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MULTIVALUED ANISOTROPIC PROBLEM WITH NEUMANN BOUNDARY CONDITION INVOLVING DIFFUSE RADON MEASURE DATA AND VARIABLE EXPONENT

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Abstract

We study a nonlinear anisotropic elliptic problem with homogeneous Neumann boundary condition governed by a general anisotropic operator with variable exponents and diffuse Radon measure data that is the Radon measure which does not charge the sets of zero $p(\cdot)$-capacity. We firstly prove the existence of renormalized solutions. Secondly, we show an equivalence between renormalized solution and entropy solution. Thirdly, we end by proving an uniqueness result of entropy solution.

Keywords: Neumann boundary, anisotropic Sobolev spaces, renormalized solution, entropy solution, maximal monotone graph, Radon diffuse measure, Marcinkiewicz spaces, $p(\cdot)$-capacity.

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1. Introduction

The study of various mathematical problems with variable exponent has received considerable attention in recent years. The interest in transposing the problems with constant exponent into new problems with variable exponents is linked to a large scale of applications that are involving some non-homogeneous materials (blood for example). Indeed, some materials cannot be modelled with sufficient accuracy using classical Lebesgue and Sobolev spaces with constant exponent.

Hence we have to allow the exponent to vary. We can refer here to electrorheological fluids (see [1, 12, 24]) thermorheological fluids, modelling of propagation of epidemic disease (see [2]), image restoration (see [10]).

The goal of this paper is to establish the existence and uniqueness of entropy solution for the following nonlinear multivalued elliptic anisotropic problem

$$
\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x_i}\right) + \beta(u) \ni \mu & \text{in } \Omega \\ \displaystyle\sum_{i=1}^{N} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \cdot \eta_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}
$$

where Ω is an open bounded domain of $\mathbb{R}^N$ $(N \geq 3)$, β a maximal monotone graph on $\mathbb{R}$ such that $0 \in \beta(0)$, μ a bounded Radon diffuse measure. We set $\overline{dom(\beta)} = [m,\ M]$ with $m \leq 0 \leq M$. Note that the space in which we work is the anisotropic Sobolev space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$, where $\overrightarrow{p}(\cdot) = (p_1(\cdot), \ldots, p_N(\cdot))$ is a vector with variable components. We denote by

$$
p_M(x) := \max(p_1(x), \ldots, p_N(x)) \quad \text{and} \quad p_m(x) := \min\left(p_1(x), \ldots, p_N(x)\right).
$$

In the classical Lebesgue and Sobolev spaces with constant exponent, many authors have studied problems with a maximal monotone graph and measure data (see [3–5, 11, 13, 17]). Note that these kind of problems have been extended in Sobolev space with variable exponents in the context of isotropic operators (see [22, 23]).

The first systematic study of anisotropic Neumann problem with variable exponents was done by Fan (see [14]). In a second time, Boureanu and Rădulescu studied anisotropic nonhomogeneous Neumann problem with obstacle (see [7]). In the two papers, the authors were interested by the existence and multiplicity results of weak solutions even if in [7], Boureanu and Rădulescu have showed some conditions under which we can get uniqueness of weak solution. When β is a power (i.e., $\beta(t) = |t|^{p_M(x)-2}t$), the study of problem (1.1) for L^1-function was done by Bonzi *et al.* (see [6]). They proved, by using the techniques of minimization, the existence of weak solution, and by approximation methods, the existence and uniqueness of entropy solution of the problem (1.1).

Ibrango and Ouaro in [16] used the technic of monotone operators in Banach spaces (see [25]) and approximation methods to get the existence and uniqueness of entropy solutions of (1.1) when μ is any L^1-function and continuous β defined in all $\mathbb{R}$.

In a recent paper, we have studied a particular instance of problem (1.1) (see [18]). More precisely, we have showed that the problem (1.1) admits a unique entropy solution for a Radon diffuse measure $\mu \in L^1(\Omega) + W^{-1,p'_m(\cdot)}(\Omega)$ and continuous β defined in all $\mathbb{R}$.

The main interest in our work is that we are dealing with general nonlinearities β.

The study of multivalued elliptic problems with measure data in the context of isotropic variable exponent was done by Arouna *et als* (see [23]) under homogeneous Neumann boundary condition. They used the argument of the decomposition of Radon diffuse measure data (more precisely, as a sum of an element in $W^{-1,p'(\cdot)}(\Omega)$ (the dual space of $W_0^{1,p(\cdot)}(\Omega)$) and a function in $L^1(\Omega)$) to prove following [17], a result of existence and uniqueness of entropy solution of the problem

$$(1.2) \qquad \begin{cases} -\nabla\cdot a(x,\nabla u)+\beta(u)\ni\mu & \text{in } \Omega \\ a(x,\nabla u)\cdot\eta=0 & \text{on } \partial\Omega, \end{cases}$$

The problem (1.1) is the anisotropic case of the non-linear isotropic problem (1.2).

In this paper, we rely our ideas on the decomposition theorem of measure done by the authors in [23], and following them we prove both the existence and uniqueness of entropy solution of the non-linear multivalued elliptic anisotropic problem (1.1).

We denote by $\mathcal{L}^N$ the N-dimensional Lebesgue measure of $\mathbb{R}^N$ and by $\mathcal{M}_b(\Omega)$ the space of bounded Radon measures in Ω, equipped with its standard norm $||\cdot||_{\mathcal{M}_b(\Omega)}$. Note that, if μ belongs to $\mathcal{M}_b(\Omega)$, then $|\mu|(\Omega)$ (the total variation of μ) is a bounded positive measure on Ω.

Given $\mu\in\mathcal{M}_b(\Omega)$, we say that μ is diffuse with respect to the capacity $W_0^{1,p(\cdot)}(\Omega)$ ($p(\cdot)$-capacity for short) if $\mu(A)=0$, for every set A such that $Cap_{p(\cdot)}(A,\Omega)=0$.

For every $A\subset\Omega$, we denote

$$S_{p(\cdot)}(A)=\left\{u\in W_0^{1,p(\cdot)}(\Omega)\cap C_0(\Omega): u=1 \text{ on } A,\ u\geq 0 \text{ on } \Omega\right\}.$$

The $p(\cdot)$-capacity of every subset A with respect to Ω is defined by

$$Cap_{p(\cdot)}(A,\Omega)=\inf_{u\in S_{p(\cdot)}(A)}\left\{\int_\Omega|\nabla u|^{p(x)}dx\right\}.$$

In the case $S_{p(\cdot)}(A)=\emptyset$, we set $Cap_{p(\cdot)}(A,\Omega)=+\infty$.

The set of bounded Radon diffuse measure in the variable exponent setting is denoted by $\mathcal{M}_b^{p(\cdot)}(\Omega)$.

Note that, since we are dealing with the Neumann boundary condition, we cannot work with the common space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. However the common space is $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$, so we cannot use directly the argument of decomposition of measure, since the second part of the measure is in $W^{-1,p'_m(\cdot)}(\Omega)$ (the dual of $W_0^{1,p_m(\cdot)}(\Omega)$).

To overcome this difficulty, we use the same ideas as authors in [23]. We consider a smooth domain Ω in order to work with the space $W_0^{1,p_m(\cdot)}(\Omega)$ and return after to the space $W^{1,p_m(\cdot)}(\Omega)$. More precisely, Ω is assumed to be a bounded domain in $\mathbb{R}^N$ with a boundary $\partial\Omega$ of class C^1. Then, it has an extension domain (see [8]), so for any fixed open bounded subset U_Ω of $\mathbb{R}^N$ such that $\overline{\Omega} \subset U_\Omega$, there exists a bounded linear operator

$$E : W^{1,p_m(\cdot)}(\Omega) \to W_0^{1,p_m(\cdot)}(U_\Omega),$$

for which

(i) $E(u) = u$ a.e. in Ω for each $u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$,

(ii) $||E(u)||_{W_0^{1,p_m(\cdot)}(U_\Omega)} \leq C||u||_{W^{1,p_m(\cdot)}(\Omega)}$, where C is a constant depending only on Ω.

