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ON PROPERTIES OF SET-VALUED INTEGRALS DRIVEN BY MARTINGALES AND SET-VALUED STOCHASTIC EQUATIONS

Wojciech Lassota and Mariusz Michta

Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra
Szafrana 4a, 65–516 Zielona Góra, Poland

e-mail: w.lassota@wmie.uz.zgora.pl
m.michta@wmie.uz.zgora.pl

Abstract

In the paper we study properties of stochastic integrals of Aumann type driven by quadratic variation process and set-valued Itô integral with respect to martingale. Next, the existence, uniqueness and convergence properties of solutions to set-valued stochastic differential equations with respect to such integrators are investigated. The results obtained in the paper generalize conclusions dealing with this topic known both in deterministic and stochastic cases.

Keywords: set-valued function, set-valued stochastic integral, set-valued stochastic differential equation.

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1. Introduction

Foundations of the theory of deterministic set-valued (or multivalued) differential equations were established by Artstein, De Blasi, Brandao Lopez Pinto and Iervolino in [4, 5, 7] and [8], and such studies were developed next by other authors (see e.g. monographs [29, 45] as well as [1, 2, 14, 13, 19, 20, 26, 27, 28, 42]). Set-valued stochastic differential equations constitute a natural extension of a deterministic case and they were developed mainly in the following two directions. First one deals with set-valued equations understood as relations in the space of nonempty, closed, convex and bounded subsets of the space of square integrable random vectors (see e.g. [21, 34, 35, 36] and recently [38]). In this case solutions are not set-valued stochastic processes. The second approach deals with set-valued differential equations driven by a Wiener process and which solutions are set-valued stochastic processes (see [24] and earlier papers [32, 33] and [39] for equations with set-valued drifts and single valued diffusion terms). The aim of our work is to extend equations studied earlier to equations both with set-valued drift and diffusion terms driven by more general integrators, i.e., quadratic variation processes and martingales, respectively. It is worth to note that in a general case set-valued Ito's integral understood as a multivalued process may be unbounded (see [37]). This constitutes its main disadvantage for applications. However, for appropriate integrands this desirable property will be satisfied and successfully applied to set-valued stochastic equations considered in the paper. For such equations we establish existence, uniqueness and other properties of their solutions. We shall also provide comments on relations of obtained properties with results dealing with these topics studied earlier, both in deterministic and stochastic case. The paper is organized as follows. In Section 2 we recall the relevant material from stochastic and set-valued analysis needed later on. In Section 3 the notions of set-valued stochastic integrals of Aumann type and Itô type are defined and some useful properties of these integrals are proved. In Section 4 generalized versions of stochastic set-valued differential equations are investigated as well as properties of their solutions.

2. Basic notions and auxiliary results

Let $I = [0, T]$ and let $\mathcal{B}(I)$ denote a Borel σ-field on I. Let $(\Omega, \mathbf{F}, \{\mathbf{F}_t\}_{t\in I}, P)$ be a complete filtered probability space satisfying the usual hypotheses, i.e., $\{\mathbf{F}_t\}_{t\in I}$ is an increasing and right continuous family of sub-σ-fields of $\mathbf{F}$. We shall assume that $\mathbf{F}_0$ contains all P-null sets. By $L^{2,d} := L^2(\Omega, \mathbf{F}, P; \mathbb{R}^d)$ we shall denote a Hilbert space of square integrable, d-dimensional ($d \geq 1$) random vectors. In the whole paper by $\mathcal{P}$ we denote a predictable σ-field on $I \times \Omega$ (see [6] for details). Let M be an $\{\mathbf{F}_t\}_{t\in I}$-adapted, continuous local martingale such that $M_0 = 0$ (see e.g. [43] for details). We shall assume that M is such that

$$\|[M, M]_T\|_{L^{2,1}} < \infty, \tag{2.1}$$

where $[M, M]$ denotes the quadratic variation process for a martingale M. For a such local martingale M, it holds $E[M, M]_t < \infty$ for all $t \in I$. Then M is a square integrable martingale such that $EM_t^2 = E[M, M]_t$ for all $t \in I$. By μ_M we shall denote the Doléans-Dade measure for M on $\mathcal{P}$ (c.f. [6]). Then for all $a \in L^2\left(I \times \Omega, \mathcal{P}, \mu_M; \mathbb{R}^d\right)$ the Itô integral $\int a_s dM_s$ exists and it satisfies

$$E\left\|\int_0^t a_s dM_s\right\|_{\mathbb{R}^d}^2 = \int_{[0,t]\times\Omega} \|a\|_{\mathbb{R}^d}^2 d\mu_M = E\left(\int_0^t \|a_s\|_{\mathbb{R}^d}^2 d[M, M]_s\right) \tag{2.2}$$

for $t \in I$. Next, since the process $[M, M]$ is adapted and it has continuous paths, it is predictable. It is also nondecreasing. Therefore, each its path induces a nonatomic measure $d[M, M]_{(\cdot)}$ on $\mathcal{B}(I)$. Next, let us define a random measure γ on a measurable space $(I, \mathcal{B}(I))$

$$\gamma(dt, \omega) := [M, M]_T (\omega)\, d[M, M]_t (\omega)\,. \tag{2.3}$$

Then integrating the measure γ with respect to P one can obtain a measure ν_M on the σ-field $\mathcal{P}$. Namely, ν_M is given by

$$\nu_M(C) := \int_\Omega \left(\int_0^T I_C(\omega, t)\gamma(dt, \omega) \right) P(d\omega) \tag{2.4}$$

for $C \in \mathcal{P}$. As a consequence we get

$$\nu_M(I \times \Omega) = E\left([M, M]_T^2\right)$$

and ν_M is a finite measure on $\mathcal{P}$ by (2.1). For any $b \in L^2\left(I \times \Omega, \mathcal{P}, \nu_M; \mathbb{R}^d\right)$ and every $t \in I$, a stochastic Lebesgue-Stieltjes integral $\int_0^t b_s d[M, M]_s$ is defined ω-by-ω and the mapping $I \ni t \to \int_0^t b_s(\omega) d[M, M]_s (\omega)$ is continuous. Moreover, for every $t \in I$ it holds

$$\left\| \int_0^t b_s d[M, M]_s \right\|_{\mathbb{R}^d}^2 \le \int_0^t \|b_s\|_{\mathbb{R}^d}^2 \gamma(ds) \qquad P\text{-a.e.}$$

and thus

$$E\left(\left\| \int_0^t b_s d[M, M]_s \right\|_{\mathbb{R}^d}^2 \right) \le \int_{[0,t]\times\Omega} \|b\|_{\mathbb{R}^d}^2 d\nu_M. \tag{2.5}$$

Finally, for $b \in L^2\left(I \times \Omega, \mathcal{P}, \nu_M; \mathbb{R}^d\right)$, the Lebesgue-Stieltjes integral process $\int_0^\cdot b_s d[M, M]_s$ is also $\{\mathbf{F}_t\}_{t\in I}$-adapted. We recall the following version of stochastic Gronwall's inequality needed in the sequel.

Theorem 1 [11]**.** *Let N, B, U be one dimensional continuous processes such that N and B are a semimartingale and a nondecreasing process, respectively, and*

$$U_t \le N_t + \int_0^t U_s dB_s \ \ P\text{-a.s. for } t \in I.$$

Then

$$U_t \le e^{B_t} \left(N_0 e^{-B_0} + \int_0^t e^{-B_s} dN_s \right) \ \ P\text{-a.e. for } t \in I.$$

For a given separable Banach space $\mathbb{X}$, by $\mathcal{K}(\mathbb{X})$ we denote the family of all nonempty and closed subsets of $\mathbb{X}$. By $\mathcal{K}^b(\mathbb{X})$ (resp. $\mathcal{K}^b_c(\mathbb{X})$) we denote the family of those elements in $\mathcal{K}(\mathbb{X})$ that are also bounded (resp. bounded and convex). The Hausdorff metric $H_{\mathbb{X}}$ in $\mathcal{K}^b(\mathbb{X})$ is defined by:

$$H_{\mathbb{X}}(C,D) := \max\left\{\bar{H}_{\mathbb{X}}(C,D), \bar{H}_{\mathbb{X}}(D,C)\right\}$$

where $\bar{H}_{\mathbb{X}}(C,D) := \sup_{c\in C} \operatorname{dist}_{\mathbb{X}}(c,D)$, $\operatorname{dist}_{\mathbb{X}}(c,D) := \inf_{d\in D} \|c-d\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{X}}$ is a norm in $\mathbb{X}$. Then $(\mathcal{K}^b(\mathbb{X}), H_{\mathbb{X}})$ is a complete metric space and $\mathcal{K}^b_c(\mathbb{X})$ is its closed subspace (see [17]). Moreover, for $A,B,C,D \in \mathcal{K}^b_c(\mathbb{X})$ it holds

$$H_{\mathbb{X}}(A+B, C+D) \le H_{\mathbb{X}}(A,C) + H_{\mathbb{X}}(B,D) \tag{2.6}$$

and

$$H_{\mathbb{X}}(A+B, C+B) = H_{\mathbb{X}}(A,C), \tag{2.7}$$

where $A+B$ denotes the Minkowski sum of A and B. For $C \in \mathcal{K}^b(\mathbb{X})$ we set

$$|||C|||_{\mathbb{X}} := \bar{H}_{\mathbb{X}}\left(C, \{0\}\right) = \sup_{c\in C} \|c\|_{\mathbb{X}}.$$

