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MULTIPLE SOLUTIONS FOR DIRICHLET IMPULSIVE FRACTIONAL DIFFERENTIAL INCLUSIONS INVOLVING THE p-LAPLACIAN WITH TWO PARAMETERS

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Abstract

In this paper, the authors establish the existence of at least three weak solutions for impulsive differential inclusions involving two parameters and the p-Laplacian and having Dirichlet boundary conditions. Their approach is based variational methods and critical point theory.

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1. Introduction

Fractional differential equations has proved to be an important tool in the modeling of dynamical systems associated with phenomena such as fractals and chaos. In fact, this branch of calculus has found its applications in various disciplines of science and engineering such as mechanics, electricity, chemistry, biology, economics, control theory, signal and image processing, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electrodynamics of complex medium, viscoelasticity and damping, control theory, wave propagation, percolation, identification, and fitting of experimental data. Fractional differential equations serve as an excellent tool for the description of hereditary properties of various materials and processes. Interest in the study of fractional-order differential equations lies in the fact that fractional-order models are found to be more accurate than the classical integer-order models; that is, there are more degrees of freedom in the fractional-order models. In consequence, the subject of fractional differential equations is gaining more and more attention; see for instance the monographs of Hilfer [16], Samko *et al.* [21], Kilbas *et al.* [22], Miller and Ross [26], Podlubny [28], and the papers [1, 2, 4–6, 9]. See also [23, 25, 30, 32] and references therein.

Impulsive boundary value problems for differential equations have been intensively studied in recent years. Such problems appear in mathematical models with sudden changes in their states such as in population dynamics, pharmacology, optimal control, etc. [24]. Impulsive problems for fractional equations are often treated by topological methods (see [3,7,8,20]. The existence of solutions of impulsive problems is often studied using variational methods and critical point theorems (see [10, 31, 34]).

In the present paper, motivated by the results of [13, 15, 29] and [33], we use two kinds of three-critical-point theorems obtained in [11] and [12] (see Theorems 2.10 and 2.11 below), to obtain sufficient conditions for the existence of at least three weak solutions to the two parameter impulsive fractional differential inclusion

$$(1)\qquad \begin{cases} {}_xD_T^{\alpha}\phi_p\big({}_0D_x^{\alpha}u(x)\big) + \phi_p(u(x)) \in \lambda F(u(x)) + \mu G(x, u(x)), \\ \text{a.e. } x \in (0,T), \quad x \neq x_j, \\ \Delta\big({}_xD_T^{\alpha-1}({}_0^cD_x^{\alpha}u)\big)(x_j) = I_j(u(x_j)), \quad j = 1, 2, \dots, m, \\ u(0) = u(T) = 0, \end{cases}$$

where $\frac{1}{p} < \alpha \leq 1$, $\lambda > 0$, $\mu \geq 0$, $T > 0$, $p > 1$, $\phi_p(x) = |x|^{p-2}x$, ${}_0D_x^{\alpha}$ and ${}_xD_T^{\alpha}$ are the left and right Riemann-Liouville fractional derivatives, respectively, and $0 = x_0 < x_1 < x_2 < \dots < x_m < x_{m+1} = T$. Here,

$$\Delta\left({}_xD_T^{\alpha-1}({}_0^cD_x^{\alpha}u)\right)(x_j) = {}_xD_T^{\alpha-1}({}_0^cD_x^{\alpha})(x_j^+) - {}_xD_T^{\alpha-1}({}_0^cD_x^{\alpha})(x_j^-)$$

where

$$ {}_xD_T^{\alpha-1}({}_0^cD_x^{\alpha})(x_j^+) = \lim_{x\to x_j^+} {}_xD_T^{\alpha-1}({}_0^cD_x^{\alpha})(x), $$

$$ {}_xD_T^{\alpha-1}({}_0^cD_x^{\alpha})(x_j^-) = \lim_{x\to x_j^-} {}_xD_T^{\alpha-1}({}_0^cD_x^{\alpha})(x), $$

and ${}_0^cD_x^{\alpha}$ is the left Caputo fractional derivatives of order α. The multifunction $F:\mathbb{R}\to 2^{\mathbb{R}}$ is assumed to satisfy:

(F1) F is upper semicontinuous with compact convex values;

(F2) $\min F$, $\max F:\mathbb{R}\to\mathbb{R}$ are Borel measurable;

(F3) There exist $a>0$ and $r>1$ such that $|\xi|\le a(1+|s|^{r-1})$ for all $s\in\mathbb{R}$ and $\xi\in F(s)$.

In addition, $G:[0,T]\times\mathbb{R}\to 2^{\mathbb{R}}$ is a multifunction satisfying:

(G1) $G(x,\cdot)$ is upper semicontinuous with compact convex values for a.e. $x\in[0,T]\setminus Q$, where $Q=\{x_1,x_2,\dots,x_m\}$.

(G2) $\min G$, $\max G:([0,T]\setminus Q)\times\mathbb{R}\to\mathbb{R}$ are Borel measurable;

(G3) There exist $a>0$ and $r>1$ such that $|\xi|\le a(1+|s|^{r-1})$ for a.e. $x\in[0,T]$, $s\in\mathbb{R}$, and $\xi\in G(x,s)$.

Finally, we ask that the impulse functions $I_i:\mathbb{R}\to\mathbb{R}$ satisfy

(I_1) $I_i\in C(\mathbb{R},\mathbb{R})$, $I_i(0)=0$, and $I_i(s)s>0$ for $s\in\mathbb{R}$ with $s\neq 0$, $i=1,2,\dots,m$.

2. Basic definitions and preliminary results

We begin this section with the following definitions.

Definition 2.1. Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order $\alpha>0$ for the function u are defined by

$$ {}_aD_t^{\alpha}u(t) := \frac{d^n}{dt^n}{}_aD_t^{\alpha-n}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1}u(s)ds $$

and

$$ {}_tD_b^{\alpha}u(t) := (-1)^n\frac{d^n}{dt^n}{}_tD_b^{\alpha-n}u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b (t-s)^{n-\alpha-1}u(s)ds, $$

respectively, for every $t\in[a,b]$, provided the right-hand sides are pointwise defined on $[a,b]$, where $n-1\le\alpha<n$, $n\in\mathbb{N}$, and $\Gamma(\alpha)$ is the standard gamma function given by

$$ \Gamma(\alpha) := \int_0^{+\infty} z^{\alpha-1}e^{-z}dz. $$

We take $AC^n([a,b],\mathbb{R})$ to be the space of functions $u:[a,b]\to\mathbb{R}$ such that $u\in C^{n-1}([a,b],\mathbb{R})$ and $u^{(n-1)}\in AC([a,b],\mathbb{R})$. Here, as usual, $C^{n-1}([a,b],\mathbb{R})$ denotes the set of $(n-1)$ times continuously differentiable functions on $[a,b]$. In particular, $AC([a,b],\mathbb{R}):=AC^1([a,b],\mathbb{R})$.

