple undirected graph $I(R)$ with vertex set, the set of all non-trivial idempotents of R and two distinct vertices x and y are adjacent if and only if $xy = 0$. The concept was first initiated by, Akbari, Habibi, Majidinya and Manaviyat [1]. They obtained some basic results of $I(M_n(R))$, for a division ring R. Influenced by the ideas of the above authors, we try to classify the finite commutative rings with unity whose idempotent graph is planar, ring graph, has genus 1 or 2 and crosscap 1.

2. Preliminaries

In this section, we recollect some definitions and theorems which are required for the subsequent sections.

Let G be a graph with n vertices and q edges. Let C be a cycle of G. We say C is a primitive cycle if it has no chords. Also a graph G has the primitive cycle property (PCP) if any two primitive cycles intersect in at most one edge. The number $frank(G)$ is called the free rank of G and it is the number of primitive cycles of G. Also the number $rank(G) = q + n - r$, is called the cycle rank of G, where r is the number of connected components of G. A graph G is called a *ring graph* if it satisfies one of the following equivalent conditions: 1. $rank(G) = frank(G)$; 2. G satisfies the PCP and G does not contain a subdivision of K_4 as a subgraph. A *split graph* is one whose vertex set can be partitioned as the disjiont union of an independent set and a clique (either of which may be empty).

A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A planar graph, which has all the vertices in the outer face of the embedding, is called an *outerplanar* graph. For non-negative integers g and k, let S_g denote the sphere with g handles and N_k denote the sphere with k crosscaps attached to it. It is well-known that every connected compact surface is homeomorphic to S_g or N_k for some non-negative integers g and k. The *genus* of a graph G, denoted by $g(G)$, is the minimum integer n such that G can be embedded in S_n. Similarly the *crosscap* (nonorientable genus) $\overline{g}(G)$ is the minimum k such that G can be embedded in N_k and G is *toroidal* if $g(G) = 1$. For details on the notion of embedding of graphs in surface, one can refer to White [14] and for graph theory definitions one can refer [4]. Also for ring theory definitions we refer [2].

The following results are useful for further reference in this paper.

Theorem 1 [14, Kuratowski's]**.** *A graph G is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

Theorem 2 [8, Theorem 1]**.** *A graph G is outerplanar if and only if it contains no subgraph homeomorphic to $K_{2,3}$ or K_4.*

Lemma 3 [14, Theorem 4.4.7]**.** $g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$ *if* $m, n \geq 2$. *In particular,* $g(K_{4,4}) = g(K_{3,n}) = 1$ *if* $n = 3, 4, 5, 6$. *Also* $g(K_{5,4}) = g(K_{6,4}) = g(K_{m,3}) = 2$ *if* $m = 7, 8, 9, 10$.

Lemma 4 [14, Theorem 4.4.7]**.** *Let m,n be positive integers. Then we have the following* $\overline{g}(K_{m,n}) = \left\lceil \frac{1}{2}(m-2)(n-2) \right\rceil$ *if* $m, n \geq 2$.

3. THE IDEMPOTENT GRAPH WITH $g(I(R)) \leq 2$

In this section, we characterize all finite commutative rings R with identity whose $I(R)$ has genus at most two. Using the Euler characteristic formula and a technique of deletion and insertion, we are able to successfully exclude some cases of higher genus.

Remark 5 [1]**.** Let R be a finite commutative ring. Then

(i) $I(R)$ is a null graph if and only if R is a field or a local ring.

(ii) $I(R)$ is a complete graph if and only if $I(R)$ is a complete graph of order 2.

In view of Remark 5, throughout this paper we assume that R is a finite commutative nonlocal ring with nonzero identity. Recall that every Artinian (finite) ring R is decomposed into Artinian local rings, i.e., $R = R_1 \times \cdots \times R_n$, where each $(R_i, \mathfrak{m}_i)$ is a local ring.

We are now in a position to classify all finite nonlocal rings such that the idempotent graph is planar. Note that, a_i's notates the non-trivial idempotents of R.

Theorem 6. *Let R be a finite commutative nonlocal ring. Then $I(R)$ planar (outerplanar) if and only if $n \leq 3$, where n is the number of distinct maximal ideals of R.*

Proof. If $n = 2$, then $I(R) \cong K_2$. When $n = 3$, then the proof follows from Figure 1.