By $cl_{\mathbb{X}}co(C)$ we will denote a closed and convex hull of the set $C \subset \mathbb{X}$, i.e., the intersection of all closed and convex subsets of $\mathbb{X}$ containing C. Then

$$\bar{H}_{\mathbb{X}}(cl_{\mathbb{X}}co(C), cl_{\mathbb{X}}co(D)) \le \bar{H}_{\mathbb{X}}(C,D) \tag{2.8}$$

and consequently

$$H_{\mathbb{X}}(cl_{\mathbb{X}}co(C), cl_{\mathbb{X}}co(D)) \le H_{\mathbb{X}}(C,D)$$

for $C,D \in \mathcal{K}^b(\mathbb{X})$. We shall also use some properties of decomposable sets and their connections with set-valued mappings. Let $(U,\mathcal{U},\mu)$ be a given finite and nonatomic measure space. Let $\mathcal{M} \subset L^p\left(U,\mathcal{U},\mu;\mathbb{R}^d\right)$, $p \ge 1$. The set $\mathcal{M}$ is said to be $\mathcal{U}$-decomposable if for every $g_1, g_2 \in \mathcal{M}$ and every $A \in \mathcal{U}$ it holds $g_1\chi_A + g_2\chi_{U\setminus A} \in \mathcal{M}$. For a nonempty set $\Phi \subset L^p\left(U,\mathcal{U},\mu;\mathbb{R}^d\right)$ by $dec_{\mathcal{U}}(\Phi)$ and $cl_{L^p}dec_{\mathcal{U}}(\Phi)$ we denote the smallest $\mathcal{U}$-decomposable and the smallest closed and $\mathcal{U}$-decomposable set, respectively, containing Φ. They are called, respectively, decomposable hull and closed decomposable hull for Φ (see [12] for details). Let $S^p_{\mathcal{U}}(G,\mu)$ denotes the set of $\mathcal{U}$-measurable and L^p-integrable selectors of G, i.e., $S^p_{\mathcal{U}}(G,\mu) = \{g \in L^p\left(U,\mathcal{U},\mu;\mathbb{R}^d\right) : g \in G \ \mu\text{-a.e.}\}$. By Theorem 3.1 in [16] there exists $\mathcal{U}$-measurable set-valued mapping $G : U \to \mathcal{K}(\mathbb{R}^d)$ such that

$$S^p_{\mathcal{U}}(G,\mu) = cl_{L^p}dec_{\mathcal{U}}(\Phi). \tag{2.9}$$

For such G we define mapping $cl_{\mathbb{R}^d}coG \to \mathcal{K}(\mathbb{R}^d)$ by the formula $(cl_{\mathbb{R}^d}coG)(u) = cl_{\mathbb{R}^d}co(G(u))$. Then $cl_{\mathbb{R}^d}coG$ is $\mathcal{U}$-measurable set-valued mapping with closed and convex values. Moreover, it holds

$$cl_{L^p}co(S^p_{\mathcal{U}}(G,\mu)) = S^p_{\mathcal{U}}(cl_{\mathbb{R}^d}coG,\mu). \tag{2.10}$$

A $\mathcal{U}$-measurable set-valued mapping $G : U \to \mathcal{K}(\mathbb{R}^d)$ is called $L^p_{\mathcal{U}}(\mu)$-integrably bounded if $|||G|||_{\mathbb{R}^d} \in L^p\left(U,\mathcal{U},\mu;\mathbb{R}^1\right)$ where $|||G|||_{\mathbb{R}^d} : U \to \mathbb{R}^1_+$ is $\mathcal{U}$-measurable function defined by $|||G|||_{\mathbb{R}^d}(u) := |||G(u)|||_{\mathbb{R}^d}$ for $u \in U$. In the sequel we shall use the following result.

Proposition 2. *For a nonempty set $\Phi \subset L^p\left(U,\mathcal{U},\mu;\mathbb{R}^d\right)$, it holds*

$$cl_{L^p}co(cl_{L^p}dec_{\mathcal{U}}(\Phi) = cl_{L^p}dec_{\mathcal{U}}(cl_{L^p}co(\Phi)). \tag{2.11}$$

Proof. Indeed, due to Theorem 3.3 in [22] the set $cl_{L^p}dec_{\mathcal{U}}(cl_{L^p}co(\Phi))$ is convex and closed. Hence we get

$$cl_{L^p}co(cl_{L^p}dec_{\mathcal{U}}(\Phi) \subset cl_{L^p}dec_{\mathcal{U}}(cl_{L^p}co(\Phi)).$$

In order to show the opposite inclusion, it suffices to claim that the set $cl_{L^p}co(cl_{L^p}dec_{\mathcal{U}}(\Phi))$ is decomposable. Let $G : U \to \mathcal{K}(\mathbb{R}^d)$ be a $\mathcal{U}$-measurable set-valued mapping such that $S^p_{\mathcal{U}}(G,\mu) = cl_{L^p}dec_{\mathcal{U}}(\Phi)$ and let $cl_{\mathbb{R}^d}coG : U \to \mathcal{K}(\mathbb{R}^d)$ be a $\mathcal{U}$-measurable set-valued mapping with closed and convex values. Then $cl_{L^p}co(cl_{L^p}dec_{\mathcal{U}}(\Phi) = cl_{L^p}co(S^p_{\mathcal{U}}(G,\mu)$ and by (2.10) it is equal to the set $S^p_{\mathcal{U}}(cl_{\mathbb{R}^d}coG,\mu)$ which is decomposable. Hence $cl_{L^p}co(cl_{L^p}dec_{\mathcal{U}}(\Phi))$ is decomposable too. Thus (2.11) follows. ∎

By $\mathcal{L}^p_{\mathcal{U}}$ we denote the space of all $L^p_{\mathcal{U}}(\mu)$-integrally bounded and $\mathcal{U}$-measurable set-valued mappings $G : U \to \mathcal{K}^b_c(\mathbb{R}^d)$. Then one can show that $\mathcal{L}^p_{\mathcal{U}}$ endowed with a metric

$$\sigma_p(G_1,G_2) = \left(\int_U H^p_{\mathbb{R}^d}(G_1,G_2)\,d\mu\right)^{\frac{1}{p}} \tag{2.12}$$

is a complete metric space. Due to properties (2.6) and (2.7) we obtain:

Proposition 3. *Let $G,G_1,G_2,\dots,G_n,F_1,F_2,\dots,F_n \in \mathcal{L}^p_{\mathcal{U}}$. Then it holds*

(i) $\sigma_p(G_1+G,G_2+G) = \sigma_p(G_1,G_2)$,

(ii) $\sigma^p_p(G_1+G_2+\dots+G_n,F_1+F_2+\dots+F_n) \le n^{p-1}\sum_{k=1}^n \sigma^p_p(G_k,F_k)$.

Finally, let us recall that a set-valued function $F \in \mathcal{L}^p_{\mathcal{U}}$ is called simple if there exists a finite $\mathcal{U}$-measurable partition $\{A_1,A_2,\dots,A_n\}$ of U and sets $X_1,X_2,\dots,X_n \in \mathcal{K}^b_c(\mathbb{R}^d)$ and such that $F(u) = \sum_{k=1}^n \chi_{A_k}(u)X_k$ for every $u \in U$.

Due to separability of the space $\left(\mathcal{K}_c^b(\mathbb{R}^d), H_{\mathbb{R}^d}\right)$ (cf. Theorem 1.30 together with Corollary 1.33 of Chapter 1 in [17]) one can show that the set of all simple set-valued functions in $\mathcal{L}_{\mathcal{U}}^p$ is dense in $\mathcal{L}_{\mathcal{U}}^p$ (cf. Section 3 in [16]). Then, for a separable measure space $(U,\mathcal{U},\mu)$ (or separable measure μ) the following result holds similarly as in a single-valued case.

Proposition 4. *For a separable measure space $(U,\mathcal{U},\mu)$ the space $\left(\mathcal{L}_{\mathcal{U}}^p, \sigma_p\right)$ is separable for every p, $1 \le p < \infty$.*

3. SET-VALUED STOCHASTIC INTEGRALS

In this part we apply notions and results of the preceding section in order to define two types of set-valued stochastic integrals. First, we consider a set-valued Itô integral driven by a square integrable martingale and next Aumann integral with respect to a quadratic variation process of such a martingale. As it was already mentioned in Introduction, such integrals are understood as set-valued random variables. We present their properties applied next to the analysis of set-valued stochastic equations. In the rest of the paper we hold up assumptions imposed on a martingale M in (2.1). We shall also use the following shorter notation: $L^2_{\mathcal{P}}(\mu_M) := L^2\left(I \times \Omega, \mathcal{P}, \mu_M; \mathbb{R}^d\right)$ and $L^2_{\mathcal{P}}(\nu_M) := L^2\left(I \times \Omega, \mathcal{P}, \nu_M; \mathbb{R}^d\right)$.

3.1. Set-valued Itô integral

Similarly as in [23] and [25] we shall introduce a generalized set-valued Ito integral applying the procedure given in (2.9). For $a \in L^2_{\mathcal{P}}(\mu_M)$ and $t \in I$, let $\mathcal{J}_t^M(a) := \int_0^t a_s dM_s$. Hence, by (2.2) and (2.5) it holds $\mathcal{J}_t^M(\mathcal{G}) := \left\{ \int_0^t a_s dM_s : a \in \mathcal{G}\right\} \subset L^{2,d}$ for any nonempty, closed and bounded set $\mathcal{G} \subset L^2_{\mathcal{P}}(\mu_M)$. Then, for $t \in I$ and for any nonempty, closed and bounded set $\mathcal{G} \subset L^2_{\mathcal{P}}(\mu_M)$ by a generalized set-valued Ito integral we call an $\mathbf{F}_t$-measurable set-valued random variable $\int_0^t \mathcal{G} dM : \Omega \to \mathcal{K}(\mathbb{R}^d)$ such that

$$S^2_{\mathbf{F}_t}\left(\int_0^t \mathcal{G} dM, P\right) := cl_{L^{2,d}} dec_{\mathbf{F}_t}\left(\mathcal{J}_t^M(\mathcal{G})\right). \tag{3.1}$$

The existence of $\int_0^t \mathcal{G} dM$ follows from Theorem 3.1 in [16]. We start with the following result that can be proved in a similar way as Lemma 3.1 in [23].