Definition 2.2. Let $\gamma\geq 0$ and $n\in\mathbb{N}$.

(i) If $\gamma\in(n-1,n)$ and $u\in AC^n([a,b],\mathbb{R})$, then the left and right Caputo fractional derivatives of order γ of the function u exist almost everywhere on $[a,b]$ and are given by

$$ {}_a^cD_t^{\gamma}u(t)=\frac{1}{\Gamma(n-\gamma)}\int_a^t(t-s)^{n-\gamma-1}u^{(n)}(s)ds $$

and

$$ {}_t^cD_b^{\gamma}u(t)=\frac{(-1)^n}{\Gamma(n-\gamma)}\int_t^b(s-t)^{n-\gamma-1}u^{(n)}(s)ds, $$

respectively, for every $t\in[a,b]$.

(ii) If $\gamma=n-1$ and $u\in AC^{n-1}([a,b],\mathbb{R})$,

$$ {}_a^cD_t^{n-1}u(t)=u^{(n-1)}(t)\quad\text{and}\quad {}_t^cD_b^{n-1}u(t)=(-1)^{(n-1)}u^{(n-1)}(t) $$

for every $t\in[a,b]$.

With these definitions, we have the following formulas for fractional integration by parts and the composition of the Riemann-Liouville fractional integration operator with the Caputo fractional differentiation operator as proved in [22] and [21].

Proposition 2.3. *If $u\in L^p([a,b],\mathbb{R})$, $v\in L^q([a,b],\mathbb{R})$ and either* (i) $p\geq 1$, $q\geq 1$, $1/p+1/q\leq 1+\gamma$ *or* (ii) $p\neq 1$, $q\neq 1$, $1/p+1/q=1+\gamma$, *then*

$$ \int_a^b[{}_aD_t^{-\gamma}u(t)]v(t)dt=\int_a^b[{}_tD_b^{-\gamma}v(t)]u(t)dt,\quad \gamma>0. \tag{2} $$

Proposition 2.4. *Let $n\in\mathbb{N}$ and $n-1<\gamma\leq n$. If $u\in AC^n([a,b],\mathbb{R})$ or $u\in C^n([a,b],\mathbb{R})$, then*

$$ {}_aD_t^{-\gamma}({}_a^cD_t^{\gamma}u(t))=u(t)-\sum_{j=0}^{n-1}\frac{u^{(j)}(a)}{j!}(t-a)^j $$

and

$$ {}_tD_b^{-\gamma}({}_t^cD_b^{\gamma}u(t))=u(t)-\sum_{j=0}^{n-1}\frac{(-1)^ju^{(j)}(b)}{j!}(b-t)^j $$

for every $t \in [a,b]$. *In particular, if* $0 < \gamma \le 1$ *and* $u \in AC([a,b],\mathbb{R})$ *or* $u \in C^1([a,b],\mathbb{R})$, *then*

$$(3)\qquad {}_aD_t^{-\gamma}({}_a^cD_t^{\gamma}u(t)) = u(t) - u(a) \quad \textit{and} \quad {}_tD_b^{-\gamma}({}_t^cD_b^{\gamma}u(t)) = u(t) - u(b).$$

Let $(X, \|\cdot\|_X)$ be a real Banach space. We denote by X^* the dual space of X, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X^* and X. A function $\varphi : X \to \mathbb{R}$ is called *locally Lipschitz* if, for all $u \in X$, there exist a neighborhood U of u and a real number $L > 0$ such that

$$|\varphi(v) - \varphi(w)| \le L\|v - w\|_X \quad \text{for all } v,\ w \in U.$$

If φ is locally Lipschitz and $u \in X$, the *generalized directional derivative* of φ at u along the direction $v \in X$ is defined by

$$\varphi^\circ(u;v) := \limsup_{w\to u,\,\tau\to 0^+} \frac{\varphi(w+\tau v) - \varphi(w)}{\tau}.$$

The *generalized gradient* of φ at u is the set

$$\partial\varphi(u) := \big\{u^* \in X^* : \langle u^*, v\rangle \le \varphi^\circ(u;v) \text{ for all } v \in X\big\}.$$

Thus, $\partial\varphi : X \to 2^{X^*}$ is a multifunction. We say that φ has a *compact gradient* if $\partial\varphi$ maps bounded subsets of X into relatively compact subsets of X^*.

Lemma 2.5 [27, Proposition 1.1]**.** *Let* $\varphi \in C^1(X)$ *be a functional. Then* φ *is locally Lipschitz,*

$$\varphi^\circ(u;v) = \langle\varphi'(u), v\rangle \textit{ for all } u,\ v \in X,$$

and

$$\partial\varphi(u) = \{\varphi'(u)\} \textit{ for all } u \in X.$$

Lemma 2.6 [27, Proposition 1.3]**.** *Let* $\varphi : X \to \mathbb{R}$ *be a locally Lipschitz functional. Then* $\varphi^\circ(u;\cdot)$ *is subadditive and positively homogeneous for all* $u \in X$, *and*

$$\varphi^\circ(u;v) \le L\|v\| \textit{ for all } u,\ v \in X,$$

with $L > 0$ *being a Lipschitz constant for* φ *around* u.

Lemma 2.7 [14]**.** *Let* $\varphi : X \to \mathbb{R}$ *be a locally Lipschitz functional. Then* $\varphi^\circ : X \times X \to \mathbb{R}$ *is upper semicontinuous, and for all* $\lambda \ge 0$ *and* $u,\ v \in X$,

$$(\lambda\varphi)^\circ(u;v) = \lambda\varphi^\circ(u;v).$$

Moreover, if $\varphi,\ \psi : X \to \mathbb{R}$ *are locally Lipschitz functionals, then*

$$(\varphi+\psi)^\circ(u;v) \le \varphi^\circ(u;v) + \psi^\circ(u;v) \textit{ for all } u,\ v \in X.$$

Lemma 2.8 [27, Proposition 1.6]**.** *Let $\varphi, \psi : X \to \mathbb{R}$ be locally Lipschitz functionals. Then*

$$\partial(\lambda\varphi)(u) = \lambda\partial\varphi(u) \text{ for all } u \in X \text{ and } \lambda \in \mathbb{R},$$

$$\partial(\varphi + \psi)(u) \subseteq \partial\varphi(u) + \partial\psi(u) \text{ for all } u \in X.$$

Lemma 2.9 [17, Proposition 1.6]**.** *Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz functional with a compact gradient. Then φ is sequentially weakly continuous.*

We say that $u \in X$ is a (*generalized*) *critical point* of a locally Lipschitz functional φ if $0 \in \partial\varphi(u)$, i.e.,

$$\varphi^\circ(u; v) \geq 0 \quad \text{for all } v \in X.$$

Our main tools are two three-critical-point theorems that we recall here in a convenient form. The first was obtained in [12] and it is a more precise version of Theorem 3.2 in [11]. The second one was established in [11]. In the first one the coercivity of the functional $\mathcal{N} - \lambda\mathcal{M}$ is required, while in the second one, a suitable sign hypothesis is assumed.

