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RANDOM INTEGRAL GUIDING FUNCTIONS WITH APPLICATION TO RANDOM DIFFERENTIAL COMPLEMENTARITY SYSTEMS

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Abstract

By applying the random topological degree we develop the methods of random smooth and nonsmooth integral guiding functions and use them for the study of random differential inclusions in finite dimensional spaces.

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Some existence theorems of random periodic solutions are presented. It is shown how the abstract results can be applied to study the random differential complementarity systems arising, in particular, from random survival models.

Keywords: random differential inclusion, random periodic solution, random differential complementarity system, random guiding function, random topological degree.

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1. Introduction

Let (Ω, Σ, μ) be a complete probability space and $I = [0, T]$. In this paper we consider the periodic problem for a random differential inclusion of the form:

$$\begin{cases} x'(\omega, t) \in F\big(\omega, t, x(\omega, t)\big), & \text{for a.e. } \ t \in I, \\ x(\omega, 0) = x(\omega, T), \end{cases} \tag{1.1}$$

for all $\omega \in \Omega$, where $F \colon \Omega \times I \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a given multivalued map.

For the study of problem (1.1) we present the method of random integral guiding functions. Let us mention that the method of guiding functions was developed by Krasnoselskii and Perov (see, e.g., [29, 30]) for the investigation of periodic oscillations in dynamical systems governed by differential equations. The notion of guiding function was then generalized in several directions and applied to various problems. Among a large number of papers on this subject let us recall: the paper of Mawhin [35] considered periodic solutions of functional differential equations; the work of Fonda [15], in which the notion of integral guiding functions was introduced and the papers [26]–[28] where this notion was essentially developeed and applied to periodic and asymptotic problems. Notice also the paper of Capietto and Zanolin [8] concerned with the periodic problem in flow-invariant Euclidean Neighborhood Retracts; the work of Górniewicz and Plaskacz [19] in which the notion of general form of guiding functions for differential inclusions was presented (see also [5, 18]); the paper of De Blasi, Górniewicz and Pianigiani [12] where the notion of a nonsmooth guiding potential for differential inclusions was introduced; the paper of Loi [33] for the method of guiding functions in infinite dimensional Hilbert spaces; the works of Kryszewski [31] and Kryszewski and Gabor [16] for the application of the method of guiding functions to bifurcation problems. The backgrounds and applications of the method of guiding functions in nonlinear analysis can be found in the recent monograph [36].

The random topological degree theory for various classes of multivalued maps was presented in the monograph of Górniewicz [18] and in the paper of Andres and

Górniewicz [2]. In the paper [2] this theory was applied to develop the method of random guiding functions for the study of random periodic solutions of random differential inclusions in finite dimensional spaces. In the present paper, based on the approach given in [2] we introduce the notions of smooth and nonsmooth random integral guiding functions and use them to prove some existence theorems of random periodic solutions to problem (1.1).

The paper is organized in the following way. In the next section we recall some notions from theory of linear Fredholm operators, multivalued analysis and theory of random topological degree. The notions of random integral guiding functions and of random nonsmooth integral guiding functions are presented in Section 3 and 4, respectively. The main results are Theorems 13 and 16. As an application, in the last section we study a random differential complementarity system of the form:

$$\begin{cases} x'(\omega,t) \in \mathfrak{F}\big(\omega,t,x(\omega,t),u(\omega,t)\big) & \text{for a.e. } t\in I,\\ K \ni u(\omega,t) \perp \mathfrak{g}(\omega,t,x(\omega,t)) + \mathfrak{f}(\omega,u(\omega,t)) \in K^*, & \text{for a.e. } t\in I,\\ x(\omega,0) = x(\omega,T), \end{cases} \tag{1.2}$$

for all $\omega\in\Omega$, where $K\subset\mathbb{R}^m$ is a closed convex cone, K^* denotes the dual cone of K, $\mathfrak{F}\colon \Omega\times I\times\mathbb{R}^n\times\mathbb{R}^m \multimap \mathbb{R}^n$, $\mathfrak{g}\colon\Omega\times I\times\mathbb{R}^n\to\mathbb{R}^m$ and $\mathfrak{f}\colon\Omega\times\mathbb{R}^m\to\mathbb{R}^m$ are given maps. Let us mention also that the differential complementarity systems were introduced by C. Henry [21] as the projected differential inclusions and they arose in many applied problems, for example, mechanical impact problem, electrical circuits with ideal diodes, Coulomb friction problems for contacting bodies, economical dynamics and related problems such as dynamic traffic network. Some results concerning the differential complementarity systems can be found in [1, 6, 7, 20, 22, 34, 38]. However, so far as we know, the random differential complementarity system (1.2) is still a new problem. In this paper, by applying the abstract result given in Section 3 we will present sufficient conditions under which problem (1.2) has a random solution. Examples arising, in particular, from random survival models are given.

2. Preliminaries and notation

2.1. Fredholm operators

Let X,Y be Banach spaces. At first, let us recall some notions from the theory of linear Fredholm operators (see, e.g., [17]).

A linear bounded operator $L\colon domL\subseteq X\to Y$ is said to be a linear Fredholm operator of index zero if

(i) $Im\,L$ is a closed subset of Y;

(ii) The spaces $Ker\,L$ and $Coker\,L$ are finite-dimensional and

$$dimKer\,L = dimCoker\,L.$$

For every linear Fredholm operator of zero index $L\colon domL \subseteq X \to Y$ there exist projections $P_L\colon X \to X$ and $Q_L\colon Y \to Y$ such that $Im\,P_L = KerL$ and $Ker\,Q_L = ImL$. If we define the operator

$$L_{P_L}\colon domL \cap KerP_L \to ImL$$

as the restriction of L on $domL \cap Ker\,P_L$, then L_{P_L} is a linear isomorphism and we can define the operator $K_{P_L}\colon ImL \to domL$ as $K_{P_L} = L_{P_L}^{-1}$. Now let $CokerL = Y/ImL$; $\Pi_L\colon Y \to CokerL$ be a canonical operator projection

$$\Pi_L(z) = z + ImL$$

and $\Lambda_L\colon CokerL \to KerL$ be a linear continuous isomorphism. Then the equation

$$Lx = y,\ y \in Y$$

is equivalent to the following one:

$$(i - P_L)x = (\Lambda_L \Pi_L + K_L)y,$$

where i is the inclusion map and $K_L\colon Y \to X$ is defined as

$$K_L = K_{P_L}(i - Q_L).$$

2.2. Multimaps and random topological degree

We recall now some notions of the theory of multivalued maps that will be used in the sequel (details can be found, for example, in [5, 18, 23, 24, 25]).

