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NECESSARY CONDITIONS OF OPTIMALITY FOR A CLASS OF STOCHASTIC DIFFERENTIAL EQUATIONS ON UMD BANACH SPACES

N.U. Ahmed

EECS, University of Ottawa
Ottawa, Canada

Abstract

In this paper we consider stochastic evolution equations on UMD-Banach spaces. In a recent paper we proved existence of optimal controls. Here in this paper we develop necessary conditions of optimality whereby one can construct the optimal controls. For illustration we use these results to treat the LQR problem in sufficient details under two sets of alternative and distinct assumptions.

Keywords: stochastic evolution equations on Banach spaces, existence of solutions, existence of optimal controls, necessary conditions of optimality.

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1. Introduction

This paper is primarily motivated by the fact that historically it is the Hilbert space setting, not the Banach space setting, that has dominated the literature on stochastic differential equations and their optimal control as seen in [6, 10] and the references therein. Many papers on optimal control of stochastic systems have been written using Hilbert space as the pivotal space [6]. Only recently we wrote one paper [3] where we considered a class of semilinear stochastic systems on a Banach space subject to additive noise only. In this paper we consider optimal control of a more general class of stochastic evolution equations on UMD Banach spaces. The fundamental questions of existence, uniqueness and regularity properties of solutions of such systems were considered in a recent paper of Neerven, Verrar and Weis [13]. Based on their results, in a recent paper [2], we considered the question of existence of optimal controls for such systems. Here in this paper we are primarily interested to develop necessary conditions of optimality. Because of the presence of unbounded drift and diffusion, there are some technical challenges we face while constructing necessary conditions of optimality.

The rest of the paper is organized as follows. In Section 2, we present the system dynamics given by a stochastic differential equation on infinite dimensional UMD Banach space and introduce the control problem considered in this paper. We state some well-known facts on γ-Radonifying operators, UMD Banach spaces and the notions of type and co-type of Banach spaces. In Section 3, we present the basic assumptions and quote a theorem on the existence and uniqueness of solutions and regularity properties thereof. Here we introduce also the class of admissible controls and conclude with a theorem stating the existence of optimal controls. In Section 4, we present some additional assumptions, in particular, differentiability properties of the drift and diffusion operators. Following this we present a proposition characterizing semi-martingales in Banach spaces. This result is then used to prove the necessary conditions of optimality. For illustration, in Section 5, we present in sufficient details the linear quadratic regulator problem (LQR) and show that under certain mild assumptions, optimality implies stability in the Lyapunov sense.

2. The system model and the control problem

The control system is given by the following stochastic differential equation,

$$dx = Axdt + F(t,x,u_t)dt + G(t,x,u_t)dW_H,\ x(0) = x_0,\ t \in I \equiv [0,T], \tag{1}$$

where A is the infinitesimal generator of a C_0-semigroup in a Banach space E, and F and G are suitable Borel measurable maps. The process $\{W_H(t), t \geq 0\}$ is an H-cylindrical Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{U}_{ad}$ denote the class of admissible controls to be precisely characterized later in the sequel. Since relaxed controls are measure valued functions (processes), for each $u \in \mathcal{U}_{ad}$, the nonlinear drift F and the diffusion G are to be understood as the following integral expressions:

$$F(t,x,u_t) \equiv \int_U F(t,x,\xi)u_t(d\xi), \quad G(t,x,u_t) \equiv \int_U G(t,x,\xi)u_t(d\xi).$$

The problem is to find a control from the admissible set $\mathcal{U}_{ad}$ that minimizes the following cost functional

$$J(u) \equiv \mathbf{E}\left\{\int_0^T \ell(t,x,u_t)dt + \Phi(x(T))\right\}, \tag{2}$$

where ℓ is also given by the following integral expression:
$\ell(t,x,u_t) \equiv \int_U \ell(t,x,\xi)u_t(d\xi)$ for $u \in \mathcal{U}_{ad}$.

In a recent paper [2] we proved existence of optimal controls. Here in this paper we wish to present necessary conditions of optimality.

Throughout the rest of the paper we assume that the operator A is the infinitesimal generator of an analytic semigroup $S(t), t \geq 0$, and without loss of generality we may also assume that $0 \in \rho(A)$, the resolvent set of A. (For example, if $\beta \in \rho(A)$ then, for $A_\beta \equiv (-\beta + A)$, $0 \in \rho(A_\beta)$, and we compensate this by adding this to the operator F in equation (1)). So $-A$ has fractional powers. For details on fractional powers of unbounded operators generating analytic semigroups see [1]. Using the fractional powers one can introduce the scale of Banach spaces

$$E_\eta \hookrightarrow E \hookrightarrow E_{-\eta},$$

for $0 \leq \eta < 1$, where $E_\eta \equiv \{x \in E : (-A)^\eta x \in E\}$. The space E_η endowed with the norm topology, $\| x \|_{E_\eta} \equiv \| (-A)^\eta x \|_E$, is a Banach space. The space $E_{-\eta}$ is the completion of E with respect to the norm topology $\| x \|_{E_{-\eta}} \equiv \| (-A)^{-\eta} x \|$.

For study of stochastic differential equations on Banach spaces we need the notion of UMD spaces. In particular, for integration of Banach space valued stochastic processes with respect to H-Wiener process (or H-Brownian motion), we need the notion of UMD spaces. For convenience of the reader we present the following definitions.

Definition 2.1 (UMD Space)**.** A Banach space E is said to be an UMD-space (Unconditional Martingale Differences) if for each $1 < p < \infty$ and every E valued L_p martingale difference sequence $\{d_i\}$, there exists a constant $\alpha > 0$ such that for any $\varepsilon \in \{-1, 1\}^n$ the following inequality holds

$$\mathbf{E} \Big\| \sum_{i=1}^n \varepsilon_i d_i \Big\|_E^p \leq \alpha^p \, \mathbf{E} \Big\| \sum_{i=1}^n d_i \Big\|_E^p$$

for every $n \in N$.

The UMD property is independent of $p \in (1, \infty)$. If E is a Hilbert space, $\alpha = 1$. Examples of UMD spaces are the classical Lebesgue spaces L_p for all $1 < p < \infty$. In general UMD spaces are also (super) reflexive Banach spaces but the converse is false. For details on UMD spaces see [8, 9, 13].

Another related notion is the type and co-type of Banach spaces. This is a generalization of the parallelogram identity that holds for Hilbert spaces. The identity is replaced by inequality.

Definition 2.2 (Type and Co-type)**.** A Banach space E is said to be of type $p \in [1, 2]$ if, and only if, there exists a constant $C > 0$ such that for any $n \in N$ and for any sequence $\{e_i\}_{i=1}^n \in E$ and any symmetric i.i.d random variables $\{\zeta_i\}$

with values $\{-1,1\}$, the following inequality holds

$$(T1): \qquad \left(\mathbf{E}\left\|\sum_{i=1}^{n}\zeta_i e_i\right\|_E^2\right)^{1/2} \leq C\left(\sum_{i=1}^{n}\|e_i\|_E^p\right)^{1/p}.$$

The smallest constant C for which the above inequality holds is called the type p-constant of E and denoted by $C_p(E)$.

In contrast, the Banach space E is said to have co-type p if the reverse inequality holds for some $p \in [2,\infty)$ and a constant $\tilde{C} > 0$. All Hilbert spaces are of type 2 and co-type 2. For more details on type and co-type see Tzafriri [15].

Another notion of crucial importance to this paper is the notion of γ-Radonifying operators contained in the space of bounded linear operators.

Definition 2.3 (γ-Radonifying Operators)**.** Let E be a Banach space and H a separable Hilbert space and $\{\gamma_n\}$ a sequence of independent standard Gaussian random variables. A bounded linear operator $L \in \mathcal{L}(H,E)$ is said to be a γ-Radonifying operator if there exists a constant $C_L > 0$, dependent only L, such that

$$\| L \|_{\gamma(H,E)}^2 \equiv \mathbf{E}\left\|\sum \gamma_n L h_n\right\|_E^2 \leq C_L < \infty$$

for every complete orthonormal system $\{h_n\}$ of H. In other words, the sum converges in the norm topology of $L_2(\Omega,E)$ independently of the choice of the orthonormal system.

In fact the space of γ-Radonifying operators is given by the completion of the space of finite rank operators $\mathcal{F}(H,E)$ with respect to the above norm topology. Thus every element of this space is a compact operator. We denote the space of γ-Radonifying operators by $\gamma(H,E) \subset \mathcal{L}(H,E)$. With respect to the norm topology $\| \cdot \|_{\gamma(H,E)}$, as defined above, $\gamma(H,E)$ is a Banach space. In case E is also a Hilbert space, the space of γ-Radonifying operators coincides with the Space of Hilbert-Schmidt operators. For more on γ-Radonifying operators see [8, 13, 14].

For study of SDE on UMD spaces, Neerven et all [13, Proposition 6.1, p. 26] introduced several special Banach spaces. Let $(S,\mathcal{B}_S,\mu) \equiv S_\mu$ be a finite measure space. The space $L_2^\gamma(S_\mu,E_\eta)$ is given by the intersection

$$L_2^\gamma(S_\mu,E_\eta) \equiv L_2(S_\mu,E_\eta) \cap \gamma(L_2(S_\mu),E_\eta).$$

Endowed with the norm topology, $\| \Psi \|_{L_2^\gamma} \equiv \| \Psi \|_{L_2(S_\mu,E_\eta)} + \| \Psi \|_{\gamma(L_2(S_\mu),E_\eta)}$, for $\Psi \in L_2^\gamma(S_\mu,E_\eta)$, the space $L_2^\gamma(S_\mu,E_\eta)$ is a Banach space. Note that the first component gives the L_2-norm in the sense of Bochner and the second one gives the γ-Radonifying norm. There are several Banach spaces on which one can consider the question of existence of solutions of the evolution equation (1).

For $\alpha \in (0, 1/2)$, $\eta \in [0, 1)$, and $1 < p < \infty$, define the space of all $\mathcal{F}_{t\geq 0}$-adapted E_η valued random processes defined on the interval $I \equiv [0, T]$ and denoted by $V^p_{\alpha,\infty}(I \times \Omega; E_\eta)$ and endowed with the following norm topology,

$$\| \varphi \|_{V^p_{\alpha,\infty}} \equiv \Big(\mathbf{E} \| \varphi \|^p_{B_0(I,E_\eta)} \Big)^{1/p} + \sup_{t\in I} \Big(\mathbf{E} \| (\mathbf{t} - \cdot)^{-\alpha} \chi_{[\mathbf{0},\mathbf{t}]}(\cdot)\varphi(\cdot) \|^{\mathbf{p}}_{\gamma(\mathbf{L_2}[\mathbf{0},\mathbf{t}],\mathbf{E}_\eta)} \Big)^{\mathbf{1/p}} \tag{3}$$

where $B_0(I, E_\eta)$ denotes the Banach space of E_η-valued path-wise bounded measurable functions furnished with the standard sup norm topology. It is known [13] that, with respect to the above norm topology, $V^p_{\alpha,\infty}$ is a Banach space which we denote by V for simplicity of notation.

3. BASIC ASSUMPTIONS AND SOME PRELIMINARIES

In this section we introduce the basic assumptions used for study of existence of solutions of the stochastic differential equation (1) and existence of optimal controls.

Basic Assumptions

(A1): The Banach space E is an UMD space of type $\tau \in [1, 2)$ and A is the infinitesimal generator of an analytic semigroup $S(t), t \geq 0$, on E.

(A2): There exist $\eta \in [0, 1), \theta_1 \in [0, 1)$ such that $F : I \times \Omega \times E_\eta \times U \longrightarrow E_{-\theta_1}$ is Borel measurable and continuous in the last two arguments and there exist constants $C_1 > 0, L_1 > 0$ such that

(1) $\| F(t, \omega, x, \xi) \|_{E_{-\theta_1}} \leq C_1(1+ \| x \|_{E_\eta})$ uniformly with respect to $\xi \in U$,

(2) $\| F(t, \omega, x, \xi) - F(t, \omega, y, \xi) \|_{E_{-\theta_1}} \leq L_1 \| x - y \|_{E_\eta}$ uniformly with respect to $\xi \in U$.

Further, for each $x \in E_\eta$, and $\xi \in U$, $(t, \omega) \longrightarrow F(t, \omega, x, \xi)$ is an $\mathcal{F}_t$-adapted $E_{-\theta_1}$ valued strongly measurable function.

(A3): There exists $\theta_2 \in [0, 1)$ such that $G : I \times \Omega \times E_\eta \times U \longrightarrow \gamma(H, E_{-\theta_2}) \subset \mathcal{L}(H, E_{-\theta_2})$ is H-strongly Borel measurable and there exist constants $C_2 > 0, L_2 > 0$ such that for every $x, y \in L^\gamma_2(I_\mu, E_\eta)$

(1) $\| G(\cdot, \omega, x, \xi) \|_{\gamma(L_2(I_\mu,H),E_{-\theta_2})} \leq C_2\big(1+ \| x \|_{L^\gamma_2(I_\mu,E_\eta)}\big)$ uniformly in $\xi \in U$,

(2) $\| G(\cdot, \omega, x, \xi)-G(\cdot, \omega, y, \xi) \|_{\gamma(L_2(I_\mu,H),E_{-\theta_2})} \leq L_2\big(\| x-y \|_{L^\gamma_2(I_\mu,E_\eta)}\big)$ uniformly in $\xi \in U$. Further, for each $x \in E_\eta, \xi \in U$, $(t, \omega) \longrightarrow G(t, \omega, x, \xi)$ is an $\mathcal{F}_t$-adapted $\gamma(H, E_{-\theta_2})$ valued H-strongly measurable function.

By use of the variation of constants formula we can write the evolution equation (1) as the following integral equation,

$$
\begin{aligned}
x(t) &= S(t)x_0 + \int_0^t S(t-s)F(s,x(s),u_s)ds \\
&+ \int_0^t S(t-s)G(s,x(s),u_s)dW_H(s),\ t \in I.
\end{aligned} \tag{4}
$$

It is well known that by a mild solution of the evolution equation (1), one means a solution of the integral equation (4) if one exists.

The existence result given below is originally due to Neerven-Verrar-Weis [12]. For convenience of the reader we present the statement of the theorem.

Theorem 3.1. *Consider the integral equation* (4) *on a Banach space E. Suppose E is a UMD space with type $\tau \in [1,2)$ and suppose the assumptions* (A1)–(A3) *hold and further the parameters $\{\tau, p, \alpha, \eta, \theta_1, \theta_2\}$ satisfy*

(i) $0 \leq \eta + \theta_1 < 3/2 - 1/\tau$,

(ii) $0 \leq \eta + \theta_2 < 1/2$,

(iii) $p > 2$, $\alpha \in (0, 1/2)$ *such that* $\eta + \theta_2 < \alpha - 1/p$.

Then, for every $x_0 \in L_p(\Omega, \mathcal{F}_0, E_\eta)$, the integral equation (4) *has a unique solution $x \in V$ and there exists a constant $\hat{C}$ such that*

$$
\| x \|_V \leq \hat{C}\big(1 + \| x_0 \|_{L_p(\Omega,\mathcal{F}_0,E_\eta)}\big). \tag{5}
$$

Proof. See [13, Theorem 6.2, p. 28].

Admissible Controls: Throughout the rest of the paper we assume that $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$ is a complete separable filtered probability space. To study optimal controls we must define the admissible class of controls. Let U be a compact Polish space and $\mathcal{M}_1(U)$ the space of regular probability measures on $\mathcal{B}(U)$, the class of Borel subsets of the set U. Let $\mathcal{G}_{t\geq 0}$ be a nondecreasing family of subsigma algebras of the sigma algebra $\mathcal{F}_{t\geq 0}$ and let $\mathcal{P}$ denote the $\mathcal{G}_t$-predictable subsigma algebra of the product sigma algebra $\mathcal{B}(I) \times \mathcal{F}$ and let ν denote the restriction of the product measure $dt \times dP$ on $\mathcal{P}$ and introduce the Lebesgue-Bochner space $L_1(\nu, C(U))$ as ν measurable Bochner integrable processes with values in $C(U)$. Since the dual of $C(U)$ is given by the space of regular Borel measures $\mathcal{M}_B(U)$ and the later space does not satisfy Radon-Nikodym property (RNP), the dual of $L_1(\nu, C(U))$ is not given by $L_\infty(\nu, \mathcal{M}_B(U))$. However, it follows from the "theory of Lifting" that the dual is given by $L^w_\infty(\nu, \mathcal{M}_B(U))$ which consists of weak star $\mathcal{P}$-measurable essentially bounded random processes with values in $\mathcal{M}_B(U)$. Let $\mathcal{M}_1(U) \subset \mathcal{M}_B(U)$ denote the space of regular Borel Probability measures. For admissible controls one may like to choose the set

$\mathcal{U}_{ad} \equiv L^w_\infty(\nu, \mathcal{M}_1(U))$. By Alaoglu's theorem this set is weak star compact. But this vague topology is rather too weak. We introduce a slightly stronger topology. Since U is a compact Polish space, $C(U)$ is a separable Banach space, and since the probability space is assumed to be separable, the Banach space $L_1(\nu, C(U))$ is separable. Therefore, it follows from Dunford-Schwartz [11, Theorem V.5.1, p. 426] that the set $L^w_\infty(\nu, \mathcal{M}_1(U))$ is metrizable. Let $\{\varphi_i\}$ be a dense set in the unit ball of $L_1(\nu, C(U))$. For $u, v \in L^w_\infty(\nu, \mathcal{M}_1(U))$ define the metric

$$d(u,v) \equiv \sum_{i=1}^{\infty} (1/2^i) \int_{I\times\Omega} \min\{1, |\varphi_i(u) - \varphi_i(v)|\}\, d\nu, \tag{6}$$

where $\varphi(u) \equiv \int_U \varphi(t,\omega,\xi) u_{t,\omega}(d\xi)$ is a ν-measurable function. This is a complete metric space. A sequence $\{u^n\} \in \mathcal{U}_{ad}$ converging in the metric topology d to u^o is equivalent to convergence of $\varphi(u^n)$ to $\varphi(u^o)$ in ν-measure on $I \times \Omega$ for any $\varphi \in L_1(\nu, C(U))$. We denote this metric space by (M, d) and note that it is a complete metric space. For the set of admissible controls we take any totally bounded subset of (M, d) and denote it again by $\mathcal{U}_{ad}$.

In the following theorem we state a recent result on the question of existence of optimal control for the problem (1)–(2).

Theorem 3.2. *Consider the control system* (1) *with the cost functional* (2) *and admissible controls* $\mathcal{U}_{ad}$ *and suppose the assumptions of Theorem* 3.1 *hold. Further, suppose* $\Phi : E_\eta \longrightarrow R$ *is lower semi-continuous and* $\ell : I \times E_\eta \times U \longrightarrow R \equiv [0, \infty]$ *is Borel measurable and lower semi-continuous in the second and continuous in the third argument on* $E_\eta \times U$ *and there exist* $g \in L_1^+(I)$ *and constants* $a, b, c \geq 0$ *such that*

$$|\ell(t,x,\xi)| \leq g(t) + a \parallel x \parallel^p_{E_\eta}, p \geq 1,\ \forall\, x \in E_\eta, \tag{7}$$

$$|\Phi(x)| \leq b + c \parallel x \parallel^p_\eta,\ \forall\, x \in E_\eta. \tag{8}$$

Then there exists an optimal control $u^o \in \mathcal{U}_{ad}$ *minimizing the cost functional* J *given by* (2).

Proof. See [2, Theorem 5.3].

4. Necessary conditions of optimality

In this section we consider the problem of characterization of optimal controls whereby one can develop an algorithm to compute the optimal or, more precisely, extremal controls. It follows from Theorem 3.2 that an optimal control exists. But it does not say how to find one. For this we need stronger regularity properties for the drift and diffusion operators $\{F, G\}$ and the cost integrands $\{\ell, \Phi\}$. For economy of notation we omit the argument ω throughout the rest of the paper. So we introduce the following assumptions:

(B1) The Banach space E is an UMD space of type $\tau \in [1,2)$ and A is the infinitesimal generator of an analytic semigroup $S(t), t \geq 0$, on E.

(B2) The operator F satisfies assumption (A2) and further it is continuously Gâteaux differentiable (in the second argument) on E_η and its Gâteaux differential, in the direction $y \in E_\eta$, is Borel measurable in all the variables and there exists a constant $b_1 > 0$ such that

$$\| DF(t,x,\xi;y) \|_{E_{-\theta_1}} \leq b_1 \| y \|_{E_\eta}, \forall\, y \in E_\eta \text{ uniformly in } (t,x,\xi) \in I \times E_\eta \times U.$$

(B3) The operator G satisfies assumption (A3) and further it is continuously Gâteaux differentiable (in the second argument) on E_η and its Gâteaux differential, in the direction $y \in E_\eta$, is Borel measurable in all the variables and there exists a constant $b_2 > 0$ such that

$$\| DG(t,x,\xi;y) \|_{\gamma(H,E_{-\theta_2})} \leq b_2 \| y \|_{E_\eta}, \forall\, y \in E_\eta \text{ uniformly on } (t,x,\xi) \in I \times E_\eta \times U.$$

(B4) The functions $\{\ell, \Phi\}$ satisfy the assumptions of Theorem 3.2 and further they are continuously Gâteaux differentiable in the state variable x with the Gâteaux differentials being Borel measurable in all the arguments and there exist $\alpha_1 \in L_q^+(I)$, $(1/p + 1/q = 1)$ and constants $\alpha_2, \beta_1, \beta_2 \geq 0$ such that

$$\| \ell_x(t,x,\xi) \|_{E_\eta^*} \leq \alpha_1(t) + \alpha_2 \| x \|_{E_\eta}^{p-1}$$
$$\| \Phi_x(x) \|_{E_\eta^*} \leq \beta_1 + \beta_2 \| x \|_{E_\eta}^{p-1}$$

We use the above assumptions to develop necessary conditions of optimality. Before we can do so we need certain basic results related to Banach space valued martingales. Let X and Y be a pair of UMD Banach spaces with the injection $Y \hookrightarrow X$ being continuous and dense. Let H be a fixed separable Hilbert space. Let $1 < p, q < \infty, 1/p + 1/q = 1$ and let $\{\phi(t), t \in I\}$ be an X valued $\mathcal{F}_t$-adapted random process satisfying

$$\mathbf{E} \int_0^t \| \phi(s) \|_X^p \, ds < \infty, \; t \in I.$$

Similarly, let $\{\Xi(t), t \in I\}$ be a $\gamma(H,Y) (\subset \mathcal{L}(H,Y))$ valued $\mathcal{F}_t$ adapted random process satisfying

$$\mathbf{E} \int_0^t \| \Xi(s) \|_{\gamma(H,Y)}^p \, ds < \infty, \; t \in I.$$

We denote the space of all $\mathcal{F}_t$-adapted random processes satisfying the above conditions by $L_p^a(\Omega, L_p(I,X))$ and $L_p^a(\Omega, L_p(I,\gamma(H,Y)))$ respectively. We emphasize that the superscript "a" stands for $\mathcal{F}_t$-measurability of its members. In other words, the elements of these spaces are $\mathcal{F}_t$ adapted. Then define the process m given by

$$m_t \equiv \int_0^t \phi(s)ds + \int_0^t \Xi(s)dW_H(s),\ t \in I,\ m_0 = 0. \tag{9}$$

The process $m \equiv \{m_t, t \in I\}$ is an X valued semi martingale so that $m \in L_p^a(\Omega, C(I,X))$. Clearly, m is dependent on the pair (ϕ, Ξ) and we may use the notation $m = m(\phi,\Xi)$ to denote this natural dependence. Let $\mathcal{SM}_p(X,Y)$ denote the class of random processes given by

$$\mathcal{SM}_p(X,Y) \equiv \Big\{ m_t(\phi,\Xi), t \in I, m_0 = 0 : \phi \in L_p^a(\Omega, L_p(I,X)) \ \ \& \\ \Xi \in L_p^a(\Omega, L_p(I,\gamma(H,Y))) \Big\}. \tag{10}$$

The random processes belonging to the class $\mathcal{SM}_p(X,Y)$ constitutes a linear vector space. We introduce a norm topology on this by defining

$$|||m|||_{X_p} \equiv \left(\mathbf{E} \sup_{t\in I} \parallel m_t \parallel_X^p\right)^{1/p} \\ \equiv \left(\mathbf{E}\int_0^T \parallel \phi(s) \parallel_X^p ds\right)^{1/p} + \left(\mathbf{E}\int_0^T \parallel \Xi(s) \parallel_{\gamma(H,Y)}^p ds\right)^{1/p}. \tag{11}$$

Clearly, this is a Banach space with respect to the above norm topology. If Y has a nontrivial type, it follows from a duality theorem [14, Theorem 10.9, p. 38–48] that $(\gamma(H,Y))^* \cong \gamma(H,Y^*)$. Since $\{X, Y, X^*, Y^*\}$ are all UMD spaces, they are all reflexive Banach spaces. So the dual of $L_p^a(\Omega, L_p(I,X))$ and $L_p(\Omega, L_p(I,\gamma(H,Y)))$ are given by $L_q^a(\Omega, L_q(I,X^*))$ and $L_q(\Omega, L_q(I,\gamma(H,Y^*)))$ respectively. Hence the topological dual of the space $\mathcal{SM}_p(X,Y)$ is given by the space of $\mathcal{F}_t$-adapted semimartingales denoted by $\mathcal{SM}_q(X^*,Y^*)$ with the dual norm $|||M|||_{X_q^*}$. The elements of this later space has the representation

$$M_t \equiv \int_0^t \psi(s)ds + \int_0^t \Sigma(s)dW_H(s), M_0 = 0,\ t \in I, \tag{12}$$

with $\psi \in L_q^a(\Omega, L_q(I,X^*))$ and $\Sigma \in L_q^a(\Omega, L_q(I,\gamma(H,Y^*))$. We may define the duality pairing between the spaces $\mathcal{SM}_p(X,Y)$ and $\mathcal{SM}_q(X^*,Y^*)$ as follows

$$< m, M >= \mathbf{E}\int_0^T < \varphi(s), \psi(s) >_{X,X^*} ds \\ + \mathbf{E}\int_0^T << \Xi(s), \Sigma(s) >>_{\gamma(H,Y),\gamma(H,Y^*)} ds \tag{13}$$

where $M \in \mathcal{SM}_q(X^*, Y^*)$ and $m \in \mathcal{SM}_p(X, Y)$. It follows from Hahn-Banach theorem that for every $m \in \mathcal{SM}_p(X, Y)$ there exists an $M \in \mathcal{SM}_q(X^*, Y^*)$ of norm one such that $< m, M > = |||m|||_{X_p}$. Most importantly, for every $m \in \mathcal{SM}_p(X, Y)$, there exists a unique pair $(\phi, \Xi) \in L_p^a(\Omega, L_p(I, X)) \times L_p^a(\Omega, L_p(I, \gamma(H, Y)))$ satisfying the representation (9). Thus we can state the following result.

Proposition 4.1. *Let $\{X, Y\}$ and their duals $\{X^*, Y^*\}$ be UMD Banach spaces with Y having a nontrivial type. For $1 < p, q < \infty$, satisfying $(1/p + 1/q = 1)$, let $L_p^a(\Omega, L_p(I, X))$ and $L_p^a(\Omega, L_p(I, \gamma(H, Y)))$ denote the Banach spaces of $\mathcal{F}_t$-adapted random processes defined on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$ with values in X and $\gamma(H, Y)$ respectively. Then the class $\mathcal{SM}_p(X, Y)$, as defined above, is a Banach space of X valued semimartingales having the unique representation* (9) *and that its topological dual is given by $\mathcal{SM}_q(X^*, Y^*)$.*

Now we are prepared to construct the necessary conditions of optimality. In Section 3 we have seen the Banach spaces $\{E_{-\theta_1}, E_{-\theta_2}\}$ appearing in the basic assumptions (A2)–(A3) for the operators F and G. We denote the dual of these spaces by $\{E^*_{-\theta_1}, E^*_{-\theta_2}\}$ respectively. In the following theorem we use the space of semimartingales $\mathcal{SM}_p(E_{-\theta_1}, E_{-\theta_2})$ and its dual $\mathcal{SM}_q(E^*_{-\theta_1}, E^*_{-\theta_2})$ to develop the necessary conditions of optimality.

Theorem 4.2. *Consider the system* (1) *with the objective functional* (2) *and suppose the assumptions of Theorem* 3.2 *and the assumptions* (B1)–(B4) *hold. Then, for the control state pair $(u^o, x^o) \in \mathcal{U}_{ad} \times L_p^a(\Omega, B_0(I, E_\eta))$ to be optimal it is necessary that there exists a $\psi \in L_q^a(\Omega, B_0\big(I, E^*_{-\theta_1}\big))$ such that*
(14)
$$dJ(u^o, u-u^o) \equiv \mathbf{E}\int_0^T \Big\{\ell(t, x^o, u-u^o) + < F(t, x^o, u-u^o), \psi >_{E_{-\theta_1}-E^*_{\theta_1}}\Big\}dt$$
$$+\,\mathbf{E}\int_0^T \Big\{<< G(t, x^o, u-u^o), (-)DG^*(x^o, u^o; \psi) >>_{\gamma(H, E_{-\theta_2}), \gamma(H, E^*_{-\theta_2})}\Big\}ds \geq 0$$

for all $u \in \mathcal{U}_{ad}$, where the triple $\{u^o, x^o, \psi\}$ satisfies respectively the following forward and backward stochastic evolution equations (*in the mild sense*):

$$dx^o = Ax^o dt + F(t, x^o, u^o)dt + G(t, x^o, u^o)dW_H(t),\ \ x_0^o = x_0; \tag{15}$$

$$\begin{aligned} -d\psi &= A^*\psi dt + DF^*(t, x^o, u^o; \psi)dt + Q_o^*(t)\psi dt + \ell_x(t, x^o, u^o)dt \\ &\quad + DG^*(t, x^o, u^o; \psi)dW_H, \quad \psi(T) = \Phi_x(x^o(T)); \end{aligned} \tag{16}$$

with the operator valued process $Q_o(t) \equiv Q(t, x^o(t), u_t^o)$ identified in the body of the proof.

Proof. Let $u^o \in \mathcal{U}_{ad}$ denote the optimal control and $u \in \mathcal{U}_{ad}$ any other control. Since the set $\mathcal{U}_{ad}$ consists of relaxed controls, it is clear that for any $\varepsilon \in [0,1]$, $u^o + \varepsilon(u - u^o) \in \mathcal{U}_{ad}$. Thus for a control u^o to be optimal it is necessary that

$$(*) \qquad J(u^o + \varepsilon(u - u^o)) - J(u^o) \geq 0, \ \forall \ u \in \mathcal{U}_{ad}, \ \text{ and } \ \varepsilon \in [0,1].$$

Let $\{x^\varepsilon, x^o\}$ denote the mild solutions of the evolution equation (1) corresponding to the controls $\{u^\varepsilon, u^o\}$ respectively. Clearly, by definition, they satisfy (respectively) the following integral equations,

$$(17) \qquad \begin{aligned} x^\varepsilon(t) &= S(t)x_0 + \int_0^t S(t-s)F(s, x^\varepsilon(s), u^\varepsilon_s)ds \\ &+ \int_0^t S(t-s)G(s, x^\varepsilon(s), u^\varepsilon_s)dW_H(s), \ t \in I, \end{aligned}$$

$$(18) \qquad \begin{aligned} x^o(t) &= S(t)x_0 + \int_0^t S(t-s)F(s, x^o(s), u^o_s)ds \\ &+ \int_0^t S(t-s)G(s, x^o(s), u^o_s)dW_H(s), \ t \in I. \end{aligned}$$

By virtue of continuity of F and G in the second argument, it is evident that, as $\varepsilon \to 0$, $x^\varepsilon \xrightarrow{s} x^o$ in V and hence in $B_0(I, E_\eta)$ P-a.s. In fact, by virtue of the assumptions (B2) and (B3), one can verify that the limit $\lim_{\varepsilon \downarrow 0}(1/\varepsilon)(x^\varepsilon(t) - x^o(t)) \equiv y(t)$ exists for all $t \in I$, P-a.s. Indeed, subtracting equation (18) from (17) term by term and letting $\varepsilon \to 0$, it is easy to verify that the process y satisfies the following integral equation on E_η,

$$(19) \qquad \begin{aligned} y(t) = &\int_0^t S(t-s)DF(s, x^o, u^o; y)ds + \int_0^t S(t-s)DG(s, x^o, u^o; y)dW_H(s) \\ &+ \int_0^t S(t-s)F(s, x^o, u - u^o)ds \\ &+ \int_0^t S(t-s)G(s, x^o, u - u^o)dW_H(s), t \in I, \end{aligned}$$

where DF and DG denote respectively the Gâteaux differentials of F and G, evaluated at $(s, x^o(s), u_s)$ $(s \in I)$ in the direction $y(s) \in E_\eta$. In other words, y is the mild solution of the following variational equation,

$$(20) \qquad \begin{aligned} dy &= Aydt + DF(t, x^o, u^o; y)dt + DG(t, x^o, u^o; y)dW_H + dm_t, \\ y(0) &= 0, \ t \in I, \end{aligned}$$

where $m \equiv \{m_t, t \geq 0\}$ is an $\mathcal{F}_t$-semimartingale given by the following expression,

$$(21) \qquad dm_t = F(t, x^o, u - u^o) \ dt + G(t, x^o, u - u^o) \ dW_H, \ m_0 = 0, \ t \in I.$$

Note that for each $t \in I$, $m_t \equiv m_t^{u-u^o} \rightarrow 0$, P-a.s, as $u \xrightarrow{d} u^o$. By virtue of the assumptions (B1)–(B3), it follows from the same procedure as used in proving theorem 3.1 (see [2] for details), that the integral equation (19) has a unique solution and hence the evolution equation (20) has a unique mild solution $y \in V$ and consequently $y \in L_p^a(\Omega, B_0(I, E_\eta))$ and therefore, $y \in B_0(I, E_\eta)$ $P-$a.s. It follows from this fact that there exists a strongly measurable linear stochastic evolution operator $\Psi(t,s), 0 \le s < t < \infty$, so that $\Psi(t,s) \in \mathcal{L}(L_p^a(\Omega, E_\eta))$ for all $0 \le s < t < \infty$, and $\lim_{t\to s} \Psi(t,s) = I_d$ (identity operator) in the strong operator topology, and that it satisfies the following operator equation,

$$\Psi(t,s) = S(t-s) + \int_s^t S(t-r)DF_0(r)\Psi(r,s)dr$$

$$+ \int_s^t S(t-r)DG_0(r)\Psi(r,s)dW_H(r), \; 0 \le s < t < \infty, \tag{22}$$

where $DF_0(r) \equiv DF(r, x^o(r), u_r^o)$ and $DG_0(r) \equiv DG_0(r, x^o(r), u_r^o)$. Hence the solution of equation (20) has the representation

$$y(t) = \int_0^t \Psi(t,s)\; dm_s, \; t \in I.$$

Thus we observe that the map $m \longrightarrow y$ is a bounded linear map from the space of semimartingales $\mathcal{SM}(E_{-\theta_1}, E_{-\theta_2})$ to the space $L_p^a(\Omega, B_0(I, E_\eta)) \subset L_p^a(\Omega, L_p(I, E_\eta))$. Considering the cost functional given by equation (2) it follows from assumption (B4) that it is Gâteaux differentiable. Taking it's Gâteaux differential at u^o in the direction $u - u^o$ (using $(*)$) we find that

$$dJ(u^o; u - u^o) = L(y) + \mathbf{E}\Big\{\int_0^T \ell(t, x^o(s), u_s - u_s^o)ds\Big\} \tag{23}$$

where the functional L is given by

$$L(y) \equiv \mathbf{E}\Big\{\int_0^T < \ell_x(t, x^o(s), u_s^o), y(s) >_{E_\eta^*, E_\eta} ds + < \Phi_x(x^o(T)), y(T) >_{E_\eta^*, E_\eta}\Big\}. \tag{24}$$

Dividing the expression $(*)$ by ε and letting $\varepsilon \downarrow 0$, it is clear from this that, for u^o to be optimal, it is necessary that the following inequality holds

$$dJ(u^o; u - u^o) = L(y) + \mathbf{E}\Big\{\int_0^T \ell(t, x^o(s), u_s - u_s^o)ds\Big\} \ge 0 \;\forall\; u \in \mathcal{U}_{ad}. \tag{25}$$

We show that the functional L is linear and continuous on $L_p^a(\Omega, L_p(I, E_\eta))$. Linearity is apparent. Using the assumption (B4) and applying Hölder inequality it is easy to verify that

$$\begin{aligned}(26)\qquad &\mathbf{E}\int_I |<\ell_x(t,x^o,u^o_t),y(t)>_{E^*_\eta,E_\eta}|dt\\ &\le \|\ell_x\|_{L_q(\Omega,L_q(I,E^*_\eta))}\|y\|_{L_p(\Omega,L_p(I,E_\eta))}<\infty,\end{aligned}$$

$$(27)\qquad \mathbf{E}|<\Phi_x(x^o(T)),y(T)>_{E^*_\eta,E_\eta}|\le \|\Phi_x\|_{L_q(\Omega,E^*_\eta)}\|y(T)\|_{L_p(\Omega,E_\eta)}<\infty.$$

Thus $y\longrightarrow L(y)$ is a continuous linear functional on $L_p(\Omega,B_0(I,E_\eta))\subset L_p(\Omega, L_p(I,E_\eta))$. Now we use Proposition 4.1 with $X=E_{-\theta_1}$ and $Y=E_{-\theta_2}$ and hence $\mathcal{SM}_p(X,Y)=\mathcal{SM}_p(E_{-\theta_1},E_{-\theta_2})$. It follows from continuity of the linear maps $m\to y$ and $y\longrightarrow L(y)$, and the assumptions (A2) and (A3) that the composition map

$$m\longrightarrow y\longrightarrow L(y)\cong\tilde{L}(m)$$

is a continuous linear functional on the space of semimartingales $\mathcal{SM}_p(E_{-\theta_1}, E_{-\theta_2})$. Hence, by duality, it follows from Proposition 4.1 that there exists an $M\in\mathcal{SM}_q(E^*_{-\theta_1},E^*_{-\theta_2})$ and a unique pair (ψ,Σ) with values $\psi\in L_q(\Omega,L_q(I,E^*_{-\theta_1}))$ and $\Sigma\in L_q(\Omega,L_q(I,\gamma(H,E^*_{-\theta_2}))$ such that

$$(28)\qquad dM_t=\psi(t)dt+\Sigma(t)dW_H(t),\ M_0=0,\ t\ge 0;$$

and that

$$\begin{aligned}L(y)\equiv\tilde{L}(m)=<m,M>\equiv\mathbf{E}\Big\{&\int_0^T<F(t,x^o(t),u_t-u^o_t),\psi(t)>_{E_{-\theta_1},E^*_{-\theta_1}}dt\\ (29)\qquad +&\int_0^T<<G(t,x^o(t),u_t-u^o_t),\Sigma(t)>>_{\gamma(H,E_{-\theta_2}),\gamma(H,E^*_{-\theta_2})}\ dt\Big\}.\end{aligned}$$

Now it remains to identify the pair $\{\psi,\Sigma\}$. Since $\theta_1\ge\theta_2\ge 0$, and $\eta\ge 0$, we have the following continuous and dense embeddings $E_\eta\hookrightarrow E_{-\theta_2}\hookrightarrow E_{-\theta_1}$ along with the duals $E^*_{-\theta_1}\hookrightarrow E^*_{-\theta_2}\hookrightarrow E^*_\eta$. As y takes its values in E_η and ψ takes values in $E^*_{-\theta_1}$, which is continuously embedded in E^*_η, the following identities hold

$$(y(t),\psi(t))_{E_\eta,E^*_\eta}=(y(t),\psi(t))_{E_\eta,E^*_{-\theta_1}}=(y(t),\psi(t))_{E_\eta,E^*_{-\theta_2}}.$$

At this point we need a stochastic differential rule similar to that of Itô. In fact, with slight modification, the Itô differential rule also holds in the Banach space settings. One only needs to replace the scalar products by appropriate duality pairings and the space of Hilbert-Schmidt operators by the space of γ-Radonifying operators. Thus taking the stochastic differential of the above expression, it follows form modified Itô formula that

$$(30)\qquad d(y(t),\psi(t))=(dy,\psi)+(y,d\psi)+<<dy,d\psi>>.$$

Using "B^*" to denote the conjugate of any linear operator B and formally substituting the differential expression for y from equation (20) into the expression (30) we obtain

$$
\begin{aligned}
& d(y(t), \psi(t)) \\
(31) \quad & = (y(t), d\psi + A^*\psi dt + DF^*(t, x^o, u^o; \psi)dt + DG^*(t, x^o, u^o; \psi)dW_H) \\
& + (F(t, x^o, u - u^o), \psi)dt + (G^*(t, x^o, u - u^o)\psi, dW_H) + << dy, d\psi >> .
\end{aligned}
$$

Examining the variational equation (20) we observe that it has two terms involving stochastic integration, while examining the above expression, we observe that the adjoint variable has one term involving stochastic integration. Thus the quadratic variation term is given by

$$
\begin{aligned}
(32