We define

$$\mathfrak{M}_b^{p_m(\cdot)}(\Omega) := \left\{\mu \in \mathcal{M}_b^{p_m(\cdot)}(U_\Omega) : \mu \text{ is concentrated on } \Omega\right\}.$$

This definition is independent of the open set U_Ω. Note that for $u \in W^{1,p_m(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $\mu \in \mathfrak{M}_b^{p_m(\cdot)}(\Omega)$, we have

$$\langle \mu, E(u)\rangle = \int_\Omega u d\mu.$$

On the other hand, as μ is diffuse, there exists (see [23]) $f \in L^1(U_\Omega)$ and $F \in (L^{p'_m(\cdot)}(U_\Omega))^N$ such that $\mu = f - div(F)$ in $\mathcal{D}'(U_\Omega)$.

Therefore, we can also write

$$\langle \mu, E(u)\rangle = \int_{U_\Omega} fE(u)dx + \int_{U_\Omega} F \cdot \nabla E(u)dx.$$

Our main results is stated as follows.

Theorem 1.1. *For any $\mu \in \mathcal{M}_b^{p_m(\cdot)}(\Omega)$, the problem* (1.1) *has at least one solution (u, w) in the sense that $(u, w) \in \mathcal{T}_{\mathcal{H}}^{1,\overrightarrow{p}(\cdot)}(\Omega) \times L^1(\Omega)$, $u \in dom(\beta)\mathcal{L}^N$ a.e. in Ω, $w \in \beta(u)\mathcal{L}^N$ a.e. in Ω, there exists $\nu \in \mathcal{M}_b^{p_m(\cdot)}(\Omega)$ such that $\nu \perp \mathcal{L}^N$, for any $h \in \mathcal{C}_c^1(\mathbb{R})$, $h(u) \in L^\infty(\Omega, d|\nu|)$, $h(u)\nu \in \mathcal{M}_b^{p_m(\cdot)}(\Omega)$,*

$$\nu^+ \textit{ is concentred on } [u = M],\ \nu^- \textit{ is concentred on } [u = m]$$

and

$$\sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}[h(u)\xi]dx + \int_\Omega wh(u)\xi dx + \int_\Omega h(u)\xi d\nu = \int_\Omega h(u)\xi d\mu, \tag{1.3}$$

for any $\xi \in W^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$. *Moreover,*

$$\lim_{n\to+\infty} \int_{[n\le|u|\le n+1]} \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx = 0. \tag{1.4}$$

The connexion between our notion of solution and the entropic formulation is given in the following Theorem. In particular, as the domain of β is bounded, this equivalent formulation is very useful for the proof of the uniqueness of solution for (1.1).

Theorem 1.2. *If* (u,w) *is a solution of* (1.1) *in the sense of Theorem* 1.1, *then* (u,w) *is a solution in the following sense: for any* $\xi \in W^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ *such that* $\xi \in dom\beta$,

$$\begin{aligned} \sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u-\xi)dx + \int_\Omega w T_k(u-\xi)dx \\ \le \int_\Omega T_k(u-\xi)d\mu, \text{ for any } k > 0. \end{aligned} \tag{1.5}$$

As the domain of β is bounded, the renormalization with the function h is not necessary in Theorem 1.1. Moreover, by using Theorem 1.2 we have the uniqueness of solution. This is summarized in the following theorem.

Theorem 1.3. *For any* $\mu \in \mathcal{M}_b^{p_m(\cdot)}(\Omega)$, *the problem* (1.1) *has at least one solution* (u,w) *in the sense that* $(u,w) \in W^{1,\vec{p}(\cdot)}(\Omega) \times L^1(\Omega)$, $u \in dom(\beta)\mathcal{L}^N$ *a.e. in* Ω, $w \in \beta(u)\mathcal{L}^N$ *a.e. in* Ω, *there exists a measure* $\nu \in \mathcal{M}_b^{p_m(\cdot)}(\Omega)$ *such that* $\nu \perp \mathcal{L}^N$,

$$\nu^+ \text{ is concentred on } [u=M], \ \nu^- \text{ is concentred on } [u=m]$$

and

$$\sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial \xi}{\partial x_i} dx + \int_\Omega w\xi dx + \int_\Omega \xi d\nu = \int_\Omega \xi d\mu, \tag{1.6}$$

for any $\xi \in W^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$. *Moreover,*

$$\nu^+ \le \mu_s \lfloor\, [u=M] \tag{1.7}$$

and

$$\mu^- \le -\mu_s \lfloor\, [u=m]. \tag{1.8}$$

Moreover the problem (1.1) *admits a unique solution in the following sense: if* (u_1,w_1) *and* (u_2,w_2) *are two solutions with the measures* ν_1 *and* ν_2 *respectively,* *we have*

$$\begin{cases} u_1 - u_2 = c & a.e.\ in\ \Omega, \\ w_1 = w_2 & a.e.\ in\ \Omega, \\ \nu_1 = \nu_2. \end{cases}$$

The rest of the paper is organized as follows. In Section 2, we introduce some fundamental preliminary results which are useful in this work. Then, we prove the Theorem 1.1 in Section 3. Finally in Section 4, we prove theorems 1.2 and 1.3.

2. Preliminary

We study problem (1.1) under the following assumptions on the data.

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$ and $\overrightarrow{p}(\cdot) = (p_1(\cdot), \ldots, p_N(\cdot))$ such that for any $i = 1, \ldots, N$, $p_i(\cdot) : \overline{\Omega} \longrightarrow \mathbb{R}$ is a continuous function with

$$1 < p_i^- := \inf_{x\in\Omega} p_i(x) \leq p_i^+ := \sup_{x\in\Omega} p_i(x) < \infty. \tag{2.1}$$

For any $i = 1, \ldots, N$, let $a_i : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function satisfying:

- there exists a positive constant C_1 such that

$$|a_i(x,\xi)| \leq C_1\left(j_i(x) + |\xi|^{p_i(x)-1}\right), \tag{2.2}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$, where j_i is a non-negative function in $L^{p_i'(\cdot)}(\Omega)$, with $\frac{1}{p_i(x)} + \frac{1}{p_i'(x)} = 1$;

- for $\xi,\ \eta\ \in \mathbb{R}$ with $\xi \neq \eta$ and for every $x \in \Omega$, there exists a positive constant C_2 such that

$$(a_i(x,\xi) - a_i(x,\eta))(\xi - \eta) \geq \begin{cases} C_2|\xi - \eta|^{p_i(x)} & \text{if } |\xi - \eta| \geq 1 \\ C_2|\xi - \eta|^{p_i^-} & \text{if } |\xi - \eta| < 1 \end{cases} \tag{2.3}$$

and,

- there exists a positive constant C_3 such that

$$a_i(x,\xi).\xi \geq C_3|\xi|^{p_i(x)}, \tag{2.4}$$

for every $\xi \in \mathbb{R}$ and almost every $x \in \Omega$.

The hypotheses on a_i are classical in the study of nonlinear problems (see [3]). Throughout this paper, we assume that

$$\frac{\overline{p}(N-1)}{N(\overline{p}-1)} < p_i^- < \frac{\overline{p}(N-1)}{N-\overline{p}}, \quad \frac{p_i^+ - p_i^- - 1}{p_i^-} < \frac{\overline{p} - N}{\overline{p}(N-1)} \tag{2.5}$$

and

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1, \tag{2.6}$$

where $\frac{N}{\overline{p}} = \sum_{i=1}^{N} \frac{1}{p_i^-}$.