Let X, Y be metric spaces. Denote

$$\begin{aligned} P(Y) &= \{\mathcal{M} \subset Y\colon \mathcal{M} \neq \emptyset\},\\ C(Y) &= \{\mathcal{M} \in P(Y)\colon \mathcal{M} \text{ is closed}\},\\ K(Y) &= \{\mathcal{M} \in P(Y)\colon \mathcal{M} \text{ is compact}\}. \end{aligned}$$

Definition 1. A multivalued map (multimap) $\mathcal{F}\colon X \to P(Y)$ is said to be:

(i) upper semicontinuous (u.s.c) if $\mathcal{F}^{-1}(V) = \{y \in X\colon \mathcal{F}(y) \cap W \neq \emptyset\}$ is a closed subset of Y for every closed set $W \subset Y$;

(ii) lower semicontinuous (l.s.c) if $\mathcal{F}^{-1}(V) = \{y \in X\colon \mathcal{F}(y) \cap V \neq \emptyset\}$ is an open subset of Y for every open set $V \subset Y$;

(iii) continuous if it is both u.s.c. and l.s.c.;

(iv) closed if its graph $\Gamma_{\mathcal{F}} = \{(y,z) \colon z \in \mathcal{F}(y)\}$ is a closed subset of $X \times Y$;

(v) compact if the set $\overline{\mathcal{F}(X)}$ is compact in Y;

(vi) completely u.s.c. if $\mathcal{F}$ is u.s.c. and the set $\mathcal{F}(U)$ is relatively compact in Y for each bounded set $U \subset X$.

Definition 2 (see [2])**.** Multimap $\mathcal{F} \colon \Omega \times X \to C(Y)$ is called a *random multioperator* if it is product-measurable (see, e.g., [9]), i.e., measurable w.r.t. $\Sigma \otimes \mathbb{B}(X)$, where $\Sigma \otimes \mathbb{B}(X)$ is the smallest σ-algebra on $\Omega \times X$ which contains all the sets $A \times B$, where $A \in \Sigma$ and $B \in \mathbb{B}(X)$ and $\mathbb{B}(X)$ denotes the Borel σ-algebra on X. If, moreover, $\mathcal{F}(\omega, \cdot) \colon X \to C(Y)$ is u.s.c. for all $\omega \in \Omega$, then $\mathcal{F}$ is called a random u-multioperator. In case $\mathcal{F}(\omega, \cdot)$ is continuous for all $\omega \in \Omega$, we will call it a random c-multioperator.

Remark 3. The notion of a single-valued random operator $f : \Omega \times X \to Y$ can be defined analogously. If, additionally, $f(\omega, \cdot) \colon X \to Y$ is continuous for all $\omega \in \Omega$, then f is called a random c-operator.

Definition 4 (see [2])**.** Let $A \subset Y$ be a closed subset and $\mathcal{F} \colon \Omega \times A \to P(Y)$ a random multioperator. *A random fixed point* ξ of $\mathcal{F}$ is a measurable map $\xi \colon \Omega \to A$ such that

$$\xi(\omega) \in \mathcal{F}(\omega, \xi(\omega)), \ \forall \omega \in \Omega.$$

Theorem 5 (see [2])**.** *Let Y be a separable Banach space, $A \subset Y$ a closed subset and $\mathcal{F} \colon \Omega \times Y \to C(Y)$ a random multioperator. If for each $\omega \in \Omega$ the set*

$$Fix\mathcal{F}_\omega := \{x \in Y \colon x \in \mathcal{F}(\omega, x)\}$$

of fixed points of $\mathcal{F}_\omega = \mathcal{F}(\omega, \cdot)$ is nonempty and closed then $\mathcal{F}$ has a random fixed point.

Definition 6. A multimap $\mathcal{F} \colon \Omega \times X \to K(Y)$ is said to be:

(a) *a random compact u-multioperator* if it is a random u-multioperator and for each $\omega \in \Omega$ the multimap $\mathcal{F}(\omega, \cdot) \colon X \to K(Y)$ is compact;

(b) *a random completely u-multioperator* if it is a random u-multioperator and for each $\omega \in \Omega$ the multimap $\mathcal{F}(\omega, \cdot) \colon X \to K(Y)$ is completely u.s.c..

Now, let Y be a separable Banach space, $Kv(Y)$ denote a collection of all nonempty compact convex subsets of Y, $U \subset Y$ an open bounded subset and $\mathcal{F} \colon \Omega \times \overline{U} \to Kv(Y)$ a random compact u-multioperator such that $x \notin \mathcal{F}(\omega, x)$ for all $x \in \partial U$ and for all $\omega \in \Omega$, where ∂U denotes the boundary of U. Then for each $\omega \in \Omega$ the topological degree of the corresponding multivalued vector field $\deg\big(i - \mathcal{F}(\omega, \cdot), \overline{U}\big)$ is well defined (see, e.g., [3, 5, 18, 23, 24, 31]. The random topological degree of $i - \mathcal{F}$ on $\overline{U}$ is defined as follows (see [2]):

$$D(i - \mathcal{F}, \overline{U}) := \Big\{ \deg\big(i - \mathcal{F}(\omega, \cdot), \overline{U}\big) \mid \omega \in \Omega \Big\}.$$

By $D(i - \mathcal{F}, \overline{U}) \neq 0$ we mean that $\deg\big(i - \mathcal{F}(\omega, \cdot), \overline{U}\big) \neq 0$ for all $\omega \in \Omega$.