Let $\mathcal{N}$ and $\mathcal{M}$ be locally Lipschitz functionals and set $J_\lambda := \mathcal{N} - \lambda\mathcal{M}$.

Theorem 2.10 [12, Theorem 3.6]**.** *Let X be a reflexive real Banach space, $\mathcal{N} : X \to \mathbb{R}$ be a coercive and sequentially weakly lower semicontinuous functional, and $\mathcal{M} : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous functional such that*

$$\mathcal{N}(0) = \mathcal{M}(0) = 0.$$

Assume that there exist $r \in \mathbb{R}$ and $\overline{x} \in X$, with $0 < r < \mathcal{N}(\overline{x})$, such that

(a$_1$) $\frac{\sup_{\mathcal{N}(x)\leq r} \mathcal{M}(x)}{r} < \frac{\mathcal{M}(\overline{x})}{\mathcal{N}(\overline{x})}$;

(a$_2$) *for each* $\lambda \in \Lambda_r := \left(\frac{\mathcal{N}(\overline{x})}{\mathcal{M}(\overline{x})}, \frac{r}{\sup_{\mathcal{N}(x)\leq r} \mathcal{M}(x)}\right)$, *the functional J_λ is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional J_λ has at least three distinct critical points in X.

Theorem 2.11 [11, Corollary 3.1]**.** *Let X be a reflexive real Banach space, $\mathcal{N} : X \to \mathbb{R}$ be a convex, coercive and sequentially weakly lower semicontinuous functional, and $\mathcal{M} : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous functional such that*

$$\inf_{u\in X} \mathcal{N}(u) = \mathcal{N}(0) = \mathcal{M}(0) = 0.$$

Assume that there exist two positive constants r_1, r_2 and $\overline{x} \in X$ with $2r_1 < \mathcal{N}(\overline{x}) < \frac{r_2}{2}$ such that (b$_1$) $\frac{\sup_{\mathcal{N}(x)<r_1}\mathcal{M}(x)}{r_1} < \frac{2}{3}\frac{\mathcal{M}(\overline{x})}{\mathcal{N}(\overline{x})}$;

(b$_2$) $\frac{\sup_{\mathcal{N}(x)<r_2}\mathcal{M}(x)}{r_2} < \frac{1}{3}\frac{\mathcal{M}(\overline{x})}{\mathcal{N}(\overline{x})}$;

(b$_3$) *for each* $\lambda \in \Lambda'_{r_1,r_2} := \left(\frac{3}{2}\frac{\mathcal{N}(\overline{x})}{\mathcal{M}(\overline{x})}, \min\left\{\frac{r_1}{\sup_{\mathcal{N}(x)<r_1}\mathcal{M}(x)}, \frac{r_2}{2\sup_{\mathcal{N}(x)<r_2}\mathcal{M}(x)}\right\}\right)$ *and for every* $x_1, x_2 \in X$ *that are local minima for the functional* J_λ *such that* $\mathcal{M}(x_1) \geq 0$ *and* $\mathcal{M}(x_2) \geq 0$*, we have* $\inf_{s\in[0,1]}\mathcal{M}(sx_1 + (1-s)x_2) \geq 0$.

Then, for each $\lambda \in \Lambda'_{r_1,r_2}$*, the functional* J_λ *admits at least three critical points which lie in* $\mathcal{N}^{-1}((-\infty, r_2))$.

In order to establish a variational structure for our problem, we need to construct appropriate function spaces. Following [19], denote by $C_0^\infty([0,T],\mathbb{R})$ the set of all functions $g \in C^\infty([0,T],\mathbb{R})$ with $g(0) = g(T) = 0$.

Definition 2.12. Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty([0,T],\mathbb{R})$ with respect to the weighted norm

$$\|u\|_{\alpha,p} := \left(\int_0^T |{}_0^cD_t^\alpha u(t)|^p dt + \int_0^T |u(t)|^p dt\right)^{1/p}, \text{ for all } u \in E_0^{\alpha,p}. \tag{4}$$

Clearly, the fractional derivative space $E_0^{\alpha,p}$ is the space of functions $u \in L^p[0,T]$ having an α-order fractional derivative ${}_0D_t^\alpha u \in L^p[0,T]$ and satisfying $u(0) = u(T) = 0$. From [19, Proposition 3.1], we know for $0 < \alpha \leq 1$, the space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.

For every $u \in E_0^\alpha$, set

$$\|u\|_{L^s} := \left(\int_0^T |u_i(t)|^s dt\right)^{1/s}, \quad s \geq 1,$$

and

$$\|u\|_\infty := \max_{t\in[0,T]} |u(t)|.$$

Remark 2.13. For any $u \in E_0^{\alpha,p}$ according to (3) and in view of the condition $u(0) = u(T) = 0$, we have ${}_0D_t^\alpha u(t) = {}_0^cD_t^\alpha u(t)$, ${}_tD_T^\alpha u(t) = {}_t^cD_T^\alpha u(t)$ for $t \in [0,T]$.

Lemma 2.14 [19]**.** *Let* $0 < \alpha \leq 1$ *and* $1 < p < \infty$*. For all* $u \in E_0^{\alpha,p}$*, we have*

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)}\left(\int_0^T |{}_0^cD_t^\alpha u(t)|^p dt\right)^{1/p}. \tag{5}$$

Moreover, If $\alpha > \frac{1}{p}$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*, then*

$$
\begin{aligned}
\|u\|_\infty &\le \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\left(\int_0^T |{}_0^cD_t^\alpha u(t)|^p dt\right)^{1/p} \\
&= \frac{1}{\Gamma(\alpha)}\frac{T^{\alpha-1/p}}{(\frac{(\alpha-1)p}{p-1}+1)^{\frac{p-1}{p}}}\left(\int_0^T |{}_0^cD_t^\alpha u(t)|^p dt\right)^{1/p}.
\end{aligned} \tag{6}
$$

As a consequence of this lemma, we can consider $E_0^{\alpha,p}$ with the norm

$$
\|u\|_{\alpha,p} := \left(\int_0^T |{}_0^cD_t^\alpha u(t)|^p dt\right)^{1/p}, \text{ for all } u \in E_0^{\alpha,p}, \tag{7}
$$

which is equivalent to (4).