Proposition 5. *For a nonempty set $\mathcal{G} \subset L^2_{\mathcal{P}}(\mu_M)$ and $t \in I$ it holds*

(a) $\mathcal{J}_t^M(cl_{L^2_{\mathcal{P}}(\mu_M)}(\mathcal{G})) = cl_{L^{2,d}}(\mathcal{J}_t^M(\mathcal{G}))$,

(b) $\mathcal{J}_t^M(cl_{L^2_{\mathcal{P}}(\mu_M)} co(\mathcal{G})) = cl_{L^{2,d}} co(\mathcal{J}_t^M(\mathcal{G}))$,

(c) $cl_{L^{2,d}} dec_{\mathbf{F}_t}(\mathcal{J}_t^M(cl_{L^2_{\mathcal{P}}(\mu_M)} co(\mathcal{G}))) = cl_{L^{2,d}} co(dec_{\mathbf{F}_t}(\mathcal{J}_t^M(\mathcal{G})))$.

Corollary 6. *For a nonempty set $\mathcal{G} \subset L^2_{\mathcal{P}}(\mu_M)$ and $t \in I$ it holds P-a.e.*

(i) $\int_0^t cl_{L^2_{\mathcal{P}}(\mu_M)}(\mathcal{G})dM = \int_0^t \mathcal{G}dM$,

(ii) $\int_0^t co(\mathcal{G})dM = cl_{\mathbb{R}^d} co\big(\int_0^t \mathcal{G}dM\big)$.

Proof. By Proposition 5 (a) we have $cl_{L^{2,d}} dec_{\mathbf{F}_t}\big(\mathcal{J}_t^M(cl_{L^2_{\mathcal{P}}(\mu_M)}(\mathcal{G}))\big) = cl_{L^{2,d}} dec_{\mathbf{F}_t}\big(\mathcal{J}_t^M(\mathcal{G})\big)$. Hence by (3.1) we get

$$S^2_{\mathbf{F}_t}\left(\int_0^t cl_{L^2_{\mathcal{P}}(\mu_M)}(\mathcal{G})dM, P\right) = S^2_{\mathbf{F}_t}\left(\int_0^t \mathcal{G}dM, P\right)$$

and thus $\int_0^t cl_{L^2_{\mathcal{P}}(\mu_M)}(\mathcal{G})dM = \int_0^t \mathcal{G}dM$ P-a.e. what proves (i). In a similar way, using (c) and (a) of the preceding corollary and (3.1) we obtain

$$S^2_{\mathbf{F}_t}\left(\int_0^t co(\mathcal{G})dM, P\right) = cl_{L^{2,d}} co\left(S^2_{\mathbf{F}_t}\left(\int_0^t \mathcal{G}dM, P\right)\right).$$

But due to (2.10) it holds $cl_{L^{2,d}} co\big(S^2_{\mathbf{F}_t}\big(\int_0^t \mathcal{G}dM, P\big)\big) = S^2_{\mathbf{F}_t}\big(cl_{\mathbb{R}^d} co\big(\int_0^t \mathcal{G}dM\big), P\big)$. Hence we get (ii). ∎

Let us note that if $G : I \times \Omega \to \mathcal{K}^b_c(\mathbb{R}^d)$ is a predictable and $L^2_{\mathcal{P}}(\mu_M)$-integrably bounded set-valued process, then by the Kuratowski and Ryll-Nardzewski selection theorem (see e.g. [22]), $S^2_{\mathcal{P}}(G, \mu_M) \neq \emptyset$ where $S^2_{\mathcal{P}}(G, \mu_M) := \{g \in L^2_{\mathcal{P}}(\mu_M) : g \in G \ \mu_M\text{-a.e.}\}$. In this case one can consider the set-valued Itô integral for G, denoted by $\int_0^t GdM$ (see [18]). It coincides with the integral in (3.1) when one takes $\mathcal{G} = S^2_{\mathcal{P}}(G, \mu_M)$. As it was mentioned in Introduction, in this case the set-valued Itô integral $\int_0^t \mathcal{G}dM$ need not be integrably bounded in $L^{2,d}$ (see [37]), even if $\mathcal{G}$ has much simpler structure than the set of selections. The example below illustrates a case of a not integrably bounded integral for countable and bounded set $\mathcal{G} \subset L^2_{\mathcal{P}}(\mu_M)$ (c.f. [37]).

Example 7. Let $I = [0,1]$, $d = 1$ and let us take $M = W$ where W is a standard one dimensional Wiener process on a given filtered probability space $(\Omega, \mathbf{F}, \{\mathbf{F}_t\}_{t \in I}, P)$. Let $\{r_n : n \geq 1\}$ be a sequence of Rademacher functions in the space $L^2([0,1])$ i.e., $r_n(t) = sign(\sin(2^n \pi t))$. For $(t, \omega) \in [0,1] \times \Omega$, let $g_n(t, \omega) = r_n(t)$. Then g_n is a predictable stochastic process for every $n \geq 1$. Let $\mathcal{G} = \{g_n : n \geq 1\}$. By (3.1), $\int_0^1 \mathcal{G}dW$ is a set-valued random variable such that

$$S^2_{\mathbf{F}_1}\left(\int_0^1 \mathcal{G}dW, P\right) := cl_{L^{2,1}} dec_{\mathbf{F}_1}\left(\mathcal{J}_1^W(\mathcal{G})\right). \tag{3.2}$$

Let $X_{n,\alpha} := \int_0^1 g_n dW$ for $n \geq 1$. Then $\{X_n : n \geq 1\} \subset \mathcal{J}_1^W(\mathcal{G})$ and $\{X_n : n \geq 1\}$ is a sequence of standard Gaussian random variables such that

$$Cov(X_n, X_m) = E\left(\left(\int_0^1 g_n dW\right)\left(\int_0^1 g_m dW\right)\right)$$
$$= \int_{[0,1]\times\Omega} g_n g_m d\mu_W = \begin{cases} 1 & \text{for } n = m \\ 0 & \text{for } n \neq m\,. \end{cases}$$

Hence, random variables $X_n, n \geq 1$ are uncorrelated and thus independent, with a common distribution $N(0,1)$. Then, due to Theorem 2.2 in [16] and (3.2) we get

$$E\left(|||\int_0^1 \mathcal{G}dW|||^2_{\mathbb{R}^1}\right) = \sup\left\{E(u^2) : u \in dec_{\mathbf{F}_1}\left(\mathcal{J}_1^W(\mathcal{G})\right)\right\}. \tag{3.3}$$

Moreover, for every $N \geq 1$ we have

$$E\left(\max\{X_i^2 : 1 \leq i \leq N\}\right) \leq \sup\left\{E(u^2) : u \in dec_{\mathbf{F}_1}\left(\mathcal{J}_1^W(\mathcal{G}_1)\right)\right\}. \tag{3.4}$$

But $X_n^2, n \geq 1$ are independent with a common cumulative distribution function (say) F of χ_1^2-distribution (central chi-squared distribution with 1-degree of freedom). Thus $0 \leq F(t) < 1$ for every $t \geq 0$ and

$$E\left(\max\left\{X_i^2 : 1 \leq i \leq N\right\}\right) = \int_0^\infty \left(1 - F^N(t)\right)dt \tag{3.5}$$

for every $N \geq 1$. Since $1 - F^N(t) \nearrow 1$ as $N \to \infty$, it follows by the Lebesgue monotone convergence theorem that $\lim_{N\to\infty}\int_0^\infty (1 - F^N(t))dt = +\infty$. Thus combining (3.3), (3.4) and (3.5) we obtain

$$E\left(|||\int_0^1 \mathcal{G}dW|||^2_{\mathbb{R}^1}\right) = +\infty.$$

This means that a set-valued random variable $\int_0^1 \mathcal{G}dW$ can not be integrably bounded.

The above example shows that some additional restrictions imposed on $\mathcal{G}$ are needed. Therefore, in the rest of the paper we restrict our attention to the case of Itô set-valued integrals $\int_0^t \mathcal{G}dM$ with $\mathcal{G} = \{g^n : n \geq 1\} \subset L^2_{\mathcal{P}}(\mu_M)$, where $\sum_{n=1}^\infty ||g^n||^2_{L^2_{\mathcal{P}}(\mu_M)} < \infty$. Similarly as in [23] we call such the set $\mathcal{G}$ absolutely summable.

