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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A FRACTIONAL p-LAPLACIAN PROBLEM IN $\mathbb{R}^N$

G. Benhamida

Laboratory of E.D.P.N.L. and H.M.
Department of Mathematics
E.N.S. Kouba, Algiers, Algeria

e-mail: benhamidag@yahoo.fr

T. Moussaoui

Laboratory of Fixed Point Theory and Applications
Department of Mathematics
E.N.S. Kouba, Algiers, Algeria

e-mail: moussaoui@ens-kouba.dz

and

Donal O'Regan

School of Mathematics, Statistics and Applied Mathematics
National University of Ireland
Galway, Ireland

e-mail: donal.oregan@nuigalway.ie

Abstract

In this article, we use the Browder-Minty theorem to establish the existence and uniqueness of a weak solution for a fractional p-Laplace equation in $\mathbb{R}^N$.

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1. Introduction

In this paper we study the fractional p-Laplacian problem

$$(1) \qquad (-\Delta)_p^s u + V(x)|u|^{p-2}u = h(x,u) \quad \text{in } \mathbb{R}^N,$$

where $p \geq 2$, $N > ps$ with $s \in (0,1)$ fixed. Here $(-\Delta)_p^s$ is the fractional p-Laplacian operator which (up to normalization factors) can be defined as

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}dy,$$

for $x \in \mathbb{R}^N$, where $B(x,\varepsilon)$ is the ball centered at $x \in \mathbb{R}^N$ with radius ε. This operator is the p-Laplacian when $s = 1$ and the Laplacian when $s = 1$, $p = 2$; for more details about this operator see [4, 5]. To study Problem (1), we require that the function V satisfies the following assumption:

$$(V_0) \quad V \in C\left(\mathbb{R}^N, (0,+\infty)\right) \cap L^\infty\left(\mathbb{R}^N\right) \text{ and } \inf_{\mathbb{R}^N} V(x) \geq \delta > 0.$$

The function $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that is nonincreasing with respect to the second variable, i.e.,

$$(h_0) \quad h(x,s_1) \leq h(x,s_2) \text{ for a.e. } x \in \mathbb{R}^N \text{ and } s_1, s_2 \in \mathbb{R} \text{ with } s_1 \geq s_2,$$

and there exist functions $a \in L^{(p_s^\star)'}(\mathbb{R}^N)$ and $b \in L^\infty(\mathbb{R}^N) \cap L^\gamma(\mathbb{R}^N)$ such that

$$(h_1) \quad |h(x,s)| \leq a(x) + b(x)|s|^q, \quad 1 < q \leq p-1,$$

where

$$\gamma = \frac{p_s^\star}{p_s^\star - (q+1)} \text{ with } p_s^\star = \frac{Np}{N-ps}.$$

Denote by $L^\nu(\Omega)$, for $\nu \in [1, p_s^\star]$, the Lebesgue space of measurable functions on Ω endowed with the norm $|u|_\nu = \left(\int_\Omega |u(x)|^\nu dx\right)^{\frac{1}{\nu}}$,which is denoted by $|\cdot|_\nu$ and $|u|_{\nu,K} = \left(\int_K |u(x)|^\nu dx\right)^{\frac{1}{\nu}}$, where K is a subset of $\mathbb{R}^N$. Recall that, for an open set Ω of $\mathbb{R}^N$, the Sobolev space $W^{s,p}(\Omega)$ is a Banach space.

In [1], the authors established the existence and uniqueness of solutions for the p-Laplacian equation in R^N, namely

$$-div\left(|\nabla u|^{p-2}\nabla u\right) + V(x)|u|^{p-2}u = h(x,u),$$

which was inspired from [2] and [6]. In our paper, we generalize the results obtained in [1] to the fractional p-Laplacian case.

The contents of the paper are as follows. In Section 2, we present preliminaries on fractional Sobolev spaces and the Minty-Browder theorem. In Section 3, we introduce a variational setting of the problem and we prove the main result.

2. Preliminaries and functional setting

In this preliminary section, for the reader's convenience, we collect some informations to be used later. Sobolev spaces of fractional order are the convenient setting for our equation; for more details we refer the reader to [4] and [5].