We also recall in this section some definitions and basic properties of anisotropic Lebesgue and Sobolev spaces.

Set

$$C_+(\overline{\Omega}) = \Big\{ p \in C(\overline{\Omega}) : \min_{x \in \overline{\Omega}} p(x) > 1 \quad a.e.\ x \in \Omega \Big\}.$$

For any $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) := \Big\{ u : u \text{ is a measurable real valued function such that } \int_\Omega |u|^{p(x)} dx < \infty \Big\},$$

endowed with the so-called Luxembourg norm

$$|u|_{p(\cdot)} := \inf \Big\{ \lambda > 0 : \int_\Omega \Big| \frac{u(x)}{\lambda} \Big|^{p(x)} dx \leq 1 \Big\}.$$

The $p(\cdot)$-modular of the space $L^{p(\cdot)}(\Omega)$ is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)} dx.$$

For any $u \in L^{p(\cdot)}(\Omega)$, the following inequalities (see [14,15]) will be used later

$$\min \Big\{ |u|^{p^-}_{p(\cdot)};\ |u|^{p^+}_{p(\cdot)} \Big\} \leq \rho_{p(\cdot)}(u) \leq \max \Big\{ |u|^{p^-}_{p(\cdot)};\ |u|^{p^+}_{p(\cdot)} \Big\}. \tag{2.7}$$

For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ in Ω, we have the Hölder type inequality

$$\Big| \int_\Omega uv dx \Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-} \Big) |u|_{p(\cdot)} |v|_{q(\cdot)}. \tag{2.8}$$

If Ω is bounded and $p,\ q \in C_+(\overline{\Omega})$ such that $p(x) \leq q(x)$ for any $x \in \Omega$, then the embedding $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous (see [20], Theorem 2.8).

Herein we need the following anisotropic Sobolev spaces (see [21]).

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) := \Big\{ u \in W_0^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i(\cdot)}(\Omega),\ i = 1, \ldots, N \Big\}$$

and

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) := \left\{ u \in L^{p_M(\cdot)}(\Omega) : \ \frac{\partial u}{\partial x_i} \in L^{p_i(\cdot)}(\Omega), \ i = 1, \dots, N \right\},$$

which are separable and reflexive Banach spaces under the norm

$$||u||_{\overrightarrow{p}(\cdot)} = \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|_{p_i(\cdot)}.$$

We introduce the numbers

$$q = \frac{N(\overline{p} - 1)}{N - 1}; \ q^* = \frac{N(\overline{p} - 1)}{N - \overline{p}} = \frac{Nq}{N - q}$$

and define $P_-^*, \ P_-^+, \ P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^* = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i^-} - 1}, \ P_-^+ = \max\left\{p_1^-, \dots, p_N^-\right\} \text{ and } P_{-,\infty} = \max\left\{P_-^+, \ P_-^*\right\}.$$

We have the following embedding results (see [20], Theorem 1).

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded open set and for all $i = 1, \dots, N$, $p_i \in L^\infty(\Omega)$, $p_i(x) \geq 1$ a.e. in Ω. Then, for any $q \in L^\infty(\Omega)$ with $q(x) \geq 1$ a.e. in Ω such that*

$$ess \inf_{x \in \Omega} (p_M(x) - q(x)) > 0,$$

we have the compact embedding

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega).$$

The following result is due to Troisi (see [26]).

Theorem 2.2. *Let $p_1, \dots, p_N \in [1, +\infty)$; $g \in W^{1,(p_1,\dots,p_N)}(\Omega)$ and*

$$q = \begin{cases} (\overline{p})^* & \text{if } (\overline{p})^* < N \\ \in [1, +\infty) & \text{if } (\overline{p})^* \geq N. \end{cases}$$

Then, there exists a constant $C_4 > 0$ depending on $N, p_1, \dots, p_N$ if $\overline{p} < N$ and also on q and $meas(\Omega)$ if $\overline{p} \geq N$ such that

$$||g||_{L^q(\Omega)} \leq C_4 \prod_{i=1}^{N} \left\| \frac{\partial g}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{\frac{1}{N}}. \tag{2.9}$$

In this paper, we will use the Marcinkiewicz space $\mathcal{M}^q(\Omega)$ $(1 < q < +\infty)$ as the set of measurable function $g : \Omega \longrightarrow \mathbb{R}$ for which the distribution

$$\lambda_g(k) = meas(\{x \in \Omega : |g(x)| > k\}),\ k \geq 0 \tag{2.10}$$

satisfies an estimate of the form

$$\lambda_g(k) \leq Ck^{-q}, \quad \text{for some finite constant } C > 0. \tag{2.11}$$

We will use the following pseudo norm in $\mathcal{M}^q(\Omega)$

$$||g||_{\mathcal{M}^q(\Omega)} := \inf\{C > 0 : \lambda_g(k) \leq Ck^{-q},\ \forall k > 0\}. \tag{2.12}$$

Finally, we use throughout the paper, the truncation function T_k, $(k > 0)$ by

$$T_k(s) = \max\{-k, \min\{k; s\}\}. \tag{2.13}$$

It is clear that $\lim_{k\to\infty} T_k(s) = s$ and $|T_k(s)| = \min\{|s|; k\}$.

Set $\mathcal{T}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ as the set of measurable functions $u : \Omega \longrightarrow \mathbb{R}$ such that $T_k(u) \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$. We define the space $\mathcal{T}_{\mathcal{H}}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ as the set of functions $u \in \mathcal{T}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ satisfying

$$u_n \longrightarrow u \text{ a.e. in } \Omega$$

and

$$\frac{\partial T_k(u_n)}{\partial x_i} \longrightarrow \frac{\partial T_k(u)}{\partial x_i} \text{ in } L^1(\Omega),\ \forall\ k > 0.$$

To give our notion of solution and the main results, we set

$$int(dom(\beta)) = (m, M) \text{ with } -\infty < m \leq 0 \leq M < +\infty.$$

For any $r \in \mathbb{R}$ and any measurable function u on Ω, $[u = 0]$, $[u \leq r]$ and $[u \geq r]$ denote, respectively the set $\{x \in \Omega : u(x) = r\}$, $\{x \in \Omega : u(x) \leq r\}$, $\{x \in \Omega : u(x) \geq r\}$.

We will use the following decomposition result of bounded Radon measure proved by Nyanquini *et al.* (see [22]).

Theorem 2.3. *Let $p(\cdot) : \overline{\Omega} \longrightarrow (1, +\infty)$ be a continuous function and $\mu \in \mathcal{M}_b(\Omega)$. Then $\mu \in \mathcal{M}_b^{p_m(\cdot)}(\Omega)$ if and only if $\mu \in L^1(\Omega) + W^{-1,p'_m(\cdot)}(\Omega)$.*

For any given $l, k > 0$, we define the function h_l by $h_l = \min\{(l+1-|r|)^+, 1\}$. Let γ be a maximal monotone operator defined on $\mathbb{R}$, we denote by γ_0 the main section of γ; i.e.,

$$\gamma_0(s) = \begin{cases} \text{minimal absolute value of } \gamma(s) & \text{if } \gamma(s) \neq \emptyset \\ +\infty & \text{if } [s,+\infty)\cap D(\gamma)=\emptyset \\ -\infty & \text{if } (-\infty,s]\cap D(\gamma)=\emptyset. \end{cases}$$

We give a useful convergence result (see [22]).