Theorem 8. *Let $\mathcal{G} = \{g^n : n \geq 1\} \subset L^2_{\mathcal{P}}(\mu_M)$ be such that $\sum_{n=1}^\infty ||g^n||^2_{L^2_{\mathcal{P}}(\mu_M)} < \infty$. Then for every $t \in I$ it hold*

(i) $\left(\int_0^t \mathcal{G}dM\right)(\cdot) = cl_{\mathbb{R}^d}\left\{\left(\int_0^t g^n_\tau dM_\tau\right)(\cdot) : n \geq 1\right\}$ *P-a.e.,*

(ii) *if* $\mathcal{H} = \{h^n : n \geq 1) \subset L^2_{\mathcal{P}}(\mu_M)$ *and* $\sum_{n=1}^{\infty} ||h^n||^2_{L^2_{\mathcal{P}}(\mu_M)} < \infty$, *then*

$$H^2_{\mathbb{R}^d}\left(\int_0^t \mathcal{G}dM, \int_0^t \mathcal{H}dM\right) \leq 2\sum_{n=1}^{\infty}\left\|\int_0^t (g^n_\tau - h^n_\tau)dM_\tau\right\|^2_{\mathbb{R}^d} \ P\text{-a.e.},$$

(iii) *for every* $s, t \in I, s < t$ *it holds* $\left(\int_0^s \mathcal{G}dM\right)(\cdot) = \left(\int_0^t \mathbb{I}_{[0,s]}\mathcal{G}dM\right)(\cdot)$ *P-a.e.*

Proof. (i) Let $t \in I$ be fixed and let us define a set-valued random variable

$$G(\omega) := cl_{\mathbb{R}^d}\left\{\left(\int_0^t g^n_\tau dM_\tau\right)(\omega) : n \geq 1\right\}.$$

Then the sequence $\left\{\int_0^t g^n_\tau dM_\tau : n \geq 1\right\} \subset S^2_{\mathbf{F}_t}\left(\int_0^t \mathcal{G}dM, P\right)$ is an $\mathbf{F}_t$-measurable Castaing representation of $\int_0^t \mathcal{G}dM$, i.e., $\left(\int_0^t \mathcal{G}dM\right)(\omega) = cl_{\mathbb{R}^d}\left\{\left(\int_0^t g^n_\tau dM_\tau\right)(\omega) : n \geq 1\right\}$ Since $\sum_{n=1}^{\infty} ||g^n||^2_{L^2_{\mathcal{P}}(\mu_M)} < \infty$, it follows by (2.2) that $\left\{\int_0^t g^n_\tau dM_\tau : n \geq 1\right\}$ is bounded in $L^{2,d}$. Thus by Remark 3.6, Chapter 2 in [22] and (3.1) we obtain

$$S^2_{\mathbf{F}_t}\left(\int_0^t \mathcal{G}dM, P\right) = cl_{L^{2,d}} dec_{\mathbf{F}_t}\left\{\mathcal{J}^M_t(g^n) : n \geq 1\right\} = S^2_{\mathbf{F}_t}(G, P).$$

Hence $G = \int_0^t \mathcal{G}dM$ P-a.e.

(ii) Similarly as in (i) we have

$$\left(\int_0^t \mathcal{H}dM\right)(\cdot) = cl_{\mathbb{R}^d}\left\{\left(\int_0^t h^n_\tau dM_\tau\right)(\cdot) : n \geq 1\right\} \text{ P-a.e.}$$

Thus

$$\begin{aligned} &H^2_{\mathbb{R}^d}\left(\int_0^t \mathcal{G}dM, \int_0^t \mathcal{H}dM\right) \\ &= \max\left\{\sup_{n\geq 1}\inf_{m\geq 1}\left\|\int_0^t (g^n_\tau - h^m_\tau)dM_\tau\right\|^2_{\mathbb{R}^d}, \sup_{m\geq 1}\inf_{n\geq 1}\left\|\int_0^t (g^n_\tau - h^m_\tau)dM_\tau\right\|^2_{\mathbb{R}^d}\right\} \\ &\leq 2\sup_{n\geq 1}\left\|\int_0^t (g^n_\tau - h^n_\tau)dM_\tau\right\|^2_{\mathbb{R}^d} \leq 2\sum_{n=1}^{\infty}\left\|\int_0^t (g^n_\tau - h^n_\tau)dM_\tau\right\|^2_{\mathbb{R}^d} \text{ P-a.e.} \end{aligned}$$

which proves (ii).

(iii) Finally, let us notice that $\chi_{[0,s]}\mathcal{G} = \{\chi_{[0,s]}g^n : n \geq 1\} \subset L^2_{\mathcal{P}}(\mu_M)$ and $\sum_{n=1}^{\infty} ||\chi_{[0,s]}g^n||^2_{L^2_{\mathcal{P}}(\mu_M)} < \infty$. Then by (i) we get

$$\left(\int_0^t \chi_{[0,s]}\mathcal{G}dM\right)(\cdot) = cl_{\mathbb{R}^d}\left\{\left(\int_0^t \chi_{[0,s]}(\tau)\, g^n_\tau dM_\tau\right)(\cdot) : n \geq 1\right\}$$

$$= cl_{\mathbb{R}^d}\left\{\left(\int_0^t \chi_{[0,s]}(\tau)\, g_\tau^n dM_\tau\right)(\cdot) : n \geq 1\right\}$$
$$= cl_{\mathbb{R}^d}\left\{\left(\int_0^s g_\tau^n dM_\tau\right)(\cdot) : n \geq 1\right\} = \left(\int_0^s \mathcal{G} dM\right)(\cdot) \text{ P-a.e.}$$

This completes the proof. ■

By Burkholder's inequality (cf. Theorem 54, Chapter IV in [43]) and (2.2) as well as Corollary 6 and (2.8) one can prove the following further properties, which imply integrably boundedness of Itô set-valued integrals for absolutely summable sets.

Corollary 9. *Let the sets* $\mathcal{G} = \{g^n : n \geq 1\}$, $\mathcal{H} = \{h^n : n \geq 1\} \subset L^2_{\mathcal{P}}(\mu_M)$ *be absolutely summable. Then for every* $t \in I$

$$E\left[\sup_{0\leq s\leq t} H^2_{\mathbb{R}^d}\left(\int_0^s \mathcal{G} dM, \int_0^s \mathcal{H} dM\right)\right] \leq 8\sum_{n=1}^{\infty}\int_{[0,t]\times\Omega} ||g^n - h^n||^2_{\mathbb{R}^d} d\mu_M$$

and

$$E\left[\sup_{0\leq s\leq t} H^2_{\mathbb{R}^d}\left(\int_0^s cl_{L^2_{\mathcal{P}}(\mu_M)} co(\mathcal{G}) dM, \int_0^s cl_{L^2_{\mathcal{P}}(\mu_M)} co(\mathcal{H}) dM\right)\right]$$
$$\leq 8\sum_{n=1}^{\infty}\int_{[0,t]\times\Omega} ||g^n - h^n||^2_{\mathbb{R}^d} d\mu_M.$$

By Theorem 8, similarly as in [25], the following result can be proved.

Theorem 10. *For an absolutely summable set* $\mathcal{G} = \{g^n : n \geq 1\} \subset L^2_{\mathcal{P}}(\mu_M)$ *a set-valued process* $\left(\int_0^t \mathcal{G} dM_\tau\right)_{t\geq 0}$ *is adapted and it has continuous (multivalued) paths.*

Let $\mathcal{L}^2_{\mathbf{F}}$ be a space of set-valued random variables with nonempty, compact and convex values in $\mathbb{R}^d$, endowed with a metric σ_2 (see (2.12)):

$$\sigma_2(G_1, G_2) = \left(\int_\Omega H^2_{\mathbb{R}^d}(G_1, G_2)\, dP\right)^{\frac{1}{2}} \tag{3.6}$$

for $G_1, G_2 \in \mathcal{L}^2_{\mathbf{F}}$. Then, we have

Corollary 11. *Under the assumptions of Theorem 10 the mapping*

$$I \ni t \to \int_0^t \mathcal{G} dM \in \mathcal{L}^2_{\mathbf{F}}$$

is σ_2*-continuous.*

3.2. Stochastic Aumann integral

Now, we shall consider a notion of stochastic Aumann integral with respect to the continuous, adapted and increasing process $[M, M]$. As in a single valued case it can be defined ω-by-ω. We assume that $F : I \times \Omega \to \mathcal{K}_c^b(\mathbb{R}^d)$ is a predictable and $L^2_{\mathcal{P}}(\nu_M)$-integrably bounded set-valued process. Hence, it possesses the predictable Castaing representation $\{a_n : n \geq 1\} \subset L^2_{\mathcal{P}}(\nu_M)$, i.e., $F(t, \omega) = cl_{\mathbb{R}^d}\{a_n(t, \omega) : n \geq 1\}$ for all $(t, \omega) \in I \times \Omega$ (see e.g. Theorem 1.0 in [16]). Hence taking (for any fixed $\omega \in \Omega$) sequences $\{a_n(\cdot, \omega) : n \geq 1\}$ of $\mathcal{B}(I)$-measurable functions, it follows again by Theorem 1.0 in [16] that a set-valued mapping $I \ni t \to F(t, \omega) \in \mathcal{K}_c^b(\mathbb{R}^d)$ is $\mathcal{B}(I)$-measurable. Since $\int_{I \times \Omega} |||F|||^2_{\mathbb{R}^d} d\nu_M < \infty$, we get $\int_0^T |||F(t, \cdot)|||^2_{\mathbb{R}^d} \gamma(dt, \cdot) < \infty$ P-a.e, where γ is a measure defined in (2.3). By the Cauchy-Schwartz inequality we have

$$\left(\int_0^T |||F(t, \cdot)|||_{\mathbb{R}^d} d[M, M]_t (\cdot) \right)^2 \leq \int_0^T |||F(t, \cdot)|||^2_{\mathbb{R}^d} \gamma(dt, \cdot) \ P\text{-a.e.}$$

Hence $\int_0^T |||F(t, \cdot)|||_{\mathbb{R}^d} d[M, M]_t (\cdot) < \infty$ P-a.e. too. Hence for P-a.e. $\omega \in \Omega$, the set-valued function $F(\cdot, \omega)$ is $L^1_{\mathcal{B}(I)}(d[M, M]_{(\cdot)} (\omega))$-integrably bounded where

$$L^1_{\mathcal{B}(I)}(d[M, M]_{(\cdot)} (\omega)) := L^1(I, \mathcal{B}(I), d[M, M]_{(\cdot)} (\omega) ; \mathbb{R}^d).$$

The above notions allows us to consider the set $S^1_{\mathcal{B}(I)}(F, d[M, M]_{(\cdot)} (\omega))$ of all $L^1_{\mathcal{B}(I)}(d[M, M]_{(\cdot)} (\omega))$-selections for $F(\cdot, \omega)$, i.e.,

$$\begin{aligned} &S^1_{\mathcal{B}(I)} \left(F, d[M, M]_{(\cdot)} (\omega)\right) \\ &:= \left\{ a \in L^1_{\mathcal{B}(I)} \left(d[M, M]_{(\cdot)} (\omega)\right) : a(t) \in F(t, \omega) \text{ for } t \in I \ d[M, M]_{(\cdot)} (\omega)\text{-a.e.} \right\}. \end{aligned}$$

Since the measure $d[M, M]_{(\cdot)} (\omega)$ is positive and nonatomic, it is possible now to define a parametrized Aumann integral of $F(\cdot, \omega)$ with respect to the measure $d[M, M]_{(\cdot)} (\omega)$ over the interval I.