Theorem 7 (see [2])**.** *If $D(i - \mathcal{F}, \overline{U}) \neq 0$, then $\mathcal{F}$ has a random fixed point in U, i.e., there exists a measurable function $\xi \colon \Omega \to U$ such that $\xi(\omega) \in \mathcal{F}(\omega, \xi(\omega))$ for all $\omega \in \Omega$.*

2.3. Complementarity problem

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$; $K \subset H$ be a closed convex subset. Let us recall that a map $g \colon K \to H$ is said to be:

(a) *continuous on finite-dimensional subspaces* if it is continuous on $K \cap H'$ for every finite-dimensional subspace H' of H with $K \cap H' \neq \emptyset$. (Notice that every continuous map from K to H is continuous on finite-dimensional subspaces)

(b) *pseudo-monotone* if for any $x, y \in K$,

$$\big\langle x - y, g(y) \big\rangle_H \geq 0 \text{ implies } \big\langle x - y, g(x) \big\rangle_H \geq 0.$$

(c) *monotone* if

$$\big\langle x - y, g(x) - g(y) \big\rangle_H \geq 0 \text{ for all } x, y \in K.$$

If the above inequality is strict whenever x and y are distinct, then g is said to be *strictly monotone.*

(d) *strongly monotone* if there exists $k > 0$ such that

$$\big\langle x - y, g(x) - g(y) \big\rangle_H \geq k \|x - y\|_H^2 \text{ for all } x, y \in K.$$

Lemma 8 (see [11, Corollary 3.1])**.** *Let K be a closed convex cone in H and $g \colon K \to H$ be a strongly monotone map which is continuous on finite-dimensional subspaces. Then there exists a unique $x^* \in K$ such that*

$$\big\langle u - x^*, g(x^*) \big\rangle_H = 0 \quad \textit{for all} \quad u \in K.$$

2.4. Notation

For simplicity, we will use the same notation $|\cdot|$ [$\langle \cdot, \cdot \rangle$] to denote the norm [resp., the inner product] in finite-dimensional spaces. By $C(I, \mathbb{R}^n)$ [$L^p(I, \mathbb{R}^n)$ ($p \geq 1$)] we denote the spaces of all continuous [respectively, p-summable] functions $u \colon I \to \mathbb{R}^n$ with usual norms:

$$\|u\|_C = \max_{t \in I} |u(t)| \text{ and } \|u\|_p = \left(\int_0^T |u(t)|^p dt \right)^{\frac{1}{p}}.$$

Consider the space of all absolutely continuous functions $u\colon I \to \mathbb{R}^n$ whose derivatives belong to $L^p(I, \mathbb{R}^n)$. It is known (see, e.g., [4]) that this space can be identified with the Sobolev space $W^{1,p}(I, \mathbb{R}^n)$ with the norm

$$\|u\|_W = \left(\|u\|_p^p + \|u'\|_p^p\right)^{\frac{1}{p}}.$$

By $W_T^{1,p}(I, \mathbb{R}^n)$ we denote the space of all functions $x \in W^{1,p}(I, \mathbb{R}^n)$ satisfying the boundary condition of periodicity $x(0) = x(T)$. Recall that (see, e.g., [4]) the embedding $W^{1,2}(I, \mathbb{R}^n) \hookrightarrow C(I, \mathbb{R}^n)$ is compact. The symbols $B_C(0, r)$ $[B_{\mathbb{R}^n}(0, r)]$ denote the closed ball of radius r centered at 0 in the space $C(I, \mathbb{R}^n)$ [respectively, $\mathbb{R}^n$].

3. RANDOM SMOOTH INTEGRAL GUIDING FUNCTIONS

Now, let us consider problem (1.1). Assume that the following hypotheses hold true.

$(F1)$ $F\colon \Omega \times I \times \mathbb{R}^n \to Kv(\mathbb{R}^n)$ is a random u-multioperator;

$(F2)$ there exists $c > 0$ such that for every $\omega \in \Omega$:

$$\|F(\omega, t, y)\| := \sup\{|z|\colon z \in F(\omega, t, y)\} \le c(1 + |y|), \quad \forall y \in \mathbb{R}^n$$

for a.e. $t \in I$.

By a *random solution* of (1.1) we mean a function $\xi\colon \Omega \times I \to \mathbb{R}^n$ such that

(1) the map $\omega \in \Omega \to \xi(\omega, \cdot) \in C(I, \mathbb{R}^n)$ is measurable;

(2) for each $\omega \in \Omega$ the function $\xi(\omega, \cdot)$ is in $W^{1,2}(I, \mathbb{R}^n)$ and satisfies

$$\begin{cases} \xi'(\omega, t) \in F\big(\omega, t, \xi(\omega, t)\big), \\ \xi(\omega, 0) = \xi(\omega, T), \end{cases}$$

for a.e. $t \in I$.

From (F1)–(F2) it follows that the superposition multioperator

$$\mathcal{P}_F\colon \Omega \times C(I, \mathbb{R}^n) \to P(L^2(I, \mathbb{R}^n)),$$

$$\mathcal{P}_F(\omega, x) = \left\{f \in L^2(I, \mathbb{R}^n)\colon f(s) \in F(\omega, s, x(s)), \quad \text{for a.e. } \ s \in I\right\},$$

is well defined. Moreover, for each $\omega \in \Omega$ the multimap $\mathcal{P}_F(\omega, \cdot)\colon C(I, \mathbb{R}^n) \to C(L^2(I, \mathbb{R}^n))$ is closed (see, e.g., [3, 5, 18, 24]).

Definition 9 (see [2, Definition 5.3])**.** A map $V\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ is called a *random potential* if the following two conditions are satisfied:

(i) $V(\cdot,x)\colon \Omega\to\mathbb{R}$ is measurable for every $x\in\mathbb{R}^n$;

(ii) $V(\omega,\cdot)\colon \mathbb{R}^n\to\mathbb{R}$ is a C^1-map for every $\omega\in\Omega$.

Definition 10 (see [2, Definition 5.4])**.** A random potential V is called a *random direct potential* if there exists $R_0>0$ such that

$$\nabla V(\omega,z)=\left(\frac{\partial V(\omega,z)}{\partial z_1},\cdots,\frac{\partial V(\omega,z)}{\partial z_n}\right)\neq 0$$

for all $(\omega,z)\in\Omega\times\mathbb{R}^n\colon |z|\geq R_0$.