Lemma 2.1. *Let $(\beta_n)_{n\geq 1}$ be a sequence of maximal monotone graphs such that $\beta_n \to \beta$ in the sense of the graph (for $(x,y)\in\beta$, there exists $(x_n,y_n)\in\beta_n$ such that $x_n \to x$ and $y_n \to y$). We consider two sequences $(z_n)_{n\geq 1}\subset L^1(\Omega)$ and $(w_n)_{n\geq 1}\subset L^1(\Omega)$. We suppose that $\forall n\geq 1, w_n\in\beta_n(z_n)$, $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$ and $z_n\to z$ in $L^1(\Omega)$. Then*

$$z \in dom(\beta).$$

3. Proof of Theorem 1.1

For every $\epsilon>0$, we consider the Yosida regularization β_ϵ of β (see [9]), given by

$$\beta_\epsilon = \frac{1}{\epsilon}\left(I-(I+\epsilon\beta)^{-1}\right).$$

Thanks to [9], there exists a non negative, convex and l.s.c. function j defined on $\mathbb{R}$ such that

$$\beta = \partial j.$$

To regularise β, we consider

$$j_\epsilon(s) = \min_{r\in\mathbb{R}}\left\{\frac{1}{2\epsilon}|s-r|^2 + j(r)\right\},\ \forall\ s\in\mathbb{R},\ \forall\epsilon\ >0.$$

By Proposition 2.11 in [9] we have

$$\begin{cases} dom(\beta)\subset dom(j)\subset \overline{dom(j)} = \overline{dom(\beta)}, \\ j_\epsilon(s) = \frac{\epsilon}{2}|\beta_\epsilon(s)|^2 + j(J_\epsilon) \text{ where } J_\epsilon := (I+\epsilon\beta)^{-1}, \\ j_\epsilon \text{ is a convex, Frechet-differentiable function and } \beta_\epsilon = \partial j_\epsilon, \\ j_\epsilon \uparrow j \text{ as } \epsilon\downarrow 0. \end{cases}$$

Moreover, for any $\epsilon>0$, β_ϵ is a non-decreasing and Lipschitz-continuous function. Since $\mu\in\mathcal{M}_b^{p_m(\cdot)}$, recall that $\mu = f - div(F)$ in $\mathcal{D}'(U_\Omega)$ with $f\in L^1(U_\Omega)$ and $F\in(L^{p'_m(\cdot)}(U_\Omega))^N$. To regularize μ, for any $\epsilon>0$, we define the function

$$f_\epsilon = T_{\frac{1}{\epsilon}}(f(x))\chi_\Omega(x).$$

Then, we consider $F_R = \chi_\Omega F$ and $\mu_\epsilon = f_\epsilon - div(F_R)$. For any $\epsilon > 0$, one has $\mu_\epsilon \in \mathfrak{M}_b^{p_m(\cdot)}(\Omega)$, $\mu_\epsilon \rightharpoonup \mu$ in $\mathcal{M}_b(U_\Omega)$ and $\mu_\epsilon \in L^\infty(\Omega)$.

We have the following lemma (see [23], Lemma 4.1).

Lemma 3.1. *The Yosida regularization β_ϵ is a surjective operator.*

Now, we consider the following approximating scheme problem

(3.1)
$$P(\beta_\epsilon, \mu_\epsilon)\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) + \beta_\epsilon(u_\epsilon) + \epsilon |u_\epsilon|^{P_M(x)-2} u_\epsilon = \mu_\epsilon & \text{in } \Omega \\ \displaystyle\sum_{i=1}^{N} a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\cdot \eta_i = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 3.1. *The problem* (3.1) *admits a unique weak solution u_ϵ in the sense that $u_\epsilon \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$, $\beta_\epsilon(u_\epsilon) \in L^1(\Omega)$ and $\forall \xi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$,*

$$\sum_{i=1}^{N} \int_\Omega a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial \xi}{\partial x_i} + \int_\Omega \beta_\epsilon(u_\epsilon)\xi dx + \epsilon \int_\Omega |u_\epsilon|^{P_M(x)-2} u_\epsilon \xi dx = \int_\Omega \mu_\epsilon \xi dx. \tag{3.2}$$

Proof. If b is a surjective, continuous and nondecreasing function with $b(0) = 0$ and $\Upsilon \in L^\infty(\Omega)$, by the Lemma 3.1 in [16] for any $k > 0$, the following problem

$$P(T_k(b), \Upsilon)\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x_i}\right) + T_k(b(u)) + \epsilon |u|^{P_M(x)-2} u = \Upsilon & \text{in } \Omega \\ \displaystyle\sum_{i=1}^{N} a_i\left(x, \frac{\partial u}{\partial x_i}\right)\cdot \eta_i = 0 & \text{on } \partial\Omega \end{cases}$$

admits at least one solution $u_\epsilon \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that $\forall\, \xi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$,

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial \xi}{\partial x_i} dx &+ \int_\Omega T_k(\beta_\epsilon(u_\epsilon))\xi dx \\ &+ \epsilon \int_\Omega |u_\epsilon|^{P_M(x)-2} u_\epsilon \xi dx = \int_\Omega \Upsilon \xi dx. \end{aligned} \tag{3.3}$$

Furthermore,

$$\forall k > ||\Upsilon||_\infty, \quad |b(u_\epsilon)| \le ||\Upsilon||_\infty \text{ a.e. in } \Omega. \tag{3.4}$$

Let us fix $k > ||\Upsilon||_\infty$, we get the existence of solution of problem $P(\beta_\epsilon, \mu_\epsilon)$. The proof of (3.4) is detailed in [16]. So we get the proof of the Theorem 3.1 by setting $\Upsilon = \mu_\epsilon$ and $b = \beta_\epsilon$. ■

We have the following results.

Proposition 3.1. *If u_ϵ is a weak solution of* (3.1), *then*

(i) *there exists* $0 < C(\mu,\Omega) < +\infty$ *such that for any* $k > 0$,

$$\sum_{i=1}^{N}\int_{\{|u_\epsilon|\le k\}}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)}dx \le kC(\mu,\Omega) \tag{3.5}$$

and

$$\int_\Omega \beta_\epsilon(u_\epsilon)T_k(u_\epsilon)dx \le kC(\mu,\Omega), \tag{3.6}$$

(ii)

$$\int_\Omega |\nabla T_k(u_\epsilon)|^{\bar{p_m}}dx \le C_5, \tag{3.7}$$

(iii) *the sequence* $(\beta_\epsilon(u_\epsilon))_{\epsilon>0}$ *is uniformly bounded in* $L^1(\Omega)$,

(iv) *for any* $k > 0$, *the sequence* $(\beta_\epsilon(T_k(u_\epsilon)))_{\epsilon>0}$ *is uniformly bounded in* $L^1(\Omega)$.

Proof. (i) Since $u_\epsilon \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$, $\xi = T_k(u_\epsilon)$ is an admissible test function (3.2), we have

$$\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial T_k(u_\epsilon)}{\partial x_i} + \int_\Omega \beta_\epsilon(u_\epsilon)T_k(u_\epsilon)dx + \epsilon\int_\Omega |u_\epsilon|^{P_M(x)-2}u_\epsilon T_k(u_\epsilon)dx = \int_\Omega T_k(u_\epsilon)d\mu_\epsilon.$$

Using the fact that $\epsilon\displaystyle\int_\Omega |u_\epsilon|^{P_M(x)-2}u_\epsilon T_k(u_\epsilon)dx \ge 0$ we get

$$\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial(u_\epsilon)}{\partial x_i} + \int_\Omega \beta_\epsilon(u_\epsilon)T_k(u_\epsilon)dx \le \int_\Omega T_k(u_\epsilon)d\mu_\epsilon. \tag{3.8}$$

Thanks to (2.4), the fact that β_ϵ, T_k are both nondecreasing and $\beta_\epsilon(0) = T_k(0) = 0$, and

$$\int_\Omega T_k(u_\epsilon)d\mu_\epsilon \le \left|\int_\Omega T_k(u_\epsilon)d\mu_\epsilon\right| \le kC(\mu,\Omega),$$

we deduce from (3.8) inequalities (3.5) and (3.6). We make the following notation.