Definition 12. By a set-valued stochastic Aumann integral over I, of a predictable and $L^2_{\mathcal{P}}(\nu_M)$-integrably bounded set-valued process F with respect to quadratic variation process $[M, M]$, we mean a set-valued mapping

$$\Omega \ni \omega \to \int_0^T F(\tau, \omega) d[M, M]_\tau (\omega) \in \mathcal{K}_c^b(\mathbb{R}^d)$$

defined as

$$\int_0^T F(\tau, \omega) d[M, M]_\tau (\omega)$$

$$:= \left\{ \int_0^T a(\tau) d[M, M]_\tau (\omega) : a \in S^1_{\mathcal{B}(I)}(F, d[M, M]_{(\cdot)} (\omega)) \right\}. \tag{3.7}$$

For every $t \in I$ and $\omega \in \Omega$ we also put $\int_0^t F(\tau,\omega)d[M,M]_\tau(\omega) := \int_0^T \mathbb{I}_{[0,t]}(\tau)$ $F(\tau,\omega)d[M,M]_\tau(\omega)$. Let $F,\ G : I \times \Omega \to \mathcal{K}_c^b(\mathbb{R}^d)$ be predictable and $L^2_{\mathcal{P}}(\nu_M)$-integrably bounded set-valued processes. Immediately, by (3.7) and Theorem 4.1 in [16] we get:

Proposition 13. *Let $F,\ G : I\times\Omega \to \mathcal{K}_c^b(\mathbb{R}^d)$ be predictable and $L^2_{\mathcal{P}}(\nu_A)$-integrably bounded set-valued processes. Then*

$$H_{\mathbb{R}^d}\left(\int_0^t F(\tau,\omega)d[M,M]_\tau(\omega)\,,\int_0^t G(\tau,\omega)d[M,M]_\tau(\omega)\right) \leq \int_0^t H_{\mathbb{R}^d}(F(\tau,\omega),G(\tau,\omega))d[M,M]_\tau(\omega) \tag{3.8}$$

for $\omega \in \Omega$ and $t \in I$.

Applying the Cauchy-Schwartz inequality, (2.3) and (2.4) to inequality (3.8) we obtain:

Corollary 14. *Under the assumptions of Proposition* 13, *it holds*

$$E\left\{H_{\mathbb{R}^d}\left(\int_0^t F(\tau,\omega)d[M,M]_\tau(\omega)\,,\int_0^t G(\tau,\omega)d[M,M]_\tau(\omega)\right)\right\} \leq \int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(F,G)d\nu_M$$

for $t \in I$.

Proposition 15. *If F is a predictable and $L^2_{\mathcal{P}}(\nu_M)$-integrably bounded set-valued process, then the set-valued process $\left\{\int_0^t F(\tau)d[M,M]_\tau : t \in I\right\}$ is $\{\mathbf{F}_t\}_{t\in I}$-adapted and it has continuous* (*multivalued*) *paths.*

Proof. Let $t \in I$ be fixed. By $\sigma(p,D) = \sup\{\langle p,d\rangle : d \in D\}$ we denote a support function of the set $D \in \mathcal{K}_c^b(\mathbb{R}^d)$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^d$. Let us consider a set-valued mapping described in (3.7). By properties of the Aumann integral (cf. Proposition 4.6.1 in [9]), for every $p \in \mathbb{R}^d$ it holds

$$\sigma\left(p,\int_0^t F(\tau,\omega)d[M,M]_\tau(\omega)\right) = \int_0^t \sigma(p,F(\tau,\omega))d[M,M]_\tau(\omega)\,. \tag{3.9}$$

Since F is predictable and $L^2_{\mathcal{P}}(\nu_M)$-integrably bounded, it follows that the real valued process $\sigma(p,F)$ is predictable and majorized by $|||F|||_{\mathbb{R}^d}$. Thus for every $p \in \mathbb{R}^d$, the real-valued stochastic process $\left\{\int_0^t \sigma(p,F(\tau,\cdot))d[M,M]_\tau(\cdot) : t \in I\right\}$ is $\{\mathbf{F}_t\}_{t\in I}$-adapted. Hence, by (3.9) it is also true for the process $\left\{\sigma\big(p,\int_0^t F(\tau,\cdot)d[M,M]_\tau(\cdot)\big) : t \in I\right\}$ for every $p \in \mathbb{R}^d$. Now by Proposition

4.3.16 in [9], we infer that the set-valued process $\left\{ \int_0^t F(\tau,\cdot)d[M,M]_\tau(\cdot) : t \in I \right\}$ is $\{\mathbf{F}_t\}_{t\in I}$-adapted. In order to prove its continuity we let $s < t$. By (3.8) we get

$$\begin{aligned}
&H_{\mathbb{R}^d}\left(\int_0^t F(\tau,\omega)d[M,M]_\tau(\omega), \int_0^s F(\tau,\omega)d[M,M]_\tau(\omega)\right) \\
&= H_{\mathbb{R}^d}\left(\int_0^T \chi_{[0,t]}(\tau)\, F(\tau,\omega)d[M,M]_\tau(\omega), \int_0^T \chi_{[0,s]}(\tau)\, F(\tau,\omega)d[M,M]_\tau(\omega)\right) \\
&\leq \int_s^t |||(F(\tau,\omega)|||_{\mathbb{R}^d} d[M,M]_\tau(\omega)
\end{aligned}$$

for $\omega \in \Omega$. Thus

$$\lim_{t\to s+} H_{\mathbb{R}^d}\left(\int_0^t F(\tau,\omega)d[M,M]_\tau(\omega), \int_0^s F(\tau,\omega)d[M,M]_\tau(\omega)\right) = 0.$$

Similarly, we have

$$\lim_{t\to s-} H_{\mathbb{R}^d}\left(\int_0^t F(\tau,\omega)d[M,M]_\tau(\omega), \int_0^s F(\tau,\omega)d[M,M]_\tau(\omega)\right) = 0$$

for $\omega \in \Omega$. Thus the set-valued process $\left\{ \int_0^t F(\tau)d[M,M]_\tau : t \in I \right\}$ has continuous paths. ■

By Corollary 14, the Aumann stochastic integral can be treated as a mapping with values in the metric space $\left(\mathcal{L}^2_{\mathbf{F}}, \sigma_2\right)$, where σ_2 is defined by (3.6). Then by the Lebesgue dominated convergence theorem we get:

Corollary 16. *If F is a predictable and $L^2_{\mathcal{P}}(\nu_M)$-integrably bounded set-valued process, then the mapping*

$$I \ni t \to \int_0^t F(\tau)d[M,M]_\tau \in \mathcal{L}^2_{\mathbf{F}}$$

is σ_2-continuous.

If $M = W$ where W is a standard one dimensional Wiener process, then $[W,W]_t = t$ and the Aumann stochastic integral described in this section reduces to the random (or parametrized) Aumann integral. Such integrals were studied in [31, 30, 32]. Other features (in particular selection properties) of parametrized Aumann integrals were studied in particular in [3, 15, 40] and [44].

4. Set-valued stochastic differential equations

In this part we apply results of preceding sections to set-valued stochastic equations. Let $\mathcal{P}^{\Omega} := \{B \times \Omega : B \in \mathcal{B}(I)\}$. Then, due to Monotone Class Theorem (see e.g. [6]) it follows that $\mathcal{P}^{\Omega}$ is a sub-σ-field of the σ-field $\mathcal{P}$. Let $\tilde{\nu}_M$ and $\tilde{\mu}_M$ denote marginal measures of ν_M and μ_M, respectively, defined on $(I, \mathcal{B}(I))$ by formulas

$$\tilde{\nu}_M(B) = \nu_M(B \times \Omega) \text{ and } \tilde{\mu}_M(B) = \mu_M(B \times \Omega) \text{ for } B \in \mathcal{B}(I).$$

Then, the function $\delta : I \to \mathbb{R}_+$ defined by

$$\delta(t) = \tilde{\nu}_M([0,t]) + \tilde{\mu}_M([0,t]) \tag{4.1}$$

is continuous and nondecreasing. Then (c.f. [36], Corollary 1) it holds

$$\int_0^t h(s)\, d\delta(s) = \int_{[0,t]\times\Omega} h(s)\left(\nu_M(ds, d\omega) + \mu_M(ds, d\omega)\right) \text{ for } t \in I. \tag{4.2}$$