Suppose that Ω is an open domain of $\mathbb{R}^N$, $s \in (0,1)$, $p \in [1,+\infty)$. Define the fractional Sobolev space $W^{s,p}(\Omega)$ by

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}} \in L^p(\Omega \times \Omega) \right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \left(\int_\Omega |u|^p dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}},$$

where the term

$$[u] = \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}},$$

is the so-called Gagliardo (semi) norm of u.

Regarding the spaces $W^{s,p}(\Omega)$ and $W^{s,p}(\mathbb{R}^N)$, we recall the following embedding theorems; see [5] for a proof of these results.

Theorem 2.1 [4, 5]**.** *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp < N$. Let $\Omega \subset \mathbb{R}^N$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $c = c(N,p,s,\Omega)$ such that, for any $u \in W^{s,p}(\Omega)$,*

$$\|u\|_{L^q(\Omega)} \leq c\|u\|_{W^{s,p}(\Omega)},$$

for any $q \in [p, p_s^\star]$; that is , the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for $q \in [p, p_s^\star]$. If, in addition, Ω is bounded, then the space $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $q \in [1, p_s^\star)$.

Theorem 2.2 ([4], p. 8 and [5], p. 555)**.** *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp < N$. Then there exists a positive constant $c = c(N,p,s)$ such that, for any $u \in W^{s,p}(\mathbb{R}^N)$,*

$$\|u\|^p_{L^{p_s^\star}(\mathbb{R}^N)} \leq c \int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy.$$

Consequently, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [p, p_s^\star]$.

We now recall the basic properties of Nemytsky operators in Lebesgue spaces.

Theorem 2.3 [8]. *Let Ω be a not necessarily bounded domain of $\mathbb{R}^N$, $p_1, p_2 \in [1, +\infty)$, and let $h : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function that satisfies the growth condition*

$$|h(x,s)| \leq a(x) + b(x)|s|^{\frac{p_1}{p_2}}, \quad x \in \Omega, \ s \in \mathbb{R},$$

where $a \in L^{p_2}(\Omega)$ and b is a non-negative function in $L^\infty(\Omega)$. Then the operator N_h from $L^{p_1}(\Omega)$ into $L^{p_2}(\Omega)$ defined by $(N_h u)(x) = h(x, u(x))$ is a bounded and continuous operator.

Now, we give the definition of a weak solution for the Problem (1). From the presence of the term $V(x)$, we denote by X the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left([u]^p_{W^{s,p}(\mathbb{R}^N)} + |u|^p_V\right)^{\frac{1}{p}}, \quad |u|^p_V = \int_{\mathbb{R}^N} V(x)|u(x)|^p dx,$$

which is an equivalent norm in $W^{s,p}(\mathbb{R}^N)$; that is

$$X = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|} = W^{s,p}(\mathbb{R}^N).$$

One can see that X is also given by

$$X = \left\{u \in L^{p_s^\star} : [u]_{W^{s,p}(\mathbb{R}^N)}, \ |u|_V < \infty\right\},$$

(see [3]). We denote by $(X', \|\cdot\|_{X'})$ the dual space of $(X, \|\cdot\|)$.

Definition 2.1. We say that $u \in X$ is a weak solution of Problem (1) if and only if

$$\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}(v(x)-v(y))dxdy$$
$$+ \int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}uvdx = \int_{\mathbb{R}^N} h(x,u)vdx,$$

for all $v \in X$.

To prove our main result, we need the following basic definitions and the Minty Browder theorem.

Definition 2.2. An operator $A : X \to X'$ that satisfies

$$\langle Au - Av, u - v\rangle \geq 0 \tag{2}$$

for any $u,\ v \in X$ is called a monotone operator; it is called a strictly monotone operator if for $u \neq v$ strict inequality holds in (2). Moreover, A is called strongly monotone if there exists $c > 0$ such that

$$\langle Au - Av, u - v\rangle \geq c\|u - v\|^2$$

for any $u,\ v \in X$.