From the above definition it follows that for a fixed $\omega\in\Omega$ the topological degree

$$\deg\big(\nabla V(\omega,\cdot),B_{\mathbb{R}^n}(0,R)\big)$$

is well-defined for all $R\geq R_0$ and it is nothing but $\deg\big(\nabla V(\omega,\cdot),B_{\mathbb{R}^n}(0,R_0)\big)$.

By a *random index ind$V$* of the random direct potential V we mean the random topological degree $D\big(\nabla V,B_{\mathbb{R}^n}(0,R_0)\big)$.

Definition 11. A random potential $V\colon\Omega\times\mathbb{R}^n\to\mathbb{R}^n$ is said to be a random integral guiding function for problem (1.1) if there exists $N>0$ such that for all $\omega\in\Omega$ for $x\in C(I,\mathbb{R}^n)$ with $\|x\|_2\geq N$ it follows that

$$\int_0^T\big\langle\nabla V(\omega,x(s)),f(s)\big\rangle dt>0\quad\forall f\in\mathcal{P}_F(\omega,x).$$

It is easy to verify the following assertion.

Lemma 12. *If $V\colon\Omega\times\mathbb{R}^n\to\mathbb{R}^n$ is a random integral guiding function for problem* (1.1)*, then it is a random direct potential, and hence there exists its random index ind$V$.*

Now we are in position to prove the main result of this section.

Theorem 13. *Let conditions* (F1)–(F2) *hold. If there exists a random integral guiding function V for problem* (1.1) *such that ind$V\neq0$, then problem* (1.1) *has a random solution.*

Proof. Define the operator $L\colon W_T^{1,2}(I,\mathbb{R}^n)\to L^2(I,\mathbb{R}^n)$, $Lx=x'$. It is well known (see, e.g., [17]) that L is a linear Fredholm operator of index zero and

$$Ker\,L\cong\mathbb{R}^n\cong Coker\,L.$$

The projection

$$\Pi_L\colon L^2(I,\mathbb{R}^n)\to\mathbb{R}^n,$$

is defined as

$$\Pi_L g = \frac{1}{T}\int_0^T g(s)\,ds$$

and the homeomorphism $\Lambda_L\colon \mathbb{R}^n \to \mathbb{R}^n$ is the identity operator. The space $L^2(I,\mathbb{R}^n)$ can be represented as

$$L^2(I,\mathbb{R}^n) = \mathcal{L}_0 \oplus \mathcal{L}_1,$$

where $\mathcal{L}_0 = Coker\, L$ and $\mathcal{L}_1 = Im\, L$.

The decomposition of an element $g \in L^2(I,\mathbb{R}^n)$ is denoted by

$$g = g^{(0)} + g^{(1)}, g^{(0)} \in \mathcal{L}_0, g^{(1)} \in \mathcal{L}_1.$$

Problem (1.1) can be substituted with the following operator inclusion

$$Lx \in \mathcal{P}_F(\omega, x),$$

or equivalently, by the fixed point problem

$$x \in \Gamma(\omega, x), \tag{3.1}$$

where the multimap $\Gamma\colon \Omega \times C(I,\mathbb{R}^n) \multimap C(I,\mathbb{R}^n)$ is defined as

$$\Gamma(\omega, x) = P_L x + (\Pi_L + K_L) \circ \mathcal{P}_F(\omega, x).$$

Let us show that Γ is measurable. In fact, it is sufficient to prove that $\mathcal{P}_F$ is measurable. To do this, notice that from $(F1)$ it follows that for a given $(\omega, x) \in \Omega \times C(I,\mathbb{R}^n)$ the multifunction $t \in I \multimap F(\omega, t, x(t))$ is measurable. Now, let us fix an arbitrary $g \in L^2(I,\mathbb{R}^n)$ and define the function

$$h_g\colon \Omega \times C(I,\mathbb{R}^n) \to [0,\infty),$$
$$h_g(\omega, x) = dist_{L^2(I,\mathbb{R}^n)}\big(g, \mathcal{P}_F(\omega, x)\big).$$

Applying [13], Proposition 3.4 (b) we conclude that

$$h_g(\omega, x) = \left(\int_0^T dist^2_{\mathbb{R}^n}(g(s), F(\omega, s, x(s)))ds\right)^{1/2}.$$

From the Fubini theorem it follows that the map h_g is measurable and hence (see, e.g., [9], Ch.III or [5], Theorem 1.5.6) the multimap $\mathcal{P}_F$ is measurable.

Now, for every $\omega \in \Omega$ let us prove that $\Gamma(\omega, \cdot)\colon C(I,\mathbb{R}^n) \multimap C(I,\mathbb{R}^n)$ is a completely u.s.c. multimap with compact, convex values. Indeed, from the fact that the operator $\Pi_L + K_L$ is linear and continuous it follows that the multimap $(\Pi_L + K_L) \circ \mathcal{P}_F(\omega, \cdot)$ is closed (see, e.g., Theorem 1.5.30 [5] or Corollary 5.1.2

[24]). Further, from $(F2)$ it follows that for every bounded subset $U \subset C(I,\mathbb{R}^n)$ the set $(\Pi_L + K_L) \circ \mathcal{P}_F(\omega, U)$ is bounded in $W_T^{1,2}(I,\mathbb{R}^n)$, and by the Sobolev embedding theorem (see, e.g., [4]) it is a relatively compact subset of $C(I,\mathbb{R}^n)$. Therefore, the multimap $(\Pi_L + K_L) \circ \mathcal{P}_F(\omega, \cdot)$ is completely u.s.c. and now the assertion follows from the fact that P_L is continuous and has a finite-dimensional range. So, Γ is a random completely u-multioperator.