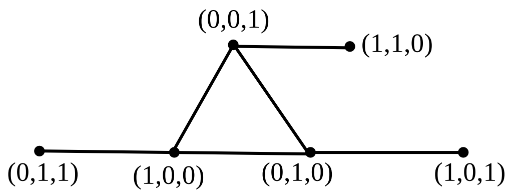

Figure 1. $I(R_1 \times R_2 \times R_3)$.

Conversely, assume that $I(R)$ is planar. Since R is finite, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is a local ring. Suppose $n > 3$. Let $\Omega = \{a_1, a_2, \ldots, a_6\}$ where, $a_1 = (1,0,0,\ldots,0)$, $a_2 = (0,1,0,\ldots,0)$, $a_3 = (1,1,0,\ldots,0)$, $a_4 = (0,0,1,\ldots,0)$, $a_5 = (0,0,0,1,0,\ldots,0)$, $a_6 = (0,0,1,1,0,\ldots,0)$. Then the subgraph induced by Ω in $I(R)$ contains $K_{3,3}$ as a subgraph and by Theorem 1, we get a contradiction. Hence $n \leq 3$. The proof for outerplanarity can be easily obtained, using a similar argument and Theorem 2. ■

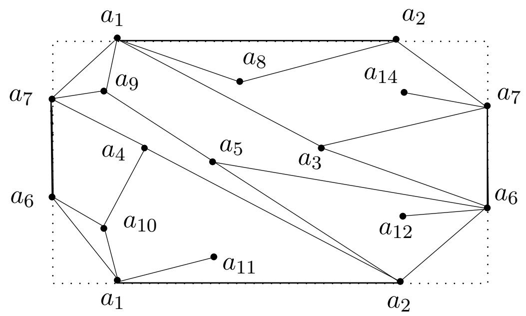

Figure 2. An embedding of $I(R_1 \times R_2 \times R_3 \times R_4)$ in S_1.

Now we characterize all finite commutative nonlocal rings R such that the idempotent graph is toroidal.

Theorem 7. *Let R be a finite commutative nonlocal ring. Then $g(I(R)) = 1$ if and only if $n = 4$, where n is the number of distinct maximal ideals of R.*

Proof. Suppose that $g(I(R)) = 1$. Since R is finite, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$is a local ring. Suppose $n > 4$. Let $B = \{a_1, a_2, \ldots, a_{10}\}$ where, $a_1 = (1,0,0,\ldots,0)$, $a_2 = (0,1,0,\ldots,0)$, $a_3 = (1,1,0,\ldots,0)$, $a_4 = (0,0,1,0,\ldots,0)$, $a_5 = (0,0,0,1,0,\ldots,0)$, $a_6 = (0,0,0,0,1,0,\ldots,0)$, $a_7 = (0,0,1,1,0,\ldots,0)$, $a_8 = (0,0,1,0,1,0,\ldots,0)$, $a_9 = (0,0,0,1,1,0,\ldots,0)$, $a_{10} = (0,0,1,1,1,0,\ldots,0)$. Then the subgraph induced by B in $I(R)$ contains $K_{3,7}$ as a subgraph and by Lemma 3, $g(I(R)) \geq 2$, a contradiction. By Theorem 6, $n = 4$.

Conversely, assume that $n = 4$. Let $C = \{a_1, a_2, \ldots, a_{14}\}$ where, $a_1 = (1,0,0,0)$, $a_2 = (0,1,0,0)$, $a_3 = (1,1,0,0)$, $a_4 = (1,0,1,0)$, $a_5 = (1,0,0,1)$, $a_6 = (0,0,1,0)$, $a_7 = (0,0,0,1)$, $a_8 = (0,0,1,1)$, $a_9 = (0,1,1,0)$, $a_{10} = (0,1,0,1)$, $a_{11} = (0,1,1,1)$, $a_{12} = (1,0,1,1)$, $a_{13} = (1,1,0,1)$, $a_{14} = (1,1,1,0)$. Then the subgraph induced by C in $I(R)$ contains $K_{3,3}$ as a subgraph. By Theorem 1, $g(I(R)) \geq 1$, whereas an embedding of $I(R)$ given in Figure 2 explicitly shows that $g(I(R)) = 1$. ■

Theorem 8. *There is no finite commutative nonlocal ring R for which $g(I(R))$ $= 2$.*