Lemma 2.15 [19]**.** *Let that $0 < \alpha \le 1$ and $1 < p < \infty$. Assume that $\alpha > \frac{1}{p}$ and the sequence $\{u_n\}$ converges weakly to u in $E_0^{\alpha,p}$, i.e., $u_n \rightharpoonup u$. Then $\{u_n\}$ converges strongly to u in $C([0,T]),\mathbb{R})$, i.e., $\|u_n - u\|_\infty \to 0$, as $n \to \infty$.*

For $v \in E_0^{\alpha,p}$, by Remark 2.13 and Definition 2.1 we have

$$
\begin{aligned}
\int_0^T [{}_xD_T^\alpha \phi_p({}_0D_x^\alpha u(x))]v(x)dx &= \int_0^T [{}_xD_T^\alpha \phi_p({}_0^cD_x^\alpha u(x))]v(x)dx \\
&= -\int_0^T v(x)d[{}_xD_T^{\alpha-1}\phi_p({}_0^cD_x^\alpha u(x))] \\
&= \int_0^T [{}_xD_T^{\alpha-1}\phi_p({}_0^cD_x^\alpha u(x))]v'(x)dx.
\end{aligned}
$$

Thus, from Proposition 2.3 and Definition 2.2 we have

$$
\begin{aligned}
\int_0^T [{}_xD_T^\alpha \phi_p({}_0D_x^\alpha u(x))]v(x)dx &= \int_0^T \phi_p({}_0^cD_x^\alpha u(x)){}_0D_x^{\alpha-1}v'(x)dx \\
&= \int_0^T \phi_p({}_0^cD_x^\alpha u(x)){}_0^cD_x^\alpha v(x)dx.
\end{aligned}
$$

Hence, we can define the weak solutions of FBVP (1) as follows.

Definition 2.16. A function $u \in X$ is a weak solution of the problem (1) if there exists $u^* \in L^p([0,T])$ (for some $p > 1$) such that

$$
\int_0^T \Big[\phi_p({}_0^cD_x^\alpha u(x)){}_0^cD_x^\alpha v(x)dx + \phi_p(u(x))v(x) - u^*(x)v(x)\Big]dx + \sum_{i=1}^m I_i(u(x_i))v(x_i) = 0
$$

for all $v \in X$ and $u^* \in \lambda F(u(x)) + \mu G(x,u(x))$ for a.e. $x \in [0,T]$.

Definition 2.17. By a solution of the impulsive differential inclusion (1) we mean a function $u : [0,T] \setminus Q \to \mathbb{R}$ in the class C^1 with $\phi_p({}_x^cD_T^\alpha u(x))$ being absolutely continuous and satisfying

$$\begin{aligned}
&{}_xD_T^\alpha \phi_p\Big({}_0D_x^\alpha u(x)\Big) + \phi_p(u(x)) = u^* \quad \text{in } [0,T] \setminus Q,\\
&\Delta({}_xD_T^{\alpha-1}({}_0^cD_x^\alpha u))(x_j) = I_j(u(x_j)), \quad j = 1, 2, \ldots, m,\\
&u(0) = u(T) = 0,
\end{aligned}$$

where $u^* \in \lambda F(u(x)) + \mu G(x, u(x))$ and $u^* \in L^p([0,T])$ (for some $p > 1$).

For a.e. $x \in [0,T]$ and all $s \in \mathbb{R}$, we introduce the Aumann-type set-valued integral

$$\int_0^s F(t)dt = \left\{ \int_0^s f(t)dt \;:\; f : \mathbb{R} \to \mathbb{R} \text{ is a measurable selection of } F \right\},$$

and we set $\mathcal{F}(u) = \int_0^T \min \int_0^u F(s)\, ds\, dx$ for all $u \in L^p([0,T])$. We also have the Aumann-type set-valued integral

$$\int_0^s G(x,t)dt = \left\{ \int_0^s g(x,t)dt \;:\; g : [0,T]\times\mathbb{R} \to \mathbb{R} \text{ is a measurable selection of } G \right\}$$

and set $\mathcal{G}(u) = \int_0^T \min \int_0^u G(x,s)\, ds\, dx$ for all $u \in L^p([0,T])$.

Lemma 2.18 [18, Lemma 3.1]**.** *The functionals $\mathcal{F}, \mathcal{G} : L^p([0,T]) \to \mathbb{R}$ are well defined and Lipschitz on any bounded subset of $L^p([0,T])$. Moreover, for all $u \in L^p([0,T])$ and all $u^* \in \partial(\mathcal{F}(u) + \mathcal{G}(u))$,*

$$u^*(x) \in F(u(x)) + G(x, u(x)) \quad \textit{for a.e. } x \in [0,T].$$

We define an energy functional for the problem (1) by

$$I_\lambda(u) = \frac{1}{p}\|u\|_{\alpha,p}^p - \lambda\mathcal{F}(u) - \mu\mathcal{G}(u) + \sum_{i=1}^m \int_0^{u(x_i)} I_i(s)ds$$

for all $u \in X$.

Lemma 2.19 [29, Lemma 4.4]**.** *The functional $I_\lambda : X \to \mathbb{R}$ is locally Lipschitz. Moreover, for each critical point $u \in X$ of I_λ, u is a weak solution of* (1).

3. Main results

To formulate our main result we need to define the following quantities. Set

$$m := \frac{1}{(\Gamma(\alpha))^p}\left(\frac{T^{\alpha p-1}}{\left(\frac{(\alpha-1)p}{p-1}+1\right)^{p-1}}\right) \tag{8}$$

and

$$\omega_\alpha := \left(\frac{2}{T}\right)^p \left(\int_0^{T/2} x^{p(1-\alpha)}dx + \int_{T/2}^T (x^{1-\alpha} - 2(x-T/2)^{1-\alpha})^p dx\right).$$

We also define

$$w(x) := \begin{cases} \frac{2\Gamma(2-\alpha)\eta}{T}x, & x \in \left[0, \frac{T}{2}\right), \\ \frac{2\Gamma(2-\alpha)\eta}{T}(T-x), & x \in \left[\frac{T}{2}, T\right]. \end{cases} \tag{9}$$