Now, fix $\omega \in \Omega$ and assume that $x_\omega \in C(I,\mathbb{R}^n)$ is a solution to the inclusion (3.1). Then there exist $f_\omega \in \mathcal{P}_F(\omega, x_\omega)$ such that

$$\begin{cases} x'_\omega(t) = f_\omega(t), \quad \text{for a.e. } t \in I, \\ x_\omega(0) = x_\omega(T). \end{cases}$$

Therefore,

$$\int_0^T \big\langle \nabla V(\omega, x_\omega(t)), f_\omega(t) \big\rangle dt = \int_0^T \big\langle \nabla V(\omega, x_\omega(t)), x'_\omega(t) \big\rangle dt = 0.$$

Consequently, $\|x_\omega\|_2 < N$. From $(F2)$ it follows that there exists $M > 0$ such that $\|x'_\omega\|_2 < M$. So, we can choose $R_1 > 0$ which does not depend on ω such that $\|x_\omega\|_C < R_1$. Let $R = R_1 + 1$. Then for any $\omega \in \Omega$ inclusion (3.1) has no solutions on $\partial B_C(0,R)$. Therefore, the topological degree $\deg(i - \Gamma(\omega,\cdot), B_C(0,R))$ is well-defined for every $\omega \in \Omega$. To evaluate this characteristic, we consider the following family of multimaps

$$\Psi_\omega \colon B_C(0,R) \times [0,1] \to Kv(C(I,\mathbb{R}^n)),$$
$$\Psi_\omega(x,\eta) = P_L x + (\Pi_L + K_L) \circ \varphi\big(\mathcal{P}_F(\omega,x), \eta\big),$$

where the map $\varphi \colon L^2(I,\mathbb{R}^n) \times [0,1] \to L^2(I,\mathbb{R}^n)$ is defined as

$$\varphi(g,\eta) = g^{(0)} + \eta g^{(1)}, \tag{3.2}$$

when $g^{(0)} \in \mathcal{L}_0$, $g^{(1)} \in \mathcal{L}_1$ and $g = g^{(0)} + g^{(1)}$. It is easy to verify that Ψ is a compact u.s.c. multimap. Let us show that

$$x \notin \Psi_\omega(x,\eta)$$

for all $(x,\eta) \in \partial B_C(0,R) \times [0,1]$. To the contrary, assume that there is $(x_*, \eta_*) \in \partial B_C(0,R) \times [0,1]$ such that $x_* \in \Psi_\omega(x_*, \eta_*)$. Then there exist $f_* \in \mathcal{P}_F(\omega, x_*)$ such that

$$\begin{cases} x'_*(t) = \varphi\big(f_*, \eta_*\big)(t) \text{ for a.e. } t \in I, \\ x_*(0) = x_*(T), \end{cases}$$

or equivalently,

$$\begin{cases} x'_* = \eta_* f_*^{(1)}, \\ 0 = f_*^{(0)}, \end{cases}$$

where $f_*^{(0)} + f_*^{(1)} = f_*$, $f_*^{(0)} \in \mathcal{L}_0$, $f_*^{(1)} \in \mathcal{L}_1$. If $\eta_* \neq 0$, then

$$\int_0^T \big\langle \nabla V(\omega, x_*(t)), f_*(t) \big\rangle dt = \frac{1}{\eta_*} \int_0^T \big\langle \nabla V(\omega, x_*(t)), x_*'(t) \big\rangle dt = 0.$$

Therefore, $\|x_*\|_2 < N$, and hence $\|x_*\|_C \leq R_1 < R$, giving a contradiction.

If $\eta_* = 0$, then $x_* \in Ker\, L$, i.e., $x_*(t) \equiv z \in \mathbb{R}^n$ for all $t \in I$. Since $\|z\|_2 > N$ we have

$$\int_0^T \big\langle \nabla V(\omega, z), \gamma(t) \big\rangle dt = T \big\langle \nabla V(\omega, z), \Pi_L \gamma \big\rangle > 0, \tag{3.3}$$

for all $\gamma \in \mathcal{P}_F(\omega, z)$. In particular,

$$0 < \big\langle \nabla V(\omega, z), \Pi_L f_* \big\rangle = \big\langle \nabla V(\omega, z), \Pi_L f_*^{(0)} \big\rangle = 0,$$

that is a contradiction.

Thus, Ψ_ω is a homotopy connecting the multimaps $\Psi_\omega(\cdot, 1) = \Gamma(\omega, \cdot)$ and

$$\Psi_\omega(\cdot, 0) = P_L + \Pi_L \circ \mathcal{P}_F(\omega, \cdot).$$

By virtue of the homotopy invariant property of the topological degree we have

$$\deg(i - \Gamma(\omega, \cdot), B_C(0, R)) = \deg(i - P_L - \Pi_L \mathcal{P}_F(\omega, \cdot), B_C(0, R)).$$

Notice that the multimap $P_L + \Pi_L \mathcal{P}_F(\omega, \cdot)$ takes values in $\mathbb{R}^n$, and hence, by the Map Restriction Property (see, e.g., [5, 24]):

$$\deg(i - P_L - \Pi_L \mathcal{P}_F(\omega, \cdot), B_C(0, R)) = \deg(i - P_L - \Pi_L \mathcal{P}_F(\omega, \cdot), B_{\mathbb{R}^n}(0, R)).$$

In the space $\mathbb{R}^n$ the vector multifield $i - P_L - \Pi_L \mathcal{P}_F(\omega, \cdot)$ has the form:

$$i - P_L - \Pi_L \mathcal{P}_F(\omega, \cdot) = -\Pi_L \mathcal{P}_F(\omega, \cdot).$$

From (3.3) it easily follows that $-\Pi_L \mathcal{P}_F(\omega, \cdot)$ and $-\nabla V(\omega, \cdot)$ are homotopic on $\partial B_{\mathbb{R}^n}(0, R)$. So we obtain

$$\begin{aligned} \deg(i - \Gamma(\omega, \cdot), B_C(0, R)) &= \deg(-\Pi_L \mathcal{P}_F(\omega, \cdot), B_{\mathbb{R}^n}(0, R)) \\ &= (-1)^n \deg\big(\nabla V(\omega, \cdot), B_{\mathbb{R}^n}(0, R)\big) \neq 0. \end{aligned}$$

Therefore, the random topological degree $D(i - \Gamma, B_C(0, R)) \neq 0$. Applying Theorem 7 we obtain that problem (1.1) has a random solution. ∎