Definition 2.3. An operator $A : X \to X'$ is demicontinuous if it maps strongly convergent sequences in X to weakly convergent sequences in X'.

Definition 2.4. An operator $A : X \to X'$ is coercive, if

$$lim_{\|u\|\to\infty} \frac{\langle Au, u\rangle}{\|u\|} = \infty.$$

Theorem 2.4 (Minty-Browder)**.** *Let X be a reflexive real Banach space and $A : X \to X'$ be an operator that is bounded, demicontinuous, coercive, and monotone on the space X. Then, the equation $Au = h$ has at least one solution $u \in X$ for each $h \in X'$. If moreover, A is a strictly monotone operator, then the equation $Au = h$ has precisely one solution $u \in X$ for every $h \in X'$.*

3. MAIN RESULT

In this section we discuss the existence of a unique weak solution for Problem (1) using the Minty-Browder theorem.

Theorem 3.1. *Assume that (V_0), (h_0) and (h_1) hold. Then Problem (1) has a unique weak solution.*

We define the operator $A : X \to X'$ by setting

$$A = L - H,$$

where

$$\langle Lu, v\rangle = \int\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}}(v(x) - v(y))dxdy$$
$$+ \int_{\mathbb{R}^N} V(x)|u|^{p-2}uvdx,$$

for $u,\ v \in X$, while

$$\langle H(u), v\rangle = \int_{\mathbb{R}^N} h(x, u)vdx$$

for $u,\ v \in X$. Our goal is to find a $u \in X$ that satisfies the equation $Au = 0$.

The proof of Theorem 3.1 is divided into several lemmas.

Lemma 3.1. *The operator A is bounded.*

Proof. First, we prove that L is bounded. Using Hölder's inequality we have for any $u, v \in X$,

$$\begin{aligned} |\langle L(u), v\rangle| \leq & \int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+ps}}|v(x)-v(y)|dxdy + \int_{\mathbb{R}^N} V(x)|u|^{p-1}|v|dx \\ \leq & \left[\int\int_{\mathbb{R}^{2N}} \left(\frac{|u(x)-u(y)|^{p-1}}{|x-y|^{\frac{(N+ps)(p-1)}{p}}}\right)^{\frac{p}{p-1}} dxdy\right]^{\frac{p-1}{p}} \\ \times & \left[\int\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} dxdy\right]^{\frac{1}{p}} \\ + & \left(\int_{\mathbb{R}^N} \left(V(x)^{\frac{p-1}{p}}|u|^{p-1}\right)^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \times \left(\int_{\mathbb{R}^N} \left(V(x)^{\frac{1}{p}}|v|\right)^p dx\right)^{\frac{1}{p}}. \end{aligned}$$

Then

$$\begin{aligned} \|L(u)\|_{X'} = & \sup_{\|v\|=1} |\langle L(u), v\rangle| \leq \sup_{\|v\|=1} \left([u]^{p-1}_{W^{s,p}(\mathbb{R}^N)}[v]_{W^{s,p}(\mathbb{R}^N)} + |u|^{p-1}_V|v|_V\right) \\ \leq & \sup_{\|v\|=1} \left[\|u\|^{p-1}\|v\|^p\right] \leq M_1^{p-1}, \end{aligned}$$

where $u \in X$ is such that $\|u\| \leq M_1$. This implies that L is bounded. Using again Hölder's inequality and Theorem 2.1, we have for any $u \in X$,

$$\begin{aligned} \|H(u)\|_{X'} = & \sup_{\|v\|=1} |\langle H(u), v\rangle| = \sup_{\|v\|=1} \left|\int_{\mathbb{R}^N} h(x,u)vdx\right| \\ \leq & \sup_{\|v\|=1} \int_{\mathbb{R}^N} |h(x,u)||v|dx \leq \sup_{\|v\|=1} \int_{\mathbb{R}^N} (a(x)+b(x)|u|^q)|v|dx \\ \leq & \sup_{\|v\|=1} \left[|a|_{(p_s^\star)'}|v|_{p_s^\star} + ||u|^q|_{\frac{p_s^\star}{q}}|b|_{\frac{p_s^\star}{p_s^\star-q-1}}|v|_{p_s^\star}\right] \leq c_1|a|_{(p_s^\star)'} + c_1^{q+1}M^q|b|_\gamma, \end{aligned}$$

where c_1 is the constant of the continuous embedding of X in $L^{p_s^\star}(\mathbb{R}^N)$, $u \in X$ and $M > 0$ is such that $\|u\| \leq M$. Thus, H is bounded, and hence A is bounded. ∎