Proof. Since R is finite, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is a local ring. By Theorems 6 and 7, $n \geq 5$. Let $E = \{a_1, a_2, \ldots, a_{30}\}$, where $a_1 = (1,0,0,0,\ldots,0)$, $a_2 = (0,1,0,0,\ldots,0)$, $a_3 = (0,0,1,0,\ldots,0)$, $a_4 = (0,0,0,1,0,\ldots,0)$, $a_5 = (0,0,0,0,1,0,\ldots,0)$, $a_6 = (1,1,0,0,\ldots,0)$, $a_7 = (1,0,1,0,\ldots,0)$, $a_8 = (1,0,0,1,0,\ldots,0)$, $a_9 = (1,0,0,0,1,0,\ldots,0)$, $a_{10} = (0,1,1,0,\ldots,0)$, $a_{11} = (0,1,0,1,0,\ldots,0)$, $a_{12} = (0,1,0,0,1,0,\ldots,0)$, $a_{13} = (0,0,1,1,0,\ldots,0)$, $a_{14} = (0,0,0,1,1,0,\ldots,0)$, $a_{15} = (0,0,1,0,1,0,\ldots,0)$, $a_{16} = (1,1,1,0,\ldots,0)$, $a_{17} = (0,1,1,1,0,\ldots,0)$, $a_{18} = (0,0,1,1,1,0,\ldots,0)$, $a_{19} = (0,1,0,1,1,0,\ldots,0)$, $a_{20} = (0,1,1,0,1,0,\ldots,0)$, $a_{21} = (1,0,1,1,0,\ldots,0)$, $a_{22} = (1,0,0,1,1,0,\ldots,0)$, $a_{23} = (1,0,1,0,1,0,\ldots,0)$, $a_{24} = (1,1,0,0,1,0,\ldots,0)$, $a_{25} = (1,1,0,1,0,\ldots,0)$, $a_{26} = (0,1,1,1,1,0,\ldots,0)$, $a_{27} = (1,0,1,1,1,0,\ldots,0)$, $a_{28} = (1,1,0,1,1,0,\ldots,0)$, $a_{29} = (1,1,1,0,1,0,\ldots,0)$, $a_{30} = (1,1,1,1,0,\ldots,0)$. Then the subgraph induced by E in $I(R)$ contains a subdivision of $K_{3,12}$ as a subgraph and by Lemma 3, $g(I(R)) \geq 3$. ■

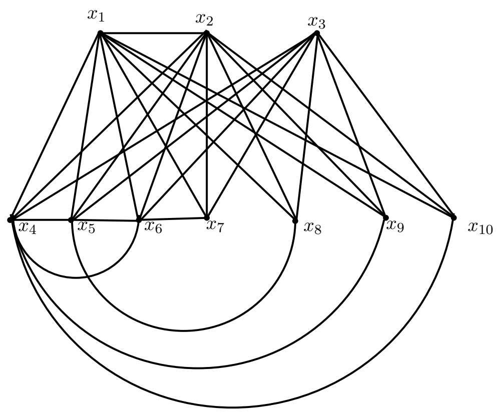

Figure 3. The subgraph induced by B.

We are now in a point to classify all finite nonlocal rings such that the idempotent graph is projective.

Theorem 9. *Let R be a finite commutative ring. Then $\overline{g}(I(R)) = 1$ if and only if $n = 4$, where n is the number of distinct maximal ideals of R.*

Proof. Suppose $\overline{g}(I(R)) = 1$. We have, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is a local ring. When $n \geq 5$, let $H = \{a_1, a_2, \ldots, a_{10}\}$ where $a_1 = (1,0,0,\ldots,0)$, $a_2 = (0,1,0,\ldots,0)$, $a_3 = (1,1,0,\ldots,0)$, $a_4 = (0,0,1,0,\ldots,0)$, $a_5 = (0,0,0,1,0,\ldots,0)$, $a_6 = (0,0,0,0,1,0,\ldots,0)$, $a_7 = (0,0,1,1,0,\ldots,0)$, $a_8 = (0,0,0,1,1,0,\ldots,0)$, $a_9 = (0,0,1,0,1,0,\ldots,0)$, $a_{10} = (0,0,1,1,1,0,\ldots,0)$. Then the subgraph induced by H in $I(R)$ contains $K_{3,7}$ as a subgraph. By Lemma 4, $\overline{g}(I(R)) \geq 3$, a contradiction. Hence $n = 4$.