4. Random nonsmooth integral guiding functions

Firstly, let us recall some notions of non-smooth analysis (see, e.g., [10]). Let $V\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz functional. For every $y_0 \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^n$ the generalized directional derivative $V^0(y_0;\nu)$ of V at the point y_0 in the direction ν is defined as

$$V^0(y_0;\nu) = \overline{\lim_{\substack{y \to y_0 \\ t \downarrow 0}}} \frac{V(y+t\nu) - V(y)}{t}. \tag{4.1}$$

By Proposition 2.1.1 [10], the functional $V^0\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is upper semicontinuous, i.e., for each sequences $(y_n,\nu_n) \in \mathbb{R}^n \times \mathbb{R}^n$, $(y_n,\nu_n) \to (y_0,\nu_0)$, the following relation holds:

$$\overline{\lim_{n\to+\infty}} V^0(y_n;\nu_n) \le V^0(y_0;\nu_0).$$

The generalized gradient $\partial V(x_0)$ of functional V at $y_0 \in \mathbb{R}^n$ is defined by:

$$\partial V(y_0) = \left\{ y \in \mathbb{R}^n \colon \langle y,\nu\rangle \le V^0(y_0;\nu) \text{ for every } \nu \in \mathbb{R}^n \right\}.$$

It is well known (see, e.g., [10]) that the multimap $\partial V\colon \mathbb{R}^n \to P(\mathbb{R}^n)$ is u.s.c. and has compact convex values. In particular, it means that for every continuous function $x\colon [0,T] \to \mathbb{R}^n$ the set $\mathcal{P}_{\partial V}(x)$ of all summable selections of the multifunction $\partial V(x(t))$ is non-empty.

A locally Lipschitz functional $V\colon \mathbb{R}^n \to \mathbb{R}$ is called *regular,* if for every $y \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^n$ there exists the directional derivative $V'(y,\nu)$ and $V'(y,\nu) = V^0(y,\nu)$. It is known (see, e.g., [10]) that locally bounded convex functionals are regular.

Lemma 14 (see [27])**.** *Let $V\colon \mathbb{R}^n \to \mathbb{R}$ be a regular functional, $x\colon [0,T] \to \mathbb{R}^n$ an absolutely continuous function. Then the function $V(x(t))$ is absolutely continuous and*

$$V(x(t)) - V(x(0)) = \int_0^t V^0(x(s),x'(s))ds,\ t \in [0,T].$$

Definition 15. A function $V\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ is said to be a random nonsmooth integral guiding function for problem (1.1), if the following conditions hold:

(i) $V(\cdot,x)\colon \Omega \to \mathbb{R}$ is measurable for every $x \in \mathbb{R}^n$;

(ii) the function $V(\omega,\cdot)$ is regular for every $\omega \in \Omega$;

(iii) there exists $N > 0$ such that for all $\omega \in \Omega$ from $x \in C(I,\mathbb{R}^n)$ with $\|x\|_2 \ge N$, it follows that

$$\int_0^T \langle \upsilon(t), f(t)\rangle dt > 0$$

for all $v \in \mathcal{P}_{\partial V}(\omega, x)$ and for all $f \in \mathcal{P}_F(\omega, x)$, where

$$\mathcal{P}_{\partial V}(\omega, x) = \big\{ v \in L^2(I, \mathbb{R}^n) \colon v(t) \in \partial V(\omega, x(t)) \ \text{ for a.e. } \ t \in I \big\}.$$

It is easy to verify that if V is the nonsmooth random guiding function for (1.1), then for every $\omega \in \Omega$ the topological degree of the multivalued vector field $\deg\big(\partial V(\omega, \cdot), B_{\mathbb{R}^n}(0, r)\big)$ is well-defined for all $r \geq \frac{N}{\sqrt{T}}$. Denote $ind\, V = D\big(\partial V, B_{\mathbb{R}^n}(0, r)\big)$. Analogously to Theorem 13 we obtain the following result.

Theorem 16. *Let conditions* (F1)–(F2) *hold. If there exists a random nonsmooth integral guiding function for problem* (1.1) *such that* $ind\, V \neq 0$*, then problem* (1.1) *has a random solution.*

5. Application to a random differential complementarity system

Consider now the differential complementarity system (1.2). Assume that

($A1$) $\mathfrak{F} \colon \Omega \times I \times \mathbb{R}^n \times \mathbb{R}^m \to Kv(\mathbb{R}^n)$ is a random c-multioperator, $\mathfrak{g} \colon \Omega \times I \times \mathbb{R}^n \to \mathbb{R}^m$ and $\mathfrak{f} \colon \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ are random c-operators;

($A2$) there exist $\alpha_{\mathfrak{F}} > 0$ and $\beta_{\mathfrak{g}} > 0$ such that

$$\|\mathfrak{F}(\omega, t, x, u)\| \leq \alpha_{\mathfrak{F}}(1 + |x| + |u|),$$

and

$$|\mathfrak{g}(\omega, t, x)| \leq \beta_{\mathfrak{g}}(1 + |x|),$$

for all $(\omega, t, x, u) \in \Omega \times I \times \mathbb{R}^n \times \mathbb{R}^m$;

($A3$) there exist the numbers $a, b > 0$ such that $|\mathfrak{f}(\omega, 0)| \leq b$ and

$$\big\langle u' - u'', \mathfrak{f}(\omega, u') - \mathfrak{f}(\omega, u'') \big\rangle \geq a|u' - u''|^2,$$

for all $u', u'' \in K$ and $\omega \in \Omega$.

Definition 17. By a solution to problem (1.2) we mean a pair (x, u) consisting of two functions $x \colon \Omega \times I \to \mathbb{R}^n$ and $u \colon \Omega \times I \to K$ such that the maps $\omega \in \Omega \to x(\omega, \cdot) \in W^{1,2}(I, \mathbb{R}^n)$ and $\omega \in \Omega \to u(\omega, \cdot) \in L^1(I, \mathbb{R}^m)$ are measurable and (x, u) satisfies (1.2).

Lemma 18. *Let conditions* ($A1$), ($A3$) *hold. Then for every* $r \in \mathbb{R}^m$*:*

(i) *the set*

$$SOL\big(K, r + \mathfrak{f}(\omega, \cdot)\big) := \big\{ v \in K \colon \big\langle v, r + \mathfrak{f}(\omega, v) \big\rangle \big\} = 0$$

consists of a unique element for each $\omega \in \Omega$*;*

(ii) $|v| \leq \frac{1}{a}(|r| + b)$ *for* $v = SOL\big(K, r + \mathfrak{f}(\omega, \cdot)\big)$ *and for all* $\omega \in \Omega$.