$\mathcal{I}_1 = \Big\{i \in \{1,\dots,N\} : |\frac{\partial u_\epsilon}{\partial x_i}| \le 1\Big\}$, $\mathcal{I}_2 = \Big\{i \in \{1,\dots,N\} : |\frac{\partial u_\epsilon}{\partial x_i}| > 1\Big\}$ and $A_h := \{|u_\epsilon| \le k\}$.

We have

$$\begin{aligned}\sum_{i=1}^{N}\int_{A_h}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)}dx &= \sum_{i\in\mathcal{I}_1}\int_{A_h}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)}dx + \sum_{i\in\mathcal{I}_2}\int_{A_h}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)}dx\\ &\geq \sum_{i\in\mathcal{I}_2}\int_{A_h}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)}dx \geq \sum_{i\in\mathcal{I}_2}\int_{A_h}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_m^-}dx\\ &\geq \sum_{i=1}^{N}\int_{A_h}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_m^-}dx - \sum_{i\in\mathcal{I}_1}\int_{A_h}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_m^-}dx\\ &\geq \sum_{i=1}^{N}\int_{A_h}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_m^-}dx - Nmeas(\Omega).\end{aligned}$$

In the other hand, there exists a constant $\alpha > 0$ such that

$$\sum_{i=1}^{N}\int_{A_h}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_m^-}dx \geq \alpha\int_{\Omega}|\nabla T_k(u_\epsilon)|^{p_m^-}.$$

We deduce from the two last inequalities that

$$\sum_{i=1}^{N}\int_{|u_\epsilon|\leq k}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)}dx \geq \alpha\int_{\Omega}|\nabla T_k(u_\epsilon)|^{p_m^-} - Nmeas(\Omega). \tag{3.9}$$

By combining (3.5) and (3.9), and setting $C_5 = \frac{1}{\alpha}\big(C(\mu,\Omega) + Nmeas(\Omega)\big)$, we get (3.7).

For the proof of (iii) and (iv) we refer to ([23], Proposition 4.3). ■

Proposition 3.2. *Let u_ϵ be a weak solution of* (3.1). *For any $k > 0$ large enough, we have*

$$meas\{|u_\epsilon| > k\} \leq \frac{C(\mu,\Omega)}{\min\{\beta_\epsilon(k), |\beta_\epsilon(-k)|\}}, \tag{3.10}$$

$$meas\left\{\left|\frac{\partial u_\epsilon}{\partial x_i}\right| > k\right\} \leq \frac{C(\mu,\Omega)}{k^{\frac{1}{(p_M)'}}} \tag{3.11}$$

and

$$meas\big\{|\nabla u_\epsilon| > k\big\} \leq \frac{C_6(k+1)}{k^{p_m^-}} + \frac{C(\mu,\Omega)}{\min\{\beta_\epsilon(k), |\beta_\epsilon(-k)|\}}, \tag{3.12}$$

where C_6 is a positive constant.

For the proof of Proposition 3.2, see [**?**, 16, 22] and [23]. We also have the following lemmas (see [6, 16, 19]).

Lemma 3.2. *For any $k > 0$, there exists some constants C_1, $C_2 > 0$ such that*

(i) $||u_\epsilon||_{\mathcal{M}^{q^*}(\Omega)} \leq C_1$,

(ii) $\left\|\frac{\partial u_\epsilon}{\partial x_i}\right\|_{\mathcal{M}^{p_i^- \frac{q}{p}}(\Omega)} \leq C_2, \quad \forall i = 1, \ldots, N.$

Lemma 3.3. *For $i = 1, \ldots, N$, as $n \to +\infty$, we have*

$$a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right) \text{ in } L^1(\Omega) \text{ a.e. } x \in \Omega. \tag{3.13}$$

We also have the following useful convergences results.

Proposition 3.3. *Let u_ϵ be a weak solution of the problem* (3.1). *Then, we have*

(i) *For any $k > 0$, the sequence $(T_k(u_\epsilon))_{\epsilon>0}$ is bounded in $W^{1,p_m^-}(\Omega)$.*

(ii) *There exists $u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) \subset \mathcal{T}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that $u \in dom(\beta)$ a.e. in Ω and*

$$u_\epsilon \longrightarrow u \text{ in measure and a.e. in } \Omega \text{ as } \epsilon \longrightarrow 0. \tag{3.14}$$

Proof. We start by the proof of (i). We have

$$\begin{aligned}\int_\Omega |T_k(u_\epsilon)|^{p_m^-} dx &\leq \int_\Omega k^{p_m^-} dx \leq \max(k^{p_m^-}, k^{p_m^+}) meas(\Omega) \\ &= C(k, p_m^-, p_m^+, \Omega).\end{aligned} \tag{3.15}$$

Thanks to (3.7) and (3.15), we get (i).

Now we prove (ii). For the proof of (3.14), we refer to [6] (see also [16, 19]). As for $k > 0$, T_k is continuous, then $T_k(u_\epsilon) \to T_k(u)$ a.e. in Ω. Finally, using Lemma 2.1 we deduce that for all $k > 0$, $T_k(u) \in dom(\beta)$ a.e. in Ω. Since $T_k(u) \in dom(\beta)$, we get $u \in dom(\beta)$ *a.e.* in Ω and as $dom(\beta)$ is bounded, then $u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)$. ■

Proposition 3.4. *Assume* (2.1)–(2.6). *If $u_\epsilon \in E$ is a weak solution of* (3.1) *then.*

(i) *For all $i = 1, \ldots, N$, $\frac{\partial u_\epsilon}{\partial x_i}$ converges in measure to the weak partial gradient of u.*

(ii) *For all $i = 1, \ldots, N$ and $k > 0$, $a_i\big(x, \frac{\partial}{\partial x_i} T_k(u_\epsilon)\big)$ converges to $a_i\big(x, \frac{\partial}{\partial x_i} T_k(u)\big)$ in $L^1(\Omega)$ strongly and in $L^{p_i'(\cdot)}(\Omega)$ weakly.*

(iii) *For $i = 1, \ldots, N$, $a_i\big(x, \frac{\partial u_\epsilon}{\partial x_i}\big) \frac{\partial u_\epsilon}{\partial x_i} \longrightarrow a_i\big(x, \frac{\partial u}{\partial x_i}\big) \frac{\partial u}{\partial x_i}$ in $L^1(\Omega)$ and a.e. $x \in \Omega$.*

Proof. For the proof of (i) and (ii) we refer to [6] (see also [16,19]).
(iii) The continuity of $a_i(x,\xi)$ with respect to $\xi \in \mathbb{R}$ gives us

$$a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right) \text{ a.e. } x \in \Omega.$$

Therefore,

$$a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial u_\epsilon}{\partial x_i} \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_i} \quad \text{a.e. } x \in \Omega.$$

Setting $y_\epsilon = a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial u_\epsilon}{\partial x_i}$ and $y = a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_i}$, for $i = 1, \dots, N$, we have

$$\begin{cases} y_\epsilon \geq 0, y_\epsilon \to y \text{ a.e. in } \Omega, \ y \in L^1(\Omega), \\ \displaystyle\int_\Omega y_\epsilon dx \to \int_\Omega y dx \end{cases}$$

and as $\int_\Omega |y_\epsilon - y| dx = 2\int_\Omega (y-y_\epsilon)^+ dx + \int_\Omega (y_\epsilon - y) dx$ and $(y-y_\epsilon)^+ \leq y$, it follows by using Lebesgue dominated convergence Theorem, that

$$\lim_{\epsilon \to 0} \int_\Omega |y_\epsilon - y| dx = 0;$$

which means that

$$a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial u_\epsilon}{\partial x_i} \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_i} \quad \text{strongly in } L^1(\Omega).$$

■

We have the following lemma.