As before, let $\mathcal{L}^2_{\mathbf{F}}$ be a space of set-valued random variables with nonempty, compact and convex values in $\mathbb{R}^d$, endowed with a metric σ_2 given by (3.6). Let F be such that $F : I \times \Omega \times \mathcal{L}^2_{\mathbf{F}} \to \mathcal{K}^b_c(\mathbb{R}^d)$, and let $\mathcal{G} = \{g_n : n \geq 1\}$ with $g_n : I \times \Omega \times \mathcal{L}^2_{\mathbf{F}} \to \mathbb{R}^d$ for $n \geq 1$. Similarly as before we use the notation: for $X : I \to \mathcal{L}^2_{\mathbf{F}}$ let $\mathcal{G} \circ X := cl_{L^2_{\mathcal{P}}(\mu_M)} co\{g_n \circ X : n \geq 1\}$ where $(g_n \circ X)(t, \omega) = g(t, \omega, X(t))$ for $(t, \omega) \in I \times \Omega$. In this part we assume that the space $(\Omega, \mathbf{F}, \{\mathbf{F}_t\}_{t\in I}, P)$ is separable. Then by Proposition 4, $\left(\mathcal{L}^2_{\mathbf{F}}, \sigma_2\right)$ is a separable and complete metric space. By $C(I, \mathcal{L}^2_{\mathbf{F}})$ we denote the space of all continuous mappings $X : I \to \mathcal{L}^2_{\mathbf{F}}$ with a metric ρ:

$$\rho(X, Y) = \sup_{t\in I} \sigma_2(X(t), Y(t)) \text{ for } X, Y \in C(I, \mathcal{L}^2_{\mathbf{F}}).$$

Then $\left(C(I, \mathcal{L}^2_{\mathbf{F}}), \rho\right)$ is a Polish metric space. We consider a set-valued stochastic equation driven by integrals established in preceding sections, i.e.,

$$X(t) = \xi + \int_0^t F(s, X(s))d[M, M]_s + \int_0^t \mathcal{G} \circ X dM, \ t \in I. \tag{4.3}$$

By a solution to equation (4.3) we mean a set-valued function $X \in C(I, \mathcal{L}^2_{\mathbf{F}})$ such that relation (4.3) is satisfied. It is unique, if $X(t) = Y(t)$ for all $t \in I$ where $Y : I \to \mathcal{L}^2_{\mathbf{F}}$ is any other solution to (4.3). By the same symbol Θ we will denote a zero element both in the space $\mathcal{L}^2_{\mathbf{F}}$ and $C(I, \mathcal{L}^2_{\mathbf{F}})$. Now, for coefficients X_0, F and $\mathcal{G}$ in (4.3) we assume the following:

(A0) $\xi \in \mathcal{L}^2_{\mathbf{F}}$ and it is $\mathbf{F}_0$-measurable, (A1) set-valued mappings $F(\cdot,\cdot,\eta)$ and $g_n(\cdot,\cdot,\eta)$ are $\mathcal{P}$-measurable for every $\eta \in \mathcal{L}^2_{\mathbf{F}}$ and $n \geq 1$,

(A2) there exist constants $C > 0$, $L > 0$ such that for every $(t,\omega) \in I \times \Omega$, and every $\eta, \eta_1, \eta_2 \in \mathcal{L}^2_{\mathbf{F}}$

$$|||F(t,\omega,\eta)|||_{\mathbb{R}^d} \leq C(1+\sigma_2(\eta,\Theta))$$

and

$$H_{\mathbb{R}^d}\left(F(t,\omega,\eta_1), F(t,\omega,\eta_2)\right) \leq L\sigma_2(\eta_1,\eta_2)$$

(A3) there exist sequences $(L_n)_{n=1}^{\infty}$, $(M_n)_{n=1}^{\infty}$ of positive numbers such that

$$\max\left\{\sum_{n=1}^{\infty} L_n^2, \sum_{n=1}^{\infty} M_n^2\right\} < \infty$$

and

$$||g_n(t,\omega,\eta)||_{\mathbb{R}^d} \leq L_n(1+\sigma_2(\eta,\Theta)),$$
$$||g_n(t,\omega,\eta_1) - g_n(t,\omega,\eta_2)||_{\mathbb{R}^d} \leq M_n\sigma_2(\eta_1,\eta_2)$$

for every $n \geq 1$, $(t,\omega) \in I \times \Omega$ and every $\eta, \eta_1, \eta_2 \in \mathcal{L}^2_{\mathbf{F}}$.

Note that since $\left(\mathcal{L}^2_{\mathbf{F}}, \sigma_2\right)$ is a separable metric space, it follows that if F and $g_n, n \geq 1$, are Carathéodory's mappings, then they are jointly measurable, i.e., $\mathcal{P} \otimes \mathcal{B}\left(\mathcal{L}^2_{\mathbf{F}}\right) | \mathcal{B}\left(\mathcal{K}^b_c(\mathbb{R}^d)\right)$ and $\mathcal{P} \otimes \mathcal{B}\left(\mathcal{L}^2_{\mathbf{F}}\right) | \mathcal{B}\left(\mathbb{R}^d\right)$ measurable, respectively (see e.g. Proposition 1.6 or Proposition 7.9 of Chapter 2 in [17]). Hence if they satisfy (A1)–(A3), then for any $X \in C\big(I, \mathcal{L}^2_{\mathbf{F}}\big)$ mappings $F \circ X$ and $g_n \circ X$ are predictable and integrably bounded. Moreover, the set $\mathcal{G} \circ X$ is absolutely summable.

Theorem 17. *Assume that* η, F *and* $\{g_n : n \geq 1\}$ *satisfy conditions* (A0)–(A3). *Then there exists a unique solution* $X \in C\big(I, \mathcal{L}^2_{\mathbf{F}}\big)$ *to equation* (4.3) *such that*

$$\rho^2(X,\Theta) \leq \left(3\sigma_2^2(\eta,\Theta) + \tilde{C}\delta(T)\right) e^{\tilde{C}\delta(T)}, \tag{4.4}$$

where $\tilde{C} = \max\left\{6C^2, 12\sum_{n=1}^{\infty} L_n^2\right\}$ *and* δ *satisfies* (4.1).

Proof. Let us fix $t \in I$. For $X, Y \in C(I, \mathcal{L}^2_{\mathbf{F}})$ let

$$\Psi(t) := \sigma_2^2\Big\{\xi + \int_0^t F(s,X(s))d[M,M]_s + \int_0^t \mathcal{G} \circ X dM,$$
$$\xi + \int_0^t F(s,Y(s))d[M,M]_s + \int_0^t \mathcal{G} \circ Y dM\Big\}.$$

Then for any $X, Y \in C\big(I, \mathcal{L}^2_{\mathbf{F}}\big)$, by Proposition 3, Corollary 14 and Theorem 8 (ii) we get

$$\begin{aligned}\Psi(t) &\leq 2\sigma_2^2\left(\int_0^t F(s, X(s))d[M,M]_s, \int_0^t F(s, Y(s))d[M,M]_s\right)\\ &+ 2\sigma_2^2\left(\int_0^t \mathcal{G}\circ X dM, (\mathcal{F})\int_0^t \mathcal{G}\circ Y dM\right)\\ &\leq 2\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(F\circ X, F\circ Y)d\nu_M + 4\sum_{n=1}^{\infty}\int_{[0,t]\times\Omega} ||g_n\circ X - g_n\circ Y||^2_{\mathbb{R}^d} d\mu_M.\end{aligned}$$

Since F and g_n are Lipschitzean, we obtain

$$\begin{aligned}\Psi(t) &\leq 2L^2\int_{[0,t]\times\Omega}\sigma_2^2(X(s), Y(s))\nu_M(ds, d\omega)\\ &+ 4\sum_{n=1}^{\infty} M_n^2\int_{[0,t]\times\Omega}\sigma_2^2(X(s), Y(s))\mu_M(ds, d\omega)\\ &\leq \bar{L}\int_{[0,t]\times\Omega}\sigma_2^2(X(s), Y(s))\left(\nu_M(ds, d\omega) + \mu_M(ds, d\omega)\right),\end{aligned}$$

where $\bar{L} = \max\{2L^2, 4\sum_{n=1}^{\infty} M_n^2\}$. Thus by (4.2) we get

$$\Psi(t) \leq \bar{L}\int_0^t \sigma_2^2(X(s), Y(s))d\delta\,(s)\,. \tag{4.5}$$

Let us consider in $C(I, \mathcal{L}^2_{\mathbf{F}})$ a new metric

$$\hat{\rho}(X, Y) = \sup_{t\in I}\left\{e^{-\bar{L}\delta(t)}\sigma_2\left(X(t), Y(t)\right)\right\}$$

for $X, Y \in C(I, \mathcal{L}^2_{\mathbf{F}})$. Notice that

$$e^{-\bar{L}\delta(T)}\rho(X, Y) \leq \hat{\rho}(X, Y) \leq \rho(X, Y).$$

Thus the metrics ρ and $\hat{\rho}$ are equivalent and therefore, $\big(C(I, \mathcal{L}^2_{\mathbf{F}}), \hat{\rho}\big)$ is a complete metric space as well. Next, let us consider a mapping $V : C(I, \mathcal{L}^2_{\mathbf{F}}) \to C(I, \mathcal{L}^2_{\mathbf{F}})$ defined as follows

$$V(X)(t) = \xi + \int_0^t F(s, X(s))d[M,M]_s + \int_0^t \mathcal{G}\circ X dM,\ t \in I.$$