Proof. From $(A1), (A3)$ it follows that for each $\omega \in \Omega$ the function $r + \mathfrak{f}(\omega, \cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is continuous and strongly monotone on K. By virtue of Lemma 8 the set $SOL\big(K, r + \mathfrak{f}(\omega, \cdot)\big)$ contains a unique element. Let $v = SOL\big(K, r + \mathfrak{f}(\omega, \cdot)\big)$. Since $\big\langle v, r + \mathfrak{f}(\omega, v)\big\rangle = 0$ we have

$$\big|\big\langle v, r\big\rangle\big| = \big|\big\langle v, \mathfrak{f}(\omega, v)\big\rangle\big|\,.$$

From $(A3)$ it follows that

$$\big\langle v, \mathfrak{f}(\omega, v) - \mathfrak{f}(\omega, 0)\big\rangle \geq a|v|^2.$$

Hence,

$$\big|\big\langle v, \mathfrak{f}(\omega, v)\big\rangle\big| \geq a|v|^2 - \big|\big\langle v, \mathfrak{f}(\omega, 0)\big\rangle\big| \geq a|v|^2 - b|v|.$$

Therefore,

$$|v|\,|r| \geq \big|\big\langle v, r\big\rangle\big| = \big|\big\langle v, \mathfrak{f}(\omega, v)\big\rangle\big| \geq a|v|^2 - b|v|.$$

So, $|v| \leq \frac{1}{a}(|r| + b)$. ∎

Define the map $\widetilde{u}\colon \Omega \times I \times \mathbb{R}^n \to \mathbb{R}^m$,

$$\widetilde{u}(\omega, t, x) = SOL\big(K, \mathfrak{g}(\omega, t, x) + \mathfrak{f}(\omega, \cdot)\big),$$

and the multimap $F\colon \Omega \times I \times \mathbb{R}^n \to Kv(\mathbb{R}^n)$,

$$F(\omega, t, x) = \mathfrak{F}\big(\omega, t, x, \widetilde{u}(\omega, t, x)\big).$$

Lemma 19. *Let conditions* $(A1)$*–*$(A3)$ *hold. Then* F *satisfies conditions* $(F1)$*–*$(F2)$.

Proof. In fact, for a fixed $(t, x) \in\ I \times \mathbb{R}^n$ from the definition of $\widetilde{u}$ we have

$$\big\langle \widetilde{u}(\omega, t, x), \mathfrak{g}(\omega, t, x) + \mathfrak{f}(\omega, \widetilde{u}(\omega, t, x))\big\rangle = 0,$$

or equivalently,

$$\widetilde{u}(\omega, t, x) = P_K\big(\widetilde{u}(\omega, t, x) - \mathfrak{g}(\omega, t, x) - \mathfrak{f}(\omega, \widetilde{u}(\omega, t, x))\big),$$

where P_K is the projection operator of $\mathbb{R}^m$ onto K. Consider the map $\mathfrak{q}\colon \Omega \times \mathbb{R}^m \to \mathbb{R}^m$,

$$\mathfrak{q}(\omega, v) = P_K\big(v - \mathfrak{g}(\omega, t, x) - \mathfrak{f}(\omega, v)\big).$$

Since $\mathfrak{g}$ and $\mathfrak{f}$ are random c-operators and P_K is continuous map, the map $\mathfrak{q}$ is also a random c-operator. Moreover, for each $\omega \in \Omega$ there exists a unique fixed point $\widetilde{u}(\omega, t, x)$ of the map $\mathfrak{q}(\omega, \cdot)$. By applying Theorem 5 we obtain that $\widetilde{u}(\cdot, t, x)\colon \Omega \to \mathbb{R}^m$ is measurable, and hence $F(\cdot, t, x)$ is measurable too.

Now, fix $\omega \in \Omega$ and let us show that $\widetilde{u}(\omega,\cdot,\cdot)\colon I \times \mathbb{R}^n \to \mathbb{R}^m$ is continuous. For this goal, let $\{(t_n, x_n)\} \subset I \times \mathbb{R}^n$ be such that $(t_n, x_n) \to (t_0, x_0)$. From $(A2)$ and Lemma 18 it follows that the set $\{\widetilde{u}(\omega, t_n, x_n)\}$ is bounded in K. W.l.o.g. assume that $\widetilde{u}(\omega, t_n, x_n) \to u_0 \in K$. Since

$$\big\langle \widetilde{u}(\omega, t_n, x_n), \mathfrak{g}(\omega, t_n, x_n) + \mathfrak{f}(\omega, \widetilde{u}(\omega, t_n, x_n))\big\rangle = 0,$$

we have

$$\big\langle u_0, \mathfrak{g}(\omega, t_0, x_0) + \mathfrak{f}(\omega, u_0)\big\rangle = 0.$$

Consequently, $u_0 = \widetilde{u}(\omega, t_0, x_0)$, i.e., $\widetilde{u}(\omega,\cdot,\cdot)$ is continuous. Hence, $F(\omega,\cdot,\cdot)$ is continuous, and, by applying Proposition 7.9 of [23] we conclude that F is a random c-multioperator. So, condition $(F1)$ holds true. Condition $(F2)$ follows easily from $(A2)$ and Lemma 18. ∎

Now, we can substitute problem (1.2) with the following equivalent problem

$$\begin{cases} x'(\omega, t) \in F(\omega, t, x(\omega, t)), & \text{for a.e. } \ t \in (0, T), \\ x(\omega, 0) = x(\omega, T). \end{cases} \tag{5.1}$$

From Theorems 13 and 16 we obtain the following existence criterion for problem (1.2).

Theorem 20. *Let conditions* $(A1)$–$(A3)$ *hold. If there exists a random integral guiding function* V *(smooth or nonsmooth) for problem* (5.1) *such that* $\operatorname{ind} V \neq 0$, *then problem* (1.2) *has a random solution.*

Consider the following application of this general principle.