Lemma 3.4. *For any $h \in C_c^1(\mathbb{R})$ and $\varphi \in W^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$, for any $i = 1, \dots, N$,*

$$\frac{\partial}{\partial x_i}(h(u_\epsilon)\varphi) \longrightarrow \frac{\partial}{\partial x_i}(h(u)\varphi) \text{ strongly in } \ L^1(\Omega) \text{ as } \epsilon \to 0, \quad \text{for any } i = 1, \dots, N.$$

Proof. For any $h \in C_c^1(\mathbb{R})$ and $\varphi \in W^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$, we have

$$\frac{\partial}{\partial x_i}[h(u_\epsilon)\varphi] = \varphi h'(u_\epsilon)\frac{\partial u_\epsilon}{\partial x_i} + h(u_\epsilon)\frac{\partial \varphi}{\partial x_i} = \varphi h'(u_\epsilon)\frac{\partial T_l(u_\epsilon)}{\partial x_i} + h(u_\epsilon)\frac{\partial \varphi}{\partial x_i}$$
$$\text{for } l > 0 \ \text{ such that } \ supp(h) \subset (-l, +l).$$

Since $\left|h(u_\epsilon)\frac{\partial \varphi}{\partial x_i}\right| \leq C(h)\left|\frac{\partial \varphi}{\partial x_i}\right| \in L^1(\Omega)$, using Lebesgue dominated convergence Theorem, we get

$$h(u_\epsilon)\frac{\partial \varphi}{\partial x_i} \to h(u)\frac{\partial \varphi}{\partial x_i} \text{ strongly in } L^1(\Omega) \text{ as } \epsilon \to 0.$$

Moreover, we have $\left|\varphi h'(u_\epsilon)\frac{\partial T_l(u_\epsilon)}{\partial x_i}\right| \leq C(h, ||\varphi||_\infty)\left|\frac{\partial T_l(u_\epsilon)}{\partial x_i}\right| \longrightarrow C(h, ||\varphi||_\infty)\left|\frac{\partial T_l(u)}{\partial x_i}\right|$ in $L^1(\Omega)$ as $\epsilon \to 0$. Then, by using generalized convergence Theorem, we deduce that

$$\varphi h'(u_\epsilon)\frac{\partial T_l(u_\epsilon)}{\partial x_i} \to \varphi h'(u)\frac{\partial T_l(u)}{\partial x_i} \text{ strongly in } L^1(\Omega) \text{ as } \epsilon \to 0. \qquad \blacksquare$$

Lemma 3.5. *For any $h \in C_c^1(\mathbb{R})$ and $\xi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$,*

$$\lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\xi d\mu_\epsilon = \int_\Omega h(u)\xi d\mu \tag{3.16}$$

and

$$\lim_{\epsilon\to 0}\epsilon\int_\Omega |u_\epsilon|^{p_M(x)-2}u_\epsilon h(u_\epsilon)\xi dx = 0. \tag{3.17}$$

Proof. • For the proof of (3.16) we refer to [23].
• For (3.17) we have

$$\begin{aligned}\left|\epsilon\int_\Omega |u_\epsilon|^{p_M(x)-2}u_\epsilon h(u_\epsilon)\xi dx\right| &\leq \epsilon\int_\Omega |u_\epsilon|^{p_M(x)-1}|h(u_\epsilon)||\xi|dx\\ &\leq \epsilon\,||\xi||_\infty\int_\Omega |u_\epsilon|^{p_M(x)-1}|h(u_\epsilon)|dx\\ &\leq \epsilon\, C(h,||\xi||_\infty)\int_\Omega |u_\epsilon|^{p_M(x)-1}dx.\end{aligned}$$

Since $|u_\epsilon|^{p_M(\cdot)-1} \in L^{p'_M(\cdot)}(\Omega)$, (3.17) follows as $\epsilon \to 0$. ■

Now, we pass to the limit in $\beta_\epsilon(u_\epsilon)$. Since, for any $k > 0$, $(h_k(u_\epsilon)z_\epsilon)_{\epsilon>0}$ is bounded in $L^1(\Omega)$, there exists $z_k \in \mathcal{M}_b(\Omega)$ such that

$$h_k(u_\epsilon)\beta_\epsilon(u_\epsilon) \rightharpoonup z_k \text{ in } \mathcal{M}_b(\Omega) \text{ as } \epsilon \to 0.$$

Moreover, for any $\xi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega \xi dz_k = \int_\Omega \xi h_k(u)d\mu - \int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}\left[h_k(u)\xi\right]dx,$$

which implies that $z_k \in \mathcal{M}_b^{p(\cdot)}(\Omega)$ and, for any $k \leq l$,

$$z_k = z_l \text{ on } [|T_k(u)| < k].$$

Let us consider the Radon measure defined by

$$\begin{cases} z = z_k, \text{ on } [|T_k(u)| < k] \text{ for } k \in \mathbb{N}^*,\\ z = 0 \text{ on } \displaystyle\bigcap_{k\in\mathbb{N}^*}[|T_k(u)| = k]. \end{cases} \tag{3.18}$$

For any $h \in \mathcal{C}_c(\mathbb{R})$, $h(u) \in L^\infty(\Omega, d|z|)$ and

$$\int_\Omega h(u)\xi dz = -\int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i}\left[h(u)\xi\right] dx + \int_\Omega h(u)\xi d\mu,$$

for any $\xi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Indeed, let $k_0 > 0$ be such that $supp(h) \subseteq [-k_0, k_0]$,

$$\begin{aligned}
\int_\Omega h(u)\xi dz = \int_\Omega h(u)\xi dz_{k_0} &= -\lim_{\epsilon\to 0}\int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \frac{\partial}{\partial x_i}\left[h(u_\epsilon)\xi\right] dx \\
&- \lim_{\epsilon\to 0}\epsilon\int_\Omega |u_\epsilon|^{p_M(x)-2}u_\epsilon h(u_\epsilon)\xi dx + \lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\xi d\mu_\epsilon \\
&= -\lim_{\epsilon\to 0}\int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial T_{k_0}(u_\epsilon)}{\partial x_i}\right) \frac{\partial}{\partial x_i}\left[h(u_\epsilon)\xi\right] dx \\
&- \lim_{\epsilon\to 0}\epsilon\int_\Omega |u_\epsilon|^{p_M(x)-2}u_\epsilon h(u_\epsilon)\xi dx + \lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\xi d\mu_\epsilon \\
&= -\int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i}\left[h(u)\xi\right] dx + \int_\Omega h(u)\xi d\mu.
\end{aligned}$$

Moreover, we have the following lemma (see [22], Lemma 4.7).