Lemma 3.2. *The operator A is demicontinuous.*

Proof. Define the functional ψ as follows:

$$\psi(u) = \frac{1}{p}\left[\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V(x)|u(x)|^p dx\right]$$

for $u \in X$. Clearly, ψ is of cass C^1 and it's derivative in the weak sense is the operator L. Then L is continuous and so, is demicontinuous. It remains to prove that H is strongly continuous. Let (u_n) be such that

$$u_n \rightharpoonup u \quad \text{in } X,$$

so (u_n) is bounded in X. Define, for $k > 0$, the set

$$B_k = \{x \in \mathbb{R}^N : |x| < k\}$$

and $\Omega_k = \mathbb{R}^N \setminus B_k$. From (h_1), Hölder's inequality and Theorem 2.1, we have

$$\begin{aligned}\left|\int_{\Omega_k}(h(x,u_n)-h(x,u))\,vdx\right| &\leq \int_{\Omega_k}|h(x,u_n)||v|dx + \int_{\Omega_k}|h(x,u)||v|dx \\ &\leq \int_{\Omega_k}(a(x)+b(x)|u_n|^q)|v|dx \\ &+ \int_{\Omega_k}(a(x)+b(x)|u|^q)|v|dx \\ &\leq 2|a|_{(p_s^\star)'}|v|_{p_s^\star} + ||u_n|^q|_{\frac{p_s^\star}{q}}|b|_\gamma|v|_{p_s^\star} + ||u|^q|_{\frac{p_s^\star}{q}}|b|_\gamma|v|_{p_s^\star} \\ &\leq 2c_1|a|_{(p_s^\star)'}\|v\| + c_1\left(||u_n|^q|_{\frac{p_s^\star}{q}} + ||u|^q|_{\frac{p_s^\star}{q}}\right)|b|_\gamma\|v\|,\end{aligned}$$

where c_1 is the constant of the continuous embedding of X in $L^{p_s^\star}(\mathbb{R}^N)$. Then, for k sufficiently large, we obtain

$$\left|\int_{\Omega_k}(h(x,u_n)-h(x,u))\,vdx\right| \to 0 \quad \text{as } n \to \infty. \tag{3}$$

Using the embedding

$$W^{s,p}(B_k) \hookrightarrow L^\nu(B_k), \ \ \forall\, \nu \in [1, p_s^\star)$$

which is compact with $\nu = q(p_s^\star)'$ (because $p_s^\star - q(p_s^\star)' = p_s^\star\left(\frac{p_s^\star-q-1}{p_s^\star-1}\right)$ and $p_s^\star - (q+1) \geq p_s^\star - p > 0$), we obtain $u_n \to u$ in $L^{q(p_s^\star)'}$. Then by Hölder's inequality and Theorem 2.3, we have

$$h(\cdot,u_n(\cdot)) \to h(\cdot,u(\cdot)) \ \text{ in } \ L^{(p_s^\star)'}(B_k),$$

so

$$\begin{aligned}\left|\int_{B_k}(h(x,u_n)-h(x,u))\,vdx\right| &\leq |\,(h(x,u_n)-h(x,u))\,|_{(p_s^\star)',B_k}|v|_{p_s^\star,B_k} \\ &\leq c|h(x,u_n)-h(x,u)|_{(p_s^\star)',B_k}\|v\|.\end{aligned}$$

As a result,

$$\int_{B_k} (h(x,u_n) - h(x,u))\, v dx \to 0, \quad \text{as} \;\; n \to \infty. \tag{4}$$

Therefore, from (3) and (4), we obtain

$$\int_{\mathbb{R}^N} (h(x,u_n) - h(x,u))\, v dx \to 0, \quad \text{as} \;\; n \to \infty.$$

Consequently, H is strongly continuous, and so A is a demicontinuous operator. ■

Lemma 3.3. *The operator A is strictly monotone.*

Proof. In this lemma, we prove that A is a strictly monotone operator. For this, we recall the following elementary inequality (see [7]). For all $x,\ y \in \mathbb{R}$ we have

$$\left(|x|^{p-2}x - |y|^{p-2}y\right)(x-y) \geq c_p|x-y|^p \quad \text{if} \quad p \geq 2, \tag{5}$$

and then

$$\begin{aligned}
&\langle L(u) - L(v), u - v\rangle \\
&= \int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y)) - |v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{N+ps}} \\
&\times \left[(u(x)-v(x)) - (u(y)-v(y))\right] dxdy \\
&+ \int_{\mathbb{R}^N} V(x)\left(|u|^{p-2}u - |v|^{p-2}v\right)(u(x)-v(x))dx.
\end{aligned}$$

Then using (5), for $u,\ v \in X$ with $u \neq v$ we obtain

$$\begin{aligned}
\langle L(u) - L(v), u - v\rangle \;\geq\; & c_p \int\int_{\mathbb{R}^{2N}} \frac{|(u(x)-v(x)) - (u(y)-v(y))|^p}{|x-y|^{N+ps}} dxdy \\
&+ \; c_p \int_{\mathbb{R}^N} V(x)|u-v|^p dx \geq c_p\|u-v\|^p.
\end{aligned}$$

Since $u \neq v$ we have $\|u-v\|^p > 0$. Then, L is strictly monotone. Next, since h is non increasing with respect to the second variable,

$$\langle H(u) - H(v), u - v\rangle \leq 0, \quad \text{for all} \;\; u,\ v \in X,$$

so A is a strictly monotone operator. ■

Lemma 3.4. *The operator A is coercive.*

Proof. From the result obtained in the previous lemma, we have for $p \geq 2$ and $u \in X$,

$$\langle A(u) - A(0), u\rangle \geq c_p \|u\|^p,$$

and then

$$\langle A(u), u\rangle \geq \langle A(0), u\rangle + c_p \|u\|^p,$$

where

$$\langle A(0), u\rangle = \langle L(0) - H(0), u\rangle = -\int_{\mathbb{R}^N} h(x,0)udx.$$

Using Theorem 2.1 and Hölder's inequality, we obtain

$$\begin{aligned}\langle A(u), u\rangle &\geq -\int_{\mathbb{R}^N} h(x,0)udx + c_p\|u\|^p \geq -\int_{\mathbb{R}^N} a(x)|u|dx + c_p\|u\|^p \\ &\geq -|a|_{(p_s^\star)'}|u|_{p_s^\star} + c_p\|u\|^p \geq -c_1|a|_{(p_s^\star)'}\|u\| + c_p\|u\|^p,\end{aligned}$$

and then

$$lim_{\|u\|\to\infty}\frac{\langle A(u), u\rangle}{\|u\|} = \infty,$$

so A is coercive. Consequently, the existence and uniqueness of weak solution for problem (1) follows from the Minty-Browder theorem. ■

Exemple 3.1. Let $s = \frac{1}{2}$, $p = 4$, $N = 3$, $a \in L^{\frac{12}{11}}(\mathbb{R}^N)$ and $b \in L^{\frac{4}{3}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Set

$$\begin{aligned}V(x) &= 2n|x| - 2n(n-1) + \delta, \quad \text{if} \quad n-1 \leq |x| \leq \frac{2n-1}{2}, \\ V(x) &= -2n|x| + 2n^2 + \delta, \quad \text{if} \quad \frac{2n-1}{2} \leq |x| \leq n,\end{aligned}$$

for $n \in \mathbb{N} \setminus \{0\}$ and $\delta > 0$. Then the problem

$$(-\Delta)_4^{\frac{1}{2}} u + V(x)u^3 = a(x) + b(x)t|t| \quad \text{in} \ \mathbb{R}^N,$$

has a unique weak solution.

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