Fixing k, $\eta > 0$ such that

$$\frac{\frac{1}{p}\omega_\alpha\eta^p + \sum_{i=1}^m \int_0^{w(x_i)} I_i(s)ds}{\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} F(s)\,ds\,dx + \int_{\frac{T}{2}}^T \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} F(s)\,ds\,dx} < \frac{\frac{1}{pm}k^p}{\sup_{|u|\le k} \min \int_0^u F(s)ds}$$

and taking

$$\lambda \in \Lambda_1 := \left(\frac{\frac{1}{p}\omega_\alpha\eta^p + \sum_{i=1}^m \int_0^{w(x_i)} I_i(s)ds}{\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} F(s)\,ds\,dx + \int_{\frac{T}{2}}^T \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} F(s)\,ds\,dx}, \frac{\frac{1}{pm}k^p}{\sup_{|u|\le k} \min \int_0^u F(s)ds}\right), \tag{10}$$

we set

$$\begin{aligned}\delta_{\lambda,G} := \min\Bigg\{ &\frac{\frac{1}{pm}k^p - \lambda \sup_{|u|\le k} \min \int_0^u F(s)ds}{\sup_{|u|\le k} \min \int_0^u G(x,s)ds}, \\ &\Bigg[\lambda\Bigg(\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} F(s)\,ds\,dx + \int_{\frac{T}{2}}^T \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} F(s)\,ds\,dx\Bigg) \\ &- \frac{1}{p}\omega_\alpha\eta^p - \sum_{i=1}^m \int_0^{w(x_i)} I_i(s)ds\Bigg] \Bigg[-\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} G(x,s)\,ds\,dx \\ &- \int_{\frac{T}{2}}^T \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} G(x,s)\,ds\,dx\Bigg]^{-1} \Bigg\}\end{aligned} \tag{11}$$

and

$$(12)\qquad \overline{\delta}_{\lambda,G} := \min\left\{\delta_{\lambda,G}, \frac{\frac{1}{p}\left(\frac{\Gamma(\alpha+1)}{T}\right)^p}{\max\left\{0, \limsup\limits_{|\xi|\to+\infty}\frac{\sup\limits_{x\in[0,T]}\min\int_0^\xi G(x,s)ds}{\xi^p}\right\}}\right\},$$

where we set $\frac{r}{0} = +\infty$.

Our first existence result is contained in the following theorem.

Theorem 3.1. *Assume that* (F_1)–(F_3) *and* (I_1) *hold and there exist positive constants* k *and* η *with*

$$(13)\qquad \left(\frac{\eta}{k}\right)^p > \frac{1}{m\omega_\alpha}$$

such that

(F_4) $\frac{\sup_{|u|\le k}\min\int_0^u F(s)ds}{\frac{1}{pm}k^p} < \frac{\int_0^{\frac{T}{2}}\min\int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x}F(s)\,ds\,dx+\int_{\frac{T}{2}}^T\min\int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)}F(s)\,ds\,dx}{\frac{1}{p}\omega_\alpha\eta^p+\sum_{i=1}^m\int_0^{w(x_i)}I_i(s)ds}$;

(F_5) $\limsup_{|\xi|\to+\infty}\frac{\min\int_0^\xi F(s)ds}{\xi^p} \le 0.$

Then, for every $\lambda \in \Lambda_1$*, where* Λ_1 *is given by* (10)*, and for every multifunction* G *satisfying* (G_1)–(G_3) *and*

(G_4) $\limsup_{|\xi|\to+\infty}\frac{\sup\limits_{x\in[0,T]}\min\int_0^\xi G(x,s)ds}{\xi^p} < +\infty,$

there exists $\overline{\delta}_{\lambda,G} > 0$ *given by* (12) *such that, for each* $\mu \in [0, \overline{\delta}_{\lambda,G})$*, the problem* (1) *admits at least three weak solutions in* X*.*

Proof. Fix λ, G, and μ as in the conclusion of this theorem. For each $u \in X$, define

$$\mathcal{N}(u) := \frac{1}{p}\|u\|_{\alpha,p}^p + \sum_{i=1}^m\int_0^{u(x_i)} I_i(s)ds,$$

and

$$\mathcal{M}(u) := \int_0^T \min\int_0^u F(s)dsdx + \frac{\mu}{\lambda}\int_0^T \min\int_0^u G(x,s)dsdx.$$

It is a simple matter to verify that $\mathcal{N}$ is sequentially weakly lower semicontinuous on X. Clearly, $\mathcal{N} \in C^1(X)$. By Lemma 2.5, $\mathcal{N}$ is locally Lipschitz on X, and by

Lemma 2.18, $\mathcal{F}$ and $\mathcal{G}$ are locally Lipschitz on $L^p([0,T])$. Therefore, $\mathcal{M}$ is locally Lipschitz on $L^p([0,T])$. Moreover, X is compactly embedded into $L^p([0,T])$, so $\mathcal{M}$ is locally Lipschitz on X. In addition, $\mathcal{M}$ is sequentially weakly upper semicontinuous. For all $u \in X$, by (I_1),

$$\int_0^{u(x_i)} I_i(s)ds > 0, \quad i = 1, 2, \dots, m.$$

Hence, we have

$$\mathcal{N}(u) = \frac{1}{p}\|u\|_{\alpha,p}^p + \sum_{i=1}^m \int_0^{u(x_i)} I_i(s)ds > \frac{1}{p}\|u\|_{\alpha,p}^p$$

for all $u \in X$. Hence, $\mathcal{N}$ is coercive and

$$\inf_{u\in X} \mathcal{N}(u) = \mathcal{N}(0) = \mathcal{M}(0) = 0.$$

Thus, the regularity assumptions $\mathcal{N}$ and $\mathcal{M}$ of Theorem 2.10 are satisfied.

We will next verify that conditions (a_1) and (a_2) of Theorem 2.10 hold. Let w be the function defined in (9). Clearly $w(0) = w(T) = 0$ and $w \in L^p([0,T])$. A direct calculation shows that

$${}_0^cD_x^\alpha w(x) = \begin{cases} \frac{2\eta}{T}x^{1-\alpha}, & x \in [0, \frac{T}{2}), \\ \frac{2\eta}{T}(x^{1-\alpha} - 2(x - T/2)^{1-\alpha}), & x \in [\frac{T}{2}, T]. \end{cases}$$

Furthermore,

$$\begin{aligned} \int_0^T |{}_0^cD_t^\alpha w(x)|^p dt &= \int_0^{T/2} (|{}_0^cD_x^\alpha w(x)|^p dt + \int_{T/2}^T (|{}_0^cD_x^\alpha w(x)|^p dt \\ &= \left(\frac{2\eta}{T}\right)^p \left\{ \int_0^{T/2} t^{p(1-\alpha)}dt + \int_{T/2}^T (x^{1-\alpha} - 2(x - T/2)^{1-\alpha})^p dt \right\} \\ &= \omega_\alpha \eta^p. \end{aligned}$$

Thus, $w \in X$, and in particular,

$$\|w\|_{\alpha,p}^p = \int_0^T |{}_0^cD_t^\alpha w(x)|^p dt = \omega_\alpha \eta^p, \tag{14}$$

so

$$\mathcal{N}(w) = \frac{1}{p}\eta^p\omega_\alpha + \sum_{i=1}^m \int_0^{w(x_i)} I_i(s)ds. \tag{15}$$