Conversely, assume that $n = 4$. Let $J = \{a_1, a_2, \ldots, a_{14}\}$ where, $a_1 = (1,0,0,0)$, $a_2 = (0,1,0,0)$, $a_3 = (1,1,0,0)$, $a_4 = (1,0,1,0)$, $a_5 = (1,0,0,1)$,

$a_6 = (0,0,1,0)$, $a_7 = (0,0,0,1)$, $a_8 = (0,0,1,1)$, $a_9 = (0,1,1,0)$, $a_{10} = (0,1,0,1)$, $a_{11} = (0,1,1,1)$, $a_{12} = (1,0,1,1)$, $a_{13} = (1,1,0,1)$, $a_{14} = (1,1,1,0)$. Then the subgraph induced by J in $I(R)$ contains $K_{3,3}$ as a subgraph. By Theorem 4, $\overline{g}(I(R)) \geq 1$, whereas an embedding of $I(R)$ given in Figure 4, explicitly shows that $\overline{g}(I(R)) = 1$. ■

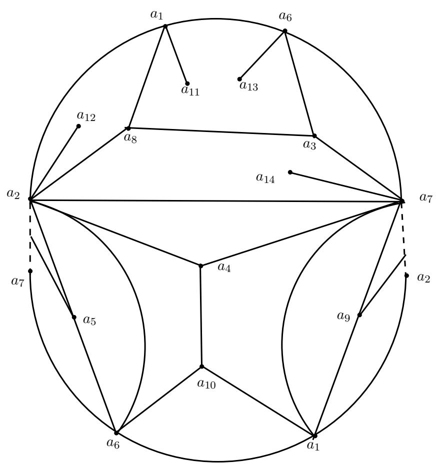

Figure 4. An embedding of $I(R_1 \times R_2 \times R_3 \times R_4)$ in N_1.

Theorem 10. *There is no finite commutative ring R for which $\overline{g}(I(R)) = 2$.*

Proof. Since R is finite, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is a local ring. By Theorem 9, $n \geq 5$. Let $K = \{a_1, a_2, \ldots, a_{10}\}$ where $a_1 = (1,0,0,\ldots,0)$, $a_2 = (0,1,0,\ldots,0)$, $a_3 = (1,1,0,\ldots,0)$, $a_4 = (0,0,1,0,\ldots,0)$, $a_5 = (0,0,0,1,0,\ldots,0)$, $a_6 = (0,0,0,0,1,0,\ldots,0)$, $a_7 = (0,0,1,1,0,\ldots,0)$, $a_8 = (0,0,0,1,1,0,\ldots,0)$, $a_9 = (0,0,1,0,1,0,\ldots,0)$, $a_{10} = (0,0,1,1,1,0,\ldots,0)$. Then the subgraph induced by K in $I(R)$ contains $K_{3,7}$ as a subgraph and by Lemma 4, $\overline{g}(I(R)) \geq 3$. ■

Theorem 11. *Let R be a finite commutative ring. Then $I(R)$ is not a split graph for $n \geq 4$, where n is the number of distinct maximal ideals of R.*

Proof. By the structure of $I(R)$, $I(R)$ contains K_n for $n \geq 4$. Consider an vertex a of $I(R)$ that is not in $V(K_n)$, which has 1 in the i,j th places and 0 in the remaining places. This vertex must adjacent with the vertex b that has 0 in the i,j th places and 1 in the remaining places. Hence the remaining vertices cannot form an independent set. Hence the theorem. ■

Theorem 12. *Let R be a finite commutative ring. Then $I(R)$ is a ring graph if and only if $n = 3$, where n is the number of distinct maximal ideals of R.*

Proof. Assume that $I(R)$ is a ring graph. Since every ring graph is planar it is enough to consider whether $I(R)$ is a ring graph for each ring in Theorem 6.

When $n = 3$, $I(R)$ has $rank(I(R)) = frank(I(R))$. Hence $I(R)$ is a ring graph. The converse is obvious. ■

4. Book thickness of $I(R)$

A standard n-book is formed by joining n half-planes, called pages, together at a common line, called spine. When embedding a graph in a book, the vertices are placed along the spine. Each edge is embedded on a single page of the book so that no two edges cross each on a page. The *book thickness* of a graph G is the smallest n, for which G has an n-book embedding. Yannakakis [10] has shown that all planar graphs have book thickness at most 4.