Lemma 3.6. *The Radon-Nikodym decomposition of the measure z given by* (3.18) *with respect to $\mathcal{L}^N$,*

$$z = w\mathcal{L}^N + \nu \quad \textit{with } \nu \perp \mathcal{L}^N$$

satisfies the following properties

$$\begin{cases} w \in \beta(u)\mathcal{L}^N - a.e. \ in \ \Omega, \ w \in L^1(\Omega), \ \nu \in \mathcal{M}_b^{p_i(\cdot)}(\Omega), \\ \nu^+ \ is \ concentrated \ on \ [u = M], \\ \nu^- \ is \ concentrated \ on \ [u = m]. \end{cases}$$

To end the proof of the Theorem 1.1, we consider $\xi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $h \in C_c^1(\mathbb{R})$. Then, we take $h(u_\epsilon)\xi$ as test function in (3.2) to get

$$\begin{aligned}
\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \frac{\partial}{\partial x_i}\left[h(u_\epsilon)\xi\right] dx + \int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)\xi dx \\
+ \epsilon\int_\Omega |u_\epsilon|^{p_M(x)-2}u_\epsilon h(u_\epsilon)\xi dx = \int_\Omega h(u_\epsilon)\xi d\mu_\epsilon.
\end{aligned} \tag{3.19}$$

The first term of (3.19) can be written as

$$\sum_{i=1}^{N}\int_{\Omega} a_i\left(x, \frac{\partial T_{l_0+1}(u_\epsilon)}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h(u_\epsilon)\xi]dx = \sum_{i=1}^{N}\int_{\Omega} a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h(u_\epsilon)\xi]dx,$$

for $l_0 > 0$ so that, by lemmas 3.4 and 3.5, we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\Omega}\sum_{i=1}^{N} a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h(u_\epsilon)\xi]dx \\
&= \lim_{\epsilon\to 0}\int_{\Omega}\sum_{i=1}^{N} a_i\left(x, \frac{\partial T_{l_0+1}(u_\epsilon)}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h(u_\epsilon)\xi]dx \\
&= \int_{\Omega}\sum_{i=1}^{N} a_i\left(x, \frac{\partial T_{l_0+1}(u)}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h(u)\xi]dx \\
&= \int_{\Omega}\sum_{i=1}^{N} a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h(u)\xi]dx.
\end{aligned}$$

Using the convergence result of lemma 3.5, we pass to limit as $\epsilon \to 0$ in (3.19), to get

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{\Omega}\beta_\epsilon(u_\epsilon)h(u_\epsilon)\xi dx &= \int_{\Omega} h(u)\xi d\mu - \sum_{i=1}^{N}\int_{\Omega} a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h(u)\xi]dx \\
&= \int_{\Omega} h(u)\xi dz = \int_{\Omega} h(u)\xi w dx + \int_{\Omega} h(u)\xi d\nu.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h(u)\xi]dx + \int_{\Omega} w h(u)\xi dx \\
&\qquad + \int_{\Omega} h(u)\xi d\nu = \int_{\Omega} h(u)\xi d\mu.
\end{aligned} \tag{3.20}$$

Letting ϵ goes to 0 in (3.19), it yields that $(u,\ w)$ is a solution of the problem (1.1). To end the proof of Theorem 1.1, we prove (1.4). We take $\xi = T_1(u_\epsilon - T_n(u_\epsilon))$ as test function in (3.2).

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial}{\partial x_i}(T_1(u_\epsilon - T_n(u_\epsilon)))dx + \int_{\Omega}\beta_\epsilon(u_\epsilon)T_1(u_\epsilon - T_n(u_\epsilon))dx \\
&\quad + \epsilon\int_{\Omega}|u_\epsilon|^{P_M(x)-2}u_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx = \int_{\Omega} T_1(u_\epsilon - T_n(u_\epsilon))d\mu_\epsilon.
\end{aligned} \tag{3.21}$$

Since $\int_\Omega \beta_\epsilon(u_\epsilon)T_1(u_\epsilon - T_n(u_\epsilon))dx + \epsilon\int_\Omega |u_\epsilon|^{P_M(x)-2}u_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx \geq 0$ and $\frac{\partial}{\partial x_i}(T_1(u_\epsilon - T_n(u_\epsilon))) = \frac{\partial u_\epsilon}{\partial x_i}\chi_{[n<|u_\epsilon|<n+1]}$, we get from (3.21),

$$\sum_{i=1}^{N}\int_{[n<|u_\epsilon|<n+1]} a_i\Big(x, \frac{\partial u_\epsilon}{\partial x_i}\Big)\frac{\partial u_\epsilon}{\partial x_i}dx \leq \int_\Omega T_1(u_\epsilon - T_n(u_\epsilon))d\mu_\epsilon. \tag{3.22}$$

We have (see [22])

$$\lim_{n\to+\infty}\lim_{\epsilon\to 0}\int_\Omega T_1(u_\epsilon - T_n(u_\epsilon))d\mu_\epsilon = 0.$$

Then, using (2.4), and letting $n\to+\infty$, $\epsilon\to 0$ respectively in (3.22), we obtain

$$\begin{aligned}&\lim_{n\to+\infty}\lim_{\epsilon\to 0}\frac{1}{C}\sum_{i=1}^{N}\int_{[n<|u_\epsilon|<n+1]}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)}dx\\ &= \lim_{n\to+\infty}\frac{1}{C}\sum_{i=1}^{N}\int_{[n<|u|<n+1]}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx \leq 0.\end{aligned} \tag{3.23}$$

Therefore, we get (1.4).

4. Proof of Theorem 1.2 and Theorem 1.3

***Proof of Theorem* 1.2.** For $n > 0$, let h_n be the function defined on $\mathbb{R}$ by $h_n(r) = \inf\{1, (n+1-|r|)^+\}$. If (u, w) is a solution of (1.1) in the sense of Theorem 1.1, we take $h = h_n$ in (1.6). We have for any $\xi \in W^{1,\overrightarrow{p}(\cdot)}(\Omega)\cap L^\infty(\Omega)$,

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}[h_n(u)\xi]dx + \int_\Omega w h_n(u)\xi dx\\ &\qquad + \int_\Omega h_n(u)\xi d\nu = \int_\Omega h_n(u)\xi d\mu.\end{aligned} \tag{4.1}$$

Since $\frac{\partial}{\partial x_i}[h_n(u)\xi] = \xi h_n'(u)\frac{\partial u}{\partial x_i} + h_n(u)\frac{\partial \xi}{\partial x_i}$, we deduce from (4.1) that

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega h_n(u)a_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\frac{\partial \xi}{\partial x_i}dx + \sum_{i=1}^{N}\int_\Omega h_n'(u)\xi a_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\frac{\partial u}{\partial x_i}dx\\ &\qquad + \int_\Omega w h_n(u)\xi dx + \int_\Omega h_n(u)\xi d\nu = \int_\Omega h_n(u)\xi d\mu.\end{aligned} \tag{4.2}$$

We have $h_n(u) \to 1$ a.e. in $\mathbb{R}$ as $n \to +\infty$. By Lebesgue dominated convergence theorem, we pass to the limit in (4.2) as $n \to +\infty$ to get

$$\sum_{i=1}^{N} \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\frac{\partial \xi}{\partial x_i}dx + \lim_{n\to+\infty} \sup \sum_{i=1}^{N} \int_\Omega h_n'(u)\xi a_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\frac{\partial u}{\partial x_i}dx + \int_\Omega w\xi dx$$
$$+\int_\Omega w\xi dx + \int_\Omega \xi d\nu = \int_\Omega \xi d\mu. \tag{4.3}$$

For the second term in (4.3), we have

$$\begin{aligned}
&\Big|\sum_{i=1}^{N} \int_\Omega h_n'(u)\xi a_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\frac{\partial u}{\partial x_i}dx\Big| \\
&\le \sum_{i=1}^{N} \int_{[n\le|u|\le n+1]} \Big|h_n'(u)\xi a_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\frac{\partial u}{\partial x_i}\Big|dx \\
&\le ||\xi||_\infty \sum_{i=1}^{N} \int_{[n\le|u|\le n+1]} \Big|a_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\frac{\partial u}{\partial x_i}\Big|dx \\
&\le C_1||\xi||_\infty \sum_{i=1}^{N} \int_{[n\le|u|\le n+1]} \Big(j_i(x)\Big|\frac{\partial u}{\partial x_i}\Big| + \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)}\Big)dx \\
&\le C_1||\xi||_\infty \sum_{i=1}^{N} \Big(\int_{[n\le|u|\le n+1]} j_i(x)\Big|\frac{\partial u}{\partial x_i}\Big|dx + \int_{[n\le|u|\le n+1]} \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)}dx\Big).
\end{aligned}$$