Let $r := \frac{1}{pm}k^p$. From condition (13), we have $\mathcal{N}(w) > r$. Also,

$$\begin{aligned}\mathcal{M}(w) :&= \int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} F(s)\,ds\,dx + \int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} F(s)\,ds\,dx \\ &+ \frac{\mu}{\lambda}\bigg(\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} G(x,s)\,ds\,dx \\ &+ \int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} G(x,s)\,ds\,dx \bigg).\end{aligned}$$

For all $u \in X$ with $\mathcal{N}(u) \le r$, and taking into account that $\|u\|_{\alpha,p}^p < pr$ and (6), we see that $|u(x)| \le k$ for all $x \in [0,T]$. Therefore,

$$\begin{aligned}\frac{\sup_{\mathcal{N}(u)\le r} \mathcal{M}(u)}{r} &= \frac{\sup_{\mathcal{N}(u)\le r} \left[\int_0^T \min \int_0^u F(s)dsdx + \frac{\mu}{\lambda}\int_0^T \min \int_0^u G(x,s)dsdx\right]}{r} \\ &\le \frac{1}{k^p} pm \sup_{|u|\le k} \min \int_0^u F(s)ds + \frac{\mu}{\lambda}\frac{1}{k^p} pm \sup_{|u|\le k} \min \int_0^u G(x,s)ds. \qquad (16)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{\mathcal{M}(w)}{\mathcal{N}(w)} &= \frac{\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} F(s)\,ds\,dx + \int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} F(s)\,ds\,dx}{\frac{1}{p}\omega_\alpha \eta^p + \sum_{i=1}^m \int_0^{w(x_i)} I_i(s)ds} \\ &+ \frac{\mu}{\lambda}\frac{\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} G(x,s)\,ds\,dx + \int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} G(x,s)\,ds\,dx}{\frac{1}{p}\omega_\alpha \eta^p + \sum_{i=1}^m \int_0^{w(x_i)} I_i(s)ds}. \qquad (17)\end{aligned}$$

Now $\mu < \delta_{\lambda,G}$ implies

$$\mu < \frac{\frac{1}{pm}k^p - \lambda \sup_{|u|\le k} \min \int_0^u F(s)ds}{\sup_{|u|\le k} \min \int_0^u G(x,s)ds},$$

and so

$$\frac{1}{k^p} pm \sup_{|u|\le k} \min \int_0^u F(s)ds + \frac{\mu}{\lambda}\frac{1}{k^p} pm \sup_{|u|\le k} \min \int_0^u G(x,s)ds < \frac{1}{\lambda}. \qquad (18)$$

Similarly,

$$\mu < \frac{\lambda\Big(\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} F(s)\,ds\,dx + \int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} F(s)\,ds\,dx\Big)}{-\Big(\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} G(x,s)\,ds\,dx + \int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} G(x,s)\,ds\,dx\Big)}$$
$$+ \frac{-\frac{1}{p}\omega_\alpha \eta^p - \sum_{i=1}^m \int_0^{w(x_i)} I_i(s)ds}{-\Big(\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} G(x,s)\,ds\,dx + \int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} G(x,s)\,ds\,dx\Big)},$$

and so

$$\frac{\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} F(s)\,ds\,dx + \int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} F(s)\,ds\,dx}{\frac{1}{p}\omega_\alpha \eta^p + \sum_{i=1}^m \int_0^{w(x_i)} I_i(s)ds} \tag{19}$$
$$+ \frac{\mu}{\lambda}\frac{\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} G(x,s)\,ds\,dx + \int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} G(x,s)\,ds\,dx}{\frac{1}{p}\omega_\alpha \eta^p + \sum_{i=1}^m \int_0^{w(x_i)} I_i(s)ds} > \frac{1}{\lambda}.$$

Hence, from (16)–(19), condition (a_1) of Theorem 2.10 is satisfied.

Since $\mu < \overline{\delta}_{\lambda,G}$, in view of (G_4), we can fix $l > 0$ such that

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in[0,T]} \min \int_0^\xi G(x,s)ds}{\xi^p} < l$$

and $\mu l < \frac{1}{p}\big(\frac{\Gamma(\alpha+1)}{T}\big)^p$. Therefore, there exists a positive constant ϱ such that

$$\min \int_0^\xi G(x,s)ds \le l\xi^p + \varrho$$

for each $(x,\xi) \in [0,T]\times\mathbb{R}$. Now, fix $0 < \varepsilon < \frac{\frac{1}{p}\left(\frac{\Gamma(\alpha+1)}{T}\right)^p - \mu l}{\lambda}$. From (F_5) there is a positive constant ϑ_ε such that

$$\min \int_0^\xi F(s)ds \le \varepsilon\xi^p + \vartheta_\varepsilon$$

for each $\xi \in \mathbb{R}$. So, for each $u \in X$,

$$J_\lambda(u) = \frac{1}{p}\|u\|_{\alpha,p}^p + \sum_{i=1}^m \int_0^{u(x_i)} I_i(s)ds - \lambda \int_0^T \min \int_0^u F(s)dsdx$$
$$- \mu \int_0^T \min \int_0^u G(x,s)dsdx$$

$$> \frac{1}{p}\|u\|^p_{\alpha,p} - \lambda\int_0^T \big(\varepsilon u^p + \vartheta_\varepsilon\big)dx - \mu\int_0^T \big(lu^p + \vartheta\big)dx$$
$$\geq \left(\frac{1}{p} - \lambda\varepsilon\left(\frac{T}{\Gamma(\alpha+1)}\right)^p - \mu l\left(\frac{T}{\Gamma(\alpha+1)}\right)^p\right)\|u\|^p_X - \lambda T\vartheta_\varepsilon - \mu T\vartheta.$$

This leads to the coercivity of J_λ and so condition (a_2) of Theorem 2.10 holds.

Since (16)–(19) imply

$$\lambda \in \Lambda_1 \subseteq \left(\frac{\mathcal{N}(w)}{\mathcal{M}(w)}, \frac{r}{\sup_{\mathcal{N}(u)\leq r}\mathcal{M}(u)}\right),$$

Theorem 2.10 ensures the existence of at least three critical points of the functional J_λ. Finally, by Lemma 2.19, the critical points of J_λ are weak solutions of the problem (1), and this completes the proof of the theorem. ■

The following result is a special case of Theorem 3.1 with $\mu = 0$.