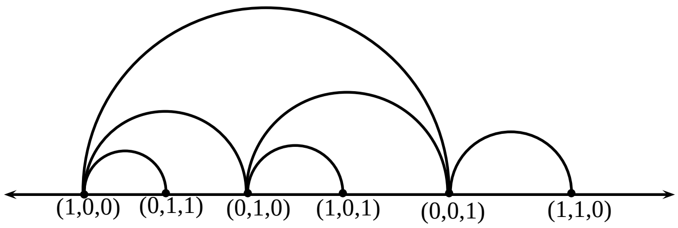

Figure 5. Book embedding of $I(R_1 \times R_2 \times R_3)$.

Theorem 13 [5, Theorem 3.4]**.** *The book thickness of the complete graph K_n is equal to $\lceil n/2 \rceil$, when $n \geq 4$.*

Theorem 14 [5, Theorem 2.5]**.** *A graph has book thickness one if and only if it is outer planar.*

Theorem 15 [5, Theorem 2.5]**.** *The book thickness of a graph is at most two if and only if it is a subgraph of a planar graph that has a Hamiltonian cycle.*

Now we classify all rings such that the book thickness of idempotent graph is 1.

Theorem 16. *Let R be a finite commutative ring. Then the book thickness of $I(R)$ is 1 if and only if $n \leq 3$, where n is the number of distinct maximal ideals of R.*

Proof. The proof is clear by Theorem 6 and Theorem 14. ■

Now we characterize all finite commutative rings R such that the book thickness of the idempotent graph is 3.

Theorem 17. *Let R be a finite commutative ring. The book thickness of $I(R)$ is 3 if and only if R has exactly 4 distinct maximal ideals.*

Proof. Suppose R has exactly 4 distinct maximal ideals, then by Theorem 7, we know that $R = R_1 \times R_2 \times R_3 \times R_4$, which is toroidal. By Theorem 15, one can note that, a two page book embedding is corresponding to a planar structure. Hence a toroidal graph has book thickness at least three. But the three pages of $I(R)$ are represented by the sets of dashed and solid edges above and below the spine in Figure 6 (For a_i, $1 \leq i \leq 14$ refer Figure 2). Hence the book thickness of $I(R)$ is 3.

Conversely assume that the book thickness of $I(R)$ is 3. Suppose that R has at least 5 distinct maximal ideals. Let $J = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_{10}, a_{11}, a_{13}, a_{18}\}$ be a subset of E in Theorem 8. Then, the subgraph induced by J in $I(R)$ must contain a subdivision (each edge should be subdivided at most once) of K_7. By [11, Theorem 3.1], we come to know that, book thickness of K_n does not change, even though we subdivide its edges at most once. Hence by Theorem 13, the book thickness of $I(R)$ is at least 4, a contradiction. ■

Corollary 18. *There exists no finite commutative ring R, for which book thickness of $I(R)$ is 2.*

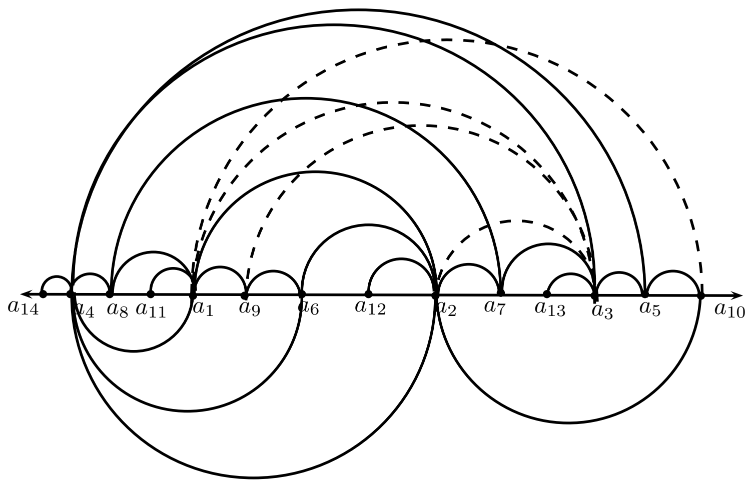

Figure 6. Book embedding of $I(R_1 \times R_2 \times R_3 \times R_4)$.

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