Example 21. Let $\Omega = [0, 1]$ and $\mathfrak{F}(\omega, t, x, u) = \mathfrak{b}(\omega, t, x) + \widetilde{\mathfrak{F}}(\omega, t, x, u)$, where $\mathfrak{b}\colon \Omega \times I \times \mathbb{R}^n \to \mathbb{R}^n$ and $\widetilde{\mathfrak{F}}\colon \Omega \times I \times \mathbb{R}^n \times \mathbb{R}^m \to Kv(\mathbb{R}^n)$. Assume that $\mathfrak{b}$ is a random c-operator, $\mathfrak{f}$ and $\mathfrak{g}$ satisfy conditions $(A1)$–$(A3)$ and $\widetilde{\mathfrak{F}}$ satisfies conditions $(A1)$–$(A2)$ with the corresponding constant $\alpha_{\widetilde{\mathfrak{F}}}$. If there exists $\lambda > e^T \alpha_{\widetilde{\mathfrak{F}}}(1 + \frac{1}{a})\beta_{\mathfrak{g}}$ such that

$$\big\langle \mathfrak{b}(\omega, t, x), x\big\rangle \geq \lambda e^{-\omega t}|x|^2, \quad \forall (\omega, t, x) \in \Omega \times I \times \mathbb{R}^n,$$

then problem (1.2) has a random solution.

In fact, by virtue of Theorem 20 it is sufficient to show that the function $V\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$,

$$V(\omega, x) = \frac{1}{2} e^{\omega T}|x|^2$$

is a random integral guiding function for problem (5.1). For each $\omega \in \Omega$ and $x \in C(I, \mathbb{R}^n)$ take $f \in \mathcal{P}_F(\omega, x)$. Then there exists measurable selection

$$\widetilde{f}(t) \in \widetilde{\mathfrak{F}}(\omega, t, x(t), \widetilde{u}(\omega, t, x(t))), \text{ for a.e. } t \in I$$

such that

$$f(t) = \mathfrak{b}(\omega, t, x(t)) + \widetilde{f}(t) \text{ for a.e. } t \in I.$$

We have

$$\begin{aligned}\int_0^T \big\langle \nabla V(\omega, x(s)), f(s)\big\rangle ds &= e^{\omega T} \int_0^T \big\langle x(s), \mathfrak{b}(\omega, s, x(s)) + \widetilde{f}(s)\big\rangle ds \\ &\geq \lambda \|x\|_2^2 - e^{\omega T} \int_0^T |x(s)||\widetilde{f}(s)| ds.\end{aligned}$$

From $(A2)$–$(A3)$ and Lemma 18(ii) it follows that

$$\begin{aligned}|\widetilde{f}(s)| &\leq \left\| \widetilde{\mathfrak{F}}\big(\omega, s, x(s), \widetilde{u}(\omega, s, x(s))\big) \right\| \\ &\leq \alpha_{\widetilde{\mathfrak{F}}}\big(1 + |x(s)| + |\widetilde{u}(\omega, s, x(s))|\big) \\ &\leq \alpha_{\widetilde{\mathfrak{F}}}\left(1 + |x(s)| + \frac{1}{a}\big(|\mathfrak{g}(\omega, s, x(s))| + b\big)\right) \\ &\leq \alpha_{\widetilde{\mathfrak{F}}}\left(1 + |x(s)| + \frac{1}{a}\big(\beta_{\mathfrak{g}}(1 + |x(s)|) + b\big)\right)\end{aligned}$$

for a.e. $s \in I$. Therefore,

$$\begin{aligned}&\int_0^T \big\langle \nabla V(\omega, x(s)), f(s)\big\rangle ds \\ &\geq \lambda \|x\|_2^2 - e^T \alpha_{\widetilde{\mathfrak{F}}} \int_0^T |x(s)| \left(1 + |x(s)| + \frac{1}{a}\big(\beta_{\mathfrak{g}}(1 + |x(s)|) + b\big)\right) ds \\ &\geq \left(\lambda - e^T \alpha_{\widetilde{\mathfrak{F}}}\left(1 + \frac{1}{a}\beta_{\mathfrak{g}}\right)\right) \|x\|_2^2 - e^T \left(\alpha_{\widetilde{\mathfrak{F}}} + \frac{1}{a}\alpha_{\widetilde{\mathfrak{F}}}(\beta_{\mathfrak{g}} + b)\right) \|x\|_2 > 0,\end{aligned}$$

provided $\|x\|_2$ is sufficiently large. Hence, V is a random integral guiding function for problem (5.1). The conclusion is easily follows from the fact that $indV \neq 0$.

As the other example we consider a control system arising from random survival models. Let us note that dynamical systems with survival probability are extensively studied recently in ecology (see, e.g., [32, 37]).

Example 22. Let $\Omega = [0, 1]$. Consider a dynamical system

$$\begin{cases} x'(\omega, t) = e^{-\omega t}\big(Ax(\omega, t) + Bu(\omega, t) + h(t)\big) & \text{for a.e. } t \in I, \\ 0 \leq u(\omega, t) \perp Cx(\omega, t) + Du(\omega, t) \geq 0, & \text{for a.e. } t \in I, \\ x(\omega, 0) = x(\omega, T), \end{cases} \tag{5.2}$$

where $h \in C(I, \mathbb{R}^n)$; $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$.

The dynamic of the model depends on the random variable ω. First equation describes the evolution of the object while the complementarity represents the relation between the control function u and the state function x. The term $e^{-\omega t}$ motivates the probability that the object exists up to time t.

Assume that

$(H1)$ there exist λ_A and λ_D such that

$$\big\langle Az, z\big\rangle \geq \lambda_A |z|^2 \text{ and } \big\langle Dv, v\big\rangle \geq \lambda_D |v|^2,$$

for all $z \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$;

$(H2)$ $\lambda_A > e^T \max\left\{\|B\|, \|h\|_C\right\}\left(1 + \frac{\|C\|}{\lambda_D}\right)$, where

$$\|B\| = \max\{|Bu|\colon |u| = 1\} \text{ and } \|C\| = \max\{|Cz|\colon |z| = 1\}.$$

From the previous example it follows that problem (5.2) has a solution.

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