Theorem 3.2. *Assume that* (F_1)–(F_3), (F_5), *and* (I_1) *hold and there exist positive constants k and η such that* (13) *and* (F_4) *hold. Then, for each $\lambda \in \Lambda_1$, where Λ_1 is given by* (10), *the problem*

$$(20)\qquad \begin{cases} {}_xD^\alpha_T\phi_p\Big({}_0D^\alpha_x u(x)\Big) + \phi_p(u(x)) \in \lambda F(u(x)), \; a.e.\; x\in(0,T), \\ x \neq x_j, \\ \Delta({}_xD^{\alpha-1}_T({}^c_0D^\alpha_x u))(x_j) = I_j(u(x_j)), \quad j = 1,2,\ldots,m, \\ u(0) = u(T) = 0, \end{cases}$$

has at least three weak solutions in X.

Next, we present a variant of Theorem 3.1 where no asymptotic condition on G such as (G_4) is required.

Fix k_1, k_2, and $d > 0$ so that

$$(21)\qquad \begin{aligned} &\frac{3\left(\frac{1}{p}\omega_\alpha\eta^p + \sum_{i=1}^m\int_0^{w(x_i)} I_i(s)ds\right)}{2\Big(\int_0^{\frac{T}{2}}\min\int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} F(s)\,ds\,dx + \int_{\frac{T}{2}}^T\min\int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} F(s)\,ds\,dx\Big)} \\ &< \min\left\{\frac{\frac{1}{pm}k_1^p}{\sup_{|u|\leq k_1}\min\int_0^u F(s)ds}, \frac{\frac{1}{2pm}k_2^p}{\sup_{|u|\leq k_2}\min\int_0^u F(s)ds}\right\}, \end{aligned}$$

choose

$$\lambda \in \Lambda_2 := \left(\frac{3\left(\frac{1}{p}u_\alpha \eta^p + \sum_{i=1}^m \int_0^{\omega(x_i)} I_i(s)ds\right)}{2\left(\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} F(s)\,ds\,dx + \int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} F(s)\,ds\,dx\right)}, \right. \tag{22}$$

$$\left. \min\left\{ \frac{\frac{1}{pm}k_1^p}{\sup_{|u|\le k_1} \min \int_0^u F(s)ds}, \frac{\frac{1}{2pm}k_2^p}{\sup_{|u|\le k_2} \min \int_0^u F(s)ds} \right\} \right), \tag{23}$$

and set

$$\delta^*_{\lambda,G} := \min\left\{ \frac{\frac{1}{pm}k_1^p - \lambda \sup_{|u|\le k_1} \min \int_0^u F(s)ds}{\sup_{|u|\le k_1} \min \int_0^u G(x,s)ds}, \frac{\frac{1}{pm}k_2^p - \lambda \sup_{|u|\le k_2} \min \int_0^u F(s)ds}{\sup_{|u|\le k_2} \min \int_0^u G(x,s)ds} \right\}. \tag{24}$$

Theorem 3.3. *Let* (F_1)–(F_3), (A_1), *and* (I_1) *hold. Assume that there exist three positive constants* k_1, k_2 *and* η *with*

$$\left(\frac{\eta}{k_1}\right)^p > \frac{2}{m\omega_\alpha} \tag{25}$$

and

$$\frac{1}{2pm}k_2^p > \frac{1}{p}w_\alpha \eta^p + \sum_{i=1}^m \int_0^{w(x_i)} I_i(s)ds \tag{26}$$

such that

(F_6) $\min \int_0^t F(s)ds \ge 0$ *for each* $t \in [0, k_2]$;

(F_7)
$$\frac{\sup_{|u|\le k_1} \min \int_0^u F(s)ds}{\frac{1}{pm}k_1^p} < \frac{2}{3}\, \frac{\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} F(s)\,ds\,dx + \int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} F(s)\,ds\,dx}{\frac{1}{p}\omega_\alpha \eta^p + \sum_{i=1}^m \int_0^{w(x_i)} I_i(s)ds};$$

(F_8)
$$\frac{\sup_{|u|\le k_2} \min \int_0^u F(s)ds}{\frac{1}{pm}k_2^p} < \frac{1}{3}\, \frac{\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} F(s)\,ds\,dx + \int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} F(s)\,ds\,dx}{\frac{1}{p}\omega_\alpha \eta^p + \sum_{i=1}^m \int_0^{w(x_i)} I_i(s)ds}.$$

Then, for each $\lambda \in \Lambda_2$, *where* Λ_2 *is given by* (22), *and for every multifunction* G *satisfying* (G_1)–(G_3) *and*

(G_5) $\min \int_0^t G(x,s)ds \ge 0$ *for each* $x \in [0,T]$ *and* $t \in \mathbb{R}$,

there exists $\delta^*_{\lambda,G}$ *given by* (24) *such that, for each* $\mu \in [0, \delta^*_{\lambda,G})$, *the problem* (1) *has at least three weak solutions* u_i, $i = 1,2,3$, *such that*

$$0 < u_i(x) < k_2 \quad \textit{for all} \quad x \in [0,T], \; i = 1,2,3.$$

Proof. Fix λ, G, and μ as in the conclusion of the theorem and take $\mathcal{N}$ and $\mathcal{M}$ as in the proof of Theorem 3.1. The regularity assumptions in Theorem 2.11 on $\mathcal{N}$ and $\mathcal{M}$ are satisfied as before. Define w as in (9), and let

$$r_1 := \frac{1}{pm} k_1^p \qquad \text{and} \qquad r_2 := \frac{1}{pm} k_2^p.$$

From conditions (25) and (26), we see that $2r_1 < \mathcal{N}(w) < \frac{r_2}{2}$. Since $\mu < \delta^*_{\lambda,G}$, we have

$$\begin{aligned}
\frac{\sup_{\mathcal{N}(u)<r_1} \mathcal{M}(u)}{r_1} &\leq \frac{1}{k_1^p} pm \sup_{|u|\leq k_1} \min \int_0^u F(s)ds + \frac{\mu}{\lambda}\frac{pm}{k_1^p} \sup_{|u|\leq k_1} \min \int_0^u G(x,s)ds \\
&< \frac{1}{\lambda} < \frac{2}{3}\left(\frac{\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} F(s)\,ds\,dx + \int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} F(s)\,ds\,dx}{\frac{1}{p}\omega_\alpha \eta^p + \sum_{i=1}^m \int_0^{w(x_i)} I_i(s)ds)} \right. \\
&+ \frac{\mu}{\lambda}\left(\frac{2\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} G(x,s)\,ds\,dx}{\frac{1}{p}\omega_\alpha \eta^p + \sum_{i=1}^m \int_0^{w(x_i)} I_i(s)ds)} \right. \\
&\left.\left. + \frac{2\int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} G(x,s)\,ds\,dx}{\frac{1}{p}\omega_\alpha \eta^p + \sum_{i=1}^m \int_0^{\omega(x_i)} I_i(s)ds)} \right)\right) = \frac{2}{3}\frac{\mathcal{M}(w)}{\mathcal{N}(w)},
\end{aligned}$$