According to (1.4), we have $\lim_{n\to+\infty} \int_{[n\le|u|\le n+1]} |\frac{\partial u}{\partial x_i}|^{p_i(x)}dx = 0$. By Young inequality, we have for any $x \in \{n \le |u| \le n+1\}$,

$$\begin{aligned}
&\int_{[n\le|u|\le n+1]} j_i(x)\Big|\frac{\partial u}{\partial x_i}\Big|dx \\
&\le \int_\Omega \frac{1}{p_i'(x)}(j_i(x)^{p_i'(x)} 1_{\{n\le|u|\le n+1\}}dx + \int_\Omega \frac{1}{p_i(x)}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)} 1_{\{n\le|u|\le n+1\}}dx \qquad (4.4)\\
&\le \int_\Omega \frac{1}{(p_i')^-}(j_i(x)^{p_i'(x)} 1_{\{n\le|u|\le n+1\}}dx + \int_\Omega \frac{1}{(p_i)^-}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)} 1_{\{n\le|u|\le n+1\}}dx.
\end{aligned}$$

Using relation (1.4) and the Lebesgue dominated convergence Theorem in (4.4) respectively, we get

$$\lim_{n\to+\infty} \int_\Omega \frac{1}{(p_i')^-}(j_i(x)^{p_i'(x)} 1_{\{n\le|u|\le n+1\}}dx = 0$$

and

$$\lim_{n\to+\infty}\int_\Omega \frac{1}{(p_i)^-}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} 1_{\{n\le|u|\le n+1\}}dx = 0.$$

Then, we set $\xi = T_k(u-\xi)$ in (4.3) to get

$$\begin{aligned}\sum_{i=1}^N \int_\Omega a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}T_k(u-\xi)dx + \int_\Omega wT_k(u-\xi)dx \\ + \int_\Omega T_k(u-\xi)d\nu = \int_\Omega T_k(u-\xi)d\mu.\end{aligned} \tag{4.5}$$

Since $\xi \in dom(\beta)$, we have

$$\begin{aligned}\int_\Omega T_k(u-\xi)d\nu &= \int_{[u=M]} T_k(u-\xi)d\nu^+ - \int_{[u=m]} T_k(u-\xi)d\nu^- \\ &= \int_{[u=M]} T_k(M-\xi)d\nu^+ - \int_{[u=M]} T_k(m-\xi)d\nu^- \ge 0.\end{aligned}$$

Then, from (4.5), (1.5) follows. ∎

***Proof of Theorem* 1.3.** The existence part of Theorem 1.3 follows from Theorem 1.1 and the fact that u is bounded. The proof of the uniqueness follows from the entropy formulation of the solution. We now prove the uniqueness of the solution. Suppose that (u_1, w_1), (u_2, w_2) are two solutions of (1.1). For u_1 we choose $\xi = u_2$ as test function in (1.5) to get

$$\sum_{i=1}^N \int_\Omega a_i\Big(x,\frac{\partial u_1}{\partial x_i}\Big)\frac{\partial}{\partial x_i}T_k(u_1-u_2)dx + \int_\Omega w_1T_k(u_1-u_2)dx \le \int_\Omega T_k(u_1-u_2)d\mu.$$

Similarly we get for u_2 by taking $\xi = u_1$ as test function in (1.5),

$$\sum_{i=1}^N \int_\Omega a_i\Big(x,\frac{\partial u_2}{\partial x_i}\Big)\frac{\partial}{\partial x_i}T_k(u_2-u_1)dx + \int_\Omega w_2T_k(u_2-u_1)dx \le \int_\Omega T_k(u_2-u_1)d\mu.$$

Adding these two last inequalities yields

$$\begin{aligned}\int_\Omega \sum_{i=1}^N \Big(a_i\Big(x,\frac{\partial u_1}{\partial x_i}\Big) - a_i\Big(x,\frac{\partial u_2}{\partial x_i}\Big)\Big)\frac{\partial}{\partial x_i}T_k(u_1-u_2)dx \\ + \int_\Omega (w_1-w_2)T_k(u_1-u_2)dx \le 0.\end{aligned} \tag{4.6}$$

Since $a_i(x,.)$ and ∂j are monotones, for any $k>0$, it follows from (4.6) that

$$\int_\Omega \sum_{i=1}^N \left(a_i\left(x, \frac{\partial u_1}{\partial x_i}\right) - a_i\left(x, \frac{\partial u_2}{\partial x_i}\right)\right)\frac{\partial}{\partial x_i}T_k(u_1-u_2)dx = 0. \tag{4.7}$$

From (4.7), it follows that there exists a constant c such that $u_1 - u_2 = c$ a.e. in Ω. At last, let us show that $w_1 = w_2$ a.e. in Ω and $\nu_1 = \nu_2$. Indeed for any $\varphi \in \mathcal{D}(\Omega)$, taking φ as test function in (1.6) for the solutions (u_1, w_1) and (u_2, w_2), after substracting of these equalities we get

$$\int_\Omega (w_1 - w_2)\varphi dx + \int_\Omega \varphi d(\nu_1 - \nu_2) = 0.$$

Hence

$$\int_\Omega w_1\varphi dx + \int_\Omega \varphi d\nu_1 = \int_\Omega w_2\varphi dx + \int_\Omega \varphi d\nu_2.$$

Therefore

$$w_1\mathcal{L}^N + \nu_1 = w_2\mathcal{L}^N + \nu_2.$$

Since the Radon-Nikodym decomposition of a measure is unique, we get $w_1 = w_2$ a.e. in Ω and $\nu_1 = \nu_2$.

To end the proof of Theorem 1.3, let us prove that (1.7) and (1.8) hold. To achieve this aim we use the following lemma (see [23] for the proof).

Lemma 4.1. *Let $\eta \in W^{1,p_m(\cdot)}(\Omega)$, $Z \in \mathcal{M}_b^{p_m(\cdot)}(\Omega)$ and $\lambda \in \mathbb{R}$ be such that*

$$\begin{cases} \eta \le \lambda \text{ a.e. in } \Omega \text{ (resp. } \eta \ge \lambda) \\ Z = -\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \text{ in } \mathcal{D}'(\Omega). \end{cases} \tag{4.8}$$

Then

$$\int_{[\eta=\lambda]} \xi dZ \ge 0, \tag{4.9}$$

(*resp.*)

$$\int_{[\eta=\lambda]} \xi dZ \le 0, \tag{4.10}$$

for any $\xi \in C_c^1(\Omega)$, $\xi \ge 0$.

By the Lemma 4.1, for any $\xi \in C_c^1(\Omega), \xi \geq 0$, we have

$$\int_{[u=M]} \xi d\nu^+ \leq \int_{[u=M]} \xi d\mu - \int_{[u=M]} \xi w dx$$

and

$$\int_{[u=m]} \xi d\nu^- \leq - \int_{[u=m]} \xi d\mu + \int_{[u=m]} \xi w dx.$$

The first inequality implies that

$$\int_\Omega \xi d\nu^+ \leq \int_\Omega \xi d\mu \lfloor [u = M] - \int_\Omega \xi w \chi_{[u=M]} dx.$$

Consequently (1.7) holds. Similarly we get (1.8). ∎

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