In order to obtain the existence and the uniqueness of solutions to (4.3) it is enough to check (by virtue of Banach Contraction Principle) that V is a contraction under the metric $\hat{\rho}$. Let $X, Y \in C(I, \mathcal{L}^2_{\mathbf{F}})$. Then by (4.5) and properties of the function δ we have

$$
\begin{aligned}
\hat{\rho}^2\left(V(X),V(Y)\right) &= \sup_{t\in I}\left\{e^{-2\bar{L}\delta(t)}\Psi(t)\right\} \\
&\leq \bar{L}\sup_{t\in I}\left\{e^{-2\bar{L}\delta(t)}\int_0^t \sigma_2^2(X(s),Y(s))d\delta\left(s\right)\right\} \\
&\leq \bar{L}\sup_{t\in I}\left\{e^{-2\bar{L}\delta(t)}\int_0^t \sigma_2^2(X(s),Y(s))d\delta\left(s\right)\right\} \\
&\leq \bar{L}\sup_{t\in I}\left\{e^{-2\bar{L}\delta(t)}\hat{\rho}^2\left(X,Y\right)\int_0^t e^{2\bar{L}\delta(s)}d\delta\left(s\right)\right\} \\
&\leq \frac{1}{2}\left(1-e^{-2\bar{L}\delta(T)}\right)\hat{\rho}^2(X,Y).
\end{aligned}
$$

Hence V is a contraction with respect to $\hat{\rho}$ and therefore, there exists a unique element $X \in C(I,\mathcal{L}^2_{\mathbf{F}})$ being a solution to equation (4.3). We show that X satisfies inequality (4.4). Again by properties stated in Proposition 3, Corollary 14 and Theorem 8 as well as by (A2) and (A3) we have

$$
\begin{aligned}
&\sigma_2^2\left(X(t),\Theta\right) \\
&\leq 3\sigma_2^2\left(\xi,\Theta\right)+3\sigma_2^2\left(\int_0^t F(s,X(s))d[M,M]_s,\Theta\right)+3\sigma_2^2\left(\int_0^t \mathcal{G}\circ X dM,\Theta\right) \\
&\leq 3\sigma_2^2\left(\xi,\Theta\right)+3\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(F\circ X,\{0\})d\nu_M+6\sum_{n=1}^{\infty}\int_{[0,t]\times\Omega}||g_n\circ X||^2_{\mathbb{R}^d}d\mu_M \\
&\leq 3\sigma_2^2\left(\xi,\Theta\right)+\tilde{C}(\nu_M\left([0,T]\times\Omega\right)+\mu_M\left([0,T]\times\Omega\right)) \\
&+\tilde{C}\int_{[0,t]\times\Omega}\sigma_2^2(x(s),\Theta)(\nu_M\left(ds,d\omega\right)+\mu_M\left(ds,d\omega\right))
\end{aligned}
$$

where $\tilde{C}=\max\left\{6C^2,12\sum_{n=1}^{\infty}L_n^2\right\}$. Hence, we infer by (4.2) that

$$\sigma_2^2\left(x(t),\Theta\right)\leq 3\sigma_2^2\left(\xi,\Theta\right)+\tilde{C}\delta(T)+\tilde{C}\int_0^t\sigma_2^2(x(s),\Theta)d\delta(s).$$

Let $U_t=\sigma_2^2\left(x(t),\Theta\right)$, $B_t=\tilde{C}\delta(t)$ and $N_t=3\sigma_2^2\left(\xi,\Theta\right)+\tilde{C}\delta(T)$ for $t\in I$. Then they are continuous deterministic processes and such that B is increasing while N is a constant semimartingale. Thus the above inequality can be rewritten as

$$U_t\leq N_t+\int_0^t U_s dB_s \text{ for } \ t\in I.$$

Thus by Theorem 1 we get

$$U_t\leq e^{B_t}\left(N_0e^{-B_0}+\int_0^t e^{-B_s}dN_s\right) \text{ for } \ t\in I.$$

Since $\int_0^t e^{-B_s} dN_s = 0$ for $t \in I$ and $B_0 = 0$, it follows that

$$U_t \leq N_0 e^{B_t} \text{ for } t \in I.$$

Hence we get (4.4) what finishes the proof. ■

Let $X_{\xi,F,\mathcal{G}}$ denote a unique solution to equation (4.3) with "coefficients" ξ, F and $\mathcal{G}$ that may change. If F and $\mathcal{G}$ are fixed, then we indicate the dependence of X only on ξ by X_ξ. We shall prove the following auxiliary result.

Proposition 18. *Let $\xi, \eta \in \mathcal{L}^2_{\mathbf{F}}$ be $\mathbf{F}_0$-measurable set-valued random variables. Suppose that $F_1, F_2 : I \times \Omega \times \mathcal{L}^2_{\mathbf{F}} \to \mathcal{K}^b_c(\mathbb{R}^d)$, $\mathcal{G}_1 = \{g_n^{(1)} : n \geq 1\}$ and $\mathcal{G}_2 = \{g_n^{(2)} : n \geq 1\}$ satisfy conditions* A(1)–A(3). *Then*

$$\begin{aligned} &\rho^2\big(X_{\xi,F_1,\mathcal{G}_1}, X_{\eta,F_2,\mathcal{G}_2}\big) \\ &\leq \bigg[3\sigma_2^2(\xi,\eta) + 6\int_{[0,T]\times\Omega} H^2_{\mathbb{R}^d}\big(F_1 \circ X_{\eta,F_2,\mathcal{G}_2}, F_2 \circ X_{\eta,F_2,\mathcal{G}_2}\big) d\nu_M \\ &+ 12\sum_{n=1}^{\infty} \int_{[0,T]\times\Omega} \Big\| g_n^{(1)} \circ X_{\eta,F_2,\mathcal{G}_2} - g_n^{(2)} \circ X_{\eta,F_2,\mathcal{G}_2} \Big\|^2_{\mathbb{R}^d} d\mu_M\bigg] e^{\tilde{L}\delta(T)} \end{aligned} \tag{4.6}$$

where $\tilde{L} = \max\{6L^2, 12\sum_{n=1}^{\infty} M_n^2\}$.

Proof. For the sake of simplicity let $X_1 = X_{\xi,F_1,\mathcal{G}_1}$ and $X_2 = X_{\eta,F_2,\mathcal{G}_2}$. Using similar argumentations and notations as in the proof of the preceding theorem, for every $t \in I$, we have

$$\begin{aligned} \sigma_2^2\left(X_1(t), X_2(t)\right) &\leq 3\sigma_2^2(\xi,\eta) + 3\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}\big(F_1 \circ X_1, F_2 \circ X_2\big) d\nu_M \\ &+ 6\sum_{n=1}^{\infty} \int_{[0,t]\times\Omega} \Big\| g_n^{(1)} \circ X_1 - g_n^{(2)} \circ X_2 \Big\|^2_{\mathbb{R}^d} d\mu_M \\ &\leq 3\,\delta_2^2(\xi,\eta) + 6\int_{[0,t]\times\Omega} H^2_{\mathbb{R}^d}(F_1 \circ X_1, F_1 \circ X_2) d\nu_M \\ &+ 6\int_{[0,T]\times\Omega} H^2_{\mathbb{R}^d}(F_1 \circ X_2, F_2 \circ X_2) d\nu_M \\ &+ 12\sum_{n=1}^{\infty} \int_{[0,t]\times\Omega} \Big\| g_n^{(1)} \circ X_1 - g_n^{(1)} \circ X_2 \Big\|^2_{\mathbb{R}^d} d\mu_M \\ &+ 12\sum_{n=1}^{\infty} \int_{[0,T]\times\Omega} \Big\| g_n^{(1)} \circ X_2 - g_n^{(2)} \circ X_2 \Big\|^2_{\mathbb{R}^d} d\mu_M. \end{aligned}$$

Let

$$N_t = 3\sigma_2^2(\xi,\eta) + 6\int_{[0,T]\times\Omega} H^2_{\mathbb{R}^d}\big(F_1 \circ X_2, F_2 \circ X_2\big) d\nu_M$$

$$+\,12\sum_{n=1}^{\infty}\int_{[0,T]\times\Omega}\left\|g_n^{(1)}\circ X_2-g_n^{(2)}\circ X_2\right\|^2_{\mathbb{R}^d}d\mu_M \text{ for } t\in I.$$

Clearly, N_t does not depend on $t\in I$. Moreover, by linear growth conditions imposed on $F_1, F_2, g_n^{(1)}, g_n^{(2)}$ in (A2) and (A3), and by (4.4), it follows that N_t is finite. Then due to Lipschitz conditions imposed on F_1 and $g_n^{(1)}$ in (A2) and (A3), one can rewrite above inequalities as

$$\begin{aligned}\sigma_2^2\left(X_1(t),X_2(t)\right) &\le N_t+6L^2\int_{[0,t]\times\Omega}\sigma_2^2(X_1(s),X_2(s))\nu_M(ds,d\omega)\\ &\quad+12\sum_{n=1}^{\infty}M_n^2\int_{[0,t]\times\Omega}\sigma_2^2(X_1(s),X_2(s))\mu_M(ds,d\omega)\\ &\le N_t+\tilde{L}\int_0^t\sigma_2^2(X_1(s),X_2(s))d\delta\left(s\right),\end{aligned}$$

where $\tilde{L}=\max\left\{6L^2,12\sum_{n=1}^{\infty}M_n^2\right\}$. Thus we have

$$U_t\le N_t+\int_0^t U_s dB_s \text{ for } t\in I,$$

where $U_t=\sigma_2^2\left(X_1(t),X_2(t)\right)$ and $B_t=\tilde{L}\delta\left(t\right)$ for $t\in I$. Let us note that processes U,N and B are continuous and deterministic. Moreover, N is a constant semimartingale, while B is nondecreasing. Thus due to Theorem 1 we obtain

$$U_t\le e^{B_t}\left(N_0e^{-B_0}+\int_0^t e^{-B_s}dN_s\right) \text{ for } t\in I.$$

Again, $\int_0^t e^{-B_s}dN_s=0$ for $t\in I$ and $B_0=0$. Therefore,

$$U_t\le N_0e^{B_t} \text{ for } t\in I$$

and thus we get inequality (4.6), which completes the proof. ■

As a consequence, we obtain the following Lipschitz type dependence of solutions to (4.3) on initial conditions.

Corollary 19. *Let $\xi,\eta\in\mathcal{L}^2_{\mathbf{F}}$ be $\mathbf{F}_0$-measurable set-valued random variables. Suppose F and $\mathcal{G}=\{g_n : n\ge 1\}$ satisfy assumptions of Theorem* 17. *Then*

$$\rho^2\left(X_\xi,X_\eta\right)\le 6\sigma_2^2(\xi,\eta)e^{\tilde{L}\delta(T)},$$

where $\tilde{L}=\max\left\{6L^2,12\sum_{n=1}^{\infty}M_n^2\right\}$.

Another consequence of Proposition 18 is a convergence property of solutions to (4.3). For this aim, apart of equation (4.3) we consider a sequence of equations (for $k = 1, 2, \dots$)

$$X(t) = \xi_k + \int_0^t F_k(s, X(s))d[M, M]_s + \int_0^t \mathcal{G}_k \circ X dM, \ t \in I$$

where $\mathcal{G}_k = \{g_n^{(k)} : n \geq 1\}$ and $\mathcal{G}_k \circ X = cl_{L^2_{\mathcal{P}}(\mu_M)} co\{g_n^{(k)} \circ X : n \geq 1\}$ for $X \in C(I, \mathcal{L}^2_{\mathbf{F}})$. As before $\mathcal{G} = \{g_n : n \geq 1\}$ and $\mathcal{G} \circ X = cl_{L^2_{\mathcal{P}}(\mu_M)} co\{g_n \circ X : n \geq 1\}$.

Corollary 20. *Let $\xi, \xi_k \in \mathcal{L}^2_{\mathbf{F}}$ be $\mathbf{F}_0$-measurable set-valued random variables such that*

$$\sigma_2(\xi_k, \xi) \to 0, \ \text{as } k \to \infty.$$

Assume that $F, \mathcal{G}$ and $F_k, \mathcal{G}_k$ satisfy conditions (A1)–(A3) ($k = 1, 2, \dots$), *and for every $U \in \mathcal{L}^2_{\mathbf{F}}$*

$$H_{\mathbb{R}^d}(F_k(\cdot, \cdot, U), F(\cdot, \cdot, U)) \to 0 \ \nu_A\text{-a.e. as } k \to \infty. \tag{4.7}$$

Moreover, let for every $n \geq 1$

$$\left\| g_n^{(k)}(\cdot, \cdot, U) - g_n(\cdot, \cdot, U) \right\|_{\mathbb{R}^d} \to 0 \ \mu_M\text{-a.e. as } k \to \infty. \tag{4.8}$$

Then

$$\rho\big(X_{\xi_k, F_k, \mathcal{G}_k}, X_{\xi, F, \mathcal{G}}\big) \to 0, \ \text{as } k \to \infty.$$

Proof. Let $X = X_{\xi, F, \mathcal{G}}$ denote a unique solution to (4.3). By Proposition 18 and assumptions, it is sufficient to show that

$$\int_{[0,T] \times \Omega} H^2_{\mathbb{R}^d}(F_k \circ X, F \circ X) d\nu_M \to 0, \text{ as } k \to \infty \tag{4.9}$$

and

$$\sum_{n=1}^{\infty} \int_{[0,T] \times \Omega} \left\| g_n^{(k)} \circ X - g_n \circ X \right\|^2_{\mathbb{R}^d} d\mu_M \to 0, \text{ as } k \to \infty. \tag{4.10}$$

We prove (4.9). By (A2) and (4.4), for every $k \geq 1$ and $(t, \omega) \in I \times \Omega$ we have

$$H^2_{\mathbb{R}^d}(F_k(t, \omega, x(t)), F(t, \omega, x(t))) \leq 8C^2(1 + \sigma_2^2(X(t), \Theta))$$
$$\leq 8C^2\left(1 + 2\left(3\sigma_2^2(\xi, \Theta) + \tilde{C}\delta(T)\right)e^{\tilde{C}\delta(T)}\right),$$

where $\tilde{C}$ and δ are as in Theorem 17. For any $k = 1, 2, \dots$, let

$$h_k(t, \omega) = H^2_{\mathbb{R}^d}(F_k(t, \omega, X(t)), F(t, \omega, X(t))).$$

Thus the functions h_k are $\mathcal{P}$-measurable and nonnegative. By the above inequality they are majorized (uniformly) by a finite constant. Thus due to (4.7) and Lebesgue dominated convergence theorem we obtain (4.9).
In order to check (4.10) we proceed in a similar way. By (4.8) and similar argumentations we have for every $n \geq 1$

$$\int_{[0,T]\times\Omega} \left\| g_n^{(k)} \circ X - g_n \circ X \right\|_{\mathbb{R}^d}^2 d\mu_M \to 0, \text{ as } k \to \infty. \tag{4.11}$$

Let $y^{(k)} : \mathbb{N} \to \mathbb{R}_+$ be a sequence defined by

$$y^{(k)}(n) = \int_{[0,T]\times\Omega} \left\| g_n^{(k)} \circ X - g_n \circ X \right\|_{\mathbb{R}^d}^2 d\mu_M \text{ for } n = 1, 2, \ldots, \ k = 1, 2, \ldots$$

By (A3) and (4.4) we get

$$y^{(k)}(n) \leq 8L_n^2 \left(1 + 2\left(3\sigma_2^2(\xi, \Theta) + \tilde{C}\delta(T)\right) e^{\tilde{C}\delta(T)}\right) \delta(T) \tag{4.12}$$

for every $n \geq 1$ and $k \geq 1$. Let us consider a counting measure ϑ on $\mathbb{N}$. Then

$$\int y^{(k)} d\vartheta = \sum_{n=1}^{\infty} \int_{[0,T]\times\Omega} \left\| g_n^{(k)} \circ X - g_n \circ X \right\|_{\mathbb{R}^d}^2 d\mu_M.$$

Thus (4.10) follows by (4.11), (4.12) and by Lebesgue dominated convergence theorem. This completes the proof. ■

Corollary 21. *Let $\xi, \xi_k \in \mathcal{L}^2_{\mathbf{F}}$ be $\mathbf{F}_0$-measurable set-valued random variables such that $\xi_1 \supset \xi_2 \supset \cdots \supset \xi$ and $\xi = \bigcap_{k\geq 1} \xi_k$ P-a.e. Assume that $F, \mathcal{G}$ and $F_k, \mathcal{G}_k$ satisfy conditions* (A1)–(A3) $(k = 1, 2, \ldots)$, *and for every $U \in \mathcal{L}^2_{\mathbf{F}}$*

$$F_1(\cdot,\cdot,U) \supset F_2(\cdot,\cdot,U) \supset \cdots \supset F(\cdot,\cdot,U) \text{ and } F(\cdot,\cdot,U) = \bigcap_{k\geq 1} F_k(\cdot,\cdot,U) \ \nu_A\text{-a.e.}$$

Moreover, let for every $n \geq 1$

$$\left\| g_n^{(k)}(\cdot,\cdot,U) - g_n(\cdot,\cdot,U) \right\|_{\mathbb{R}^d} \to 0 \ \mu_M\text{-a.e. as } k \to \infty.$$

Then

$$\rho\left(X_{\xi_k, F_k, \mathcal{G}_k}, X_{\xi, F, \mathcal{G}}\right) \to 0, \text{ as } k \to \infty.$$

Proof. By Proposition 2.4.5 in [10] we get $H_{\mathbb{R}^d}(\xi_k, \xi) \to 0$ P-a.e. and for every $U \in \mathcal{L}^2_{\mathbf{F}}$, $H_{\mathbb{R}^d}(F_k(\cdot,\cdot,U), F(\cdot,\cdot,U)) \to 0$ ν_M-a.e. as $k \to \infty$. Then by the Lebesgue dominated convergence theorem we also have that $\sigma_2(\xi_k, \xi) \to 0$, as $k \to \infty$. Hence by Corollary 20 it follows a required convergence for a sequence of solutions. ■

Remark 22. Let us note that equation (4.3) extends equations studied earlier. In particular, for M being a Wiener process W, (4.3) reduces to the equation

$$X(t) = \eta + \int_0^t F(s, X(s))ds + \int_0^t \mathcal{G} \circ X dW_s, \ t \in I$$

studied recently in [24]. Next if $\mathcal{G} = \{g\}$ then we obtain a class of equations with single valued diffusion terms, i.e.,

$$X(t) = \eta + \int_0^t F(s, X(s))ds + \int_0^t g(s, X(s))dW_s, \ t \in I.$$

Such equations were considered among others in [32, 33] and [39]. Furthermore, if $\eta \in \mathcal{K}_c^b(\mathbb{R}^d)$, $F : I \times \mathcal{K}_c^b(\mathbb{R}^d) \to \mathcal{K}_c^b(\mathbb{R}^d)$ and $g \equiv 0$ then we arrive to the deterministic set-valued differential equation with Hukuhara's derivative D_H:

$$D_H X(t) = F(t, X(t)) \text{ and } X(0) = \eta.$$

Such equations were widely studied in various contexts in the literature (see e.g. [4, 5, 7, 13, 14, 19, 20, 26, 27, 28, 29, 41, 42, 45]).

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