and similarly,

$$\begin{aligned}
2\frac{\sup_{\mathcal{N}(u)<r_2} \mathcal{M}(u)}{r_2} &\leq 2\frac{pm}{k_2^p} \sup_{|u|\leq k_2} \min \int_0^u F(s)ds \\
&+ \frac{\mu}{\lambda}\frac{pm}{k_2^p} \sup_{|u|\leq k_2} \min \int_0^u G(x,s)ds < \frac{1}{\lambda} \\
&< \frac{2}{3}\left(\frac{\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} F(s)\,ds\,dx + \int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} F(s)\,ds\,dx}{\frac{1}{p}\omega_\alpha \eta^p + \sum_{i=1}^m \int_0^{w(x_i)} I_i(s)ds)} \right. \\
&+ \frac{\mu}{\lambda}\left(\frac{2\int_0^{\frac{T}{2}} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}x} G(x,s)\,ds\,dx}{\frac{1}{p}\omega_\alpha \eta^p + \sum_{i=1}^m \int_0^{w(x_i)} I_i(s)ds)} \right. \\
&\left.\left. + \frac{2\int_{\frac{T}{2}}^{T} \min \int_0^{\frac{2\Gamma(2-\alpha)\eta}{T}(T-x)} G(x,s)\,ds\,dx}{\frac{1}{p}\omega_\alpha \eta^p + \sum_{i=1}^m \int_0^{w(x_i)} I_i(s)ds} \right)\right) = \frac{2}{3}\frac{\mathcal{M}(w)}{\mathcal{N}(w)}.
\end{aligned}$$

Therefore, conditions (b_1) and (b_2) of Theorem 2.11 are satisfied.

To show that the functional J_λ satisfies condition (b_3) of Theorem 2.11, let u^* and u^{**} be two local minima for J_λ. Then u^* and u^{**} are critical points of

J_λ, and so by Lemma 2.19, they are weak solutions of the problem (1). From (F_6) and (G_5) and the Maximum Principle, we have $u^*(x) \geq 0$ and $u^{**}(x) \geq 0$ for every $x \in [0,T]$. It follows that $su^* + (1-s)u^{**} \geq 0$ for all $s \in [0,1]$, and consequently $\mathcal{M}\big(su^* + (1-s)u^{**}\big) \geq 0$ for all $s \in [0,1]$. Thus, Theorem 2.11 ensures at least three weak solutions whose norm in X is less than $2k_2$. Hence, the Strong Maximum Principle and Lemma 2.14 ensure the conclusion. ■

We next give an example to illustrate Theorem 3.3.

Example 3.4. Let $p = 4$, $\alpha = 0.8$, $x_1 = \frac{1}{2}$, $x_2 = \frac{7}{4}$, $I_1(s) = I_2(s) = 2s$ for all $s \in \mathbb{R}$, and consider the problem

$$\begin{cases} {}_xD_2^{0.8}\phi_4\Big({}_0D_x^{0.8}u(x)\Big) + \phi_4(u(x)) \in \lambda F(u(x)) + \mu G(x, u(x)), \\ \text{ a.e. } x \in (0,2),\ x \neq x_j, \\ \Delta({}_xD_2^{-0.2}({}_0^cD_x^{0.8}))(x_j) = I_j(u(x_j)), \quad j = 1, 2, \\ u(0) = u(2) = 0, \end{cases} \tag{27}$$

where for all $s \in \mathbb{R}$,

$$F(s) = \begin{cases} \{0\}, & \text{if } |s| < 2^{-1/3}, \\ [0,1], & \text{if } |s| = 2^{-1/3}, \\ \{s - 2^{-1/3} + 1\}, & \text{if } s > 2^{-1/3}, \\ \{s + 2^{-1/3} + 1\}, & \text{if } s < -2^{-1/3}. \end{cases}$$

Clearly the assumptions (F_1)–(F_3), (A_1), and (I_1) are satisfied. By choosing $k_1 = 2^{-\frac{1}{3}}$, $k_2 = 8$ and $\eta = 1$, we can easily see that conditions (25), (26), and (F_6) are satisfied. Moreover, simple calculations show that

$$\sup_{|u| \leq 2^{-\frac{1}{3}}} \min \int_0^u F(s)ds = \sup_{|u| \leq 8} \min \int_0^u F(s)ds = 0$$

and

$$\begin{aligned}
&\frac{\int_0^1 \min \int_0^{\Gamma(2-0.8)x} F(s)\,ds\,dx + \int_1^2 \min \int_0^{\Gamma(2-0.8)(2-x)} F(s)\,ds\,dx}{\frac{1}{4}\omega_{0.8} + \sum_{i=1}^2 \int_0^{u(x_i)} I_i(s)ds} \\
&\simeq \frac{\int_0^1 \min \int_0^{0.9182x} F(s)\,ds\,dx + \int_1^2 \min \int_0^{0.9182(2-x)} F(s)\,ds\,dx}{0.25\omega_{0.8} + \int_0^{0.4591} 2sds + \int_0^{0.2296} 2sds} \\
&\simeq \frac{1}{0.25w_{0.8} + 0.2634} 2\bigg(\int_0^1 \min \int_0^{0.9182x} F(s)\,ds\,dx\bigg) \\
&\simeq \frac{2}{0.25w_{0.8} + 0.2634}\bigg(\int_0^{2^{-1/3}} \int_0^{0.9182x} \max F(s)\,ds\,dx \\
&+ \int_{2^{-1/3}}^1 \int_0^x \max F(s)\,ds\,dx\bigg) > 0,
\end{aligned}$$

so (F_7) and (F_8) hold. Hence, for any multifunction G satisfying (G_1)–(G_3), by Theorem 3.3 the problem (27) admits at least three weak solutions u_i, $i = 1, 2, 3$ in $X := E_0^{0.8,4}$, such that

$$0 < u_i(x) < 8, \text{ for all } x \in [0, 2], \ i = 1, 2, 3,$$

for λ and μ lying in appropriate intervals.

The following result is a special case of Theorem 3.3 with $\mu = 0$.

Theorem 3.5. *Assume that* (F_1)–(F_3) *and* (I_1) *hold and there exist positive constants* k_1, k_2, *and* η *such that conditions* (25), (26), *and* (F_6)–(F_8) *hold. Then, for each* $\lambda \in \Lambda_2$, *where* Λ_2 *is given by* (22), *the problem* (20) *has at least three weak solutions* u_i, $i = 1, 2, 3$, *in* X, *such that*

$$0 < u_i(x) < k_2 \textit{ for all } x \in [0, T], \ i = 1, 2, 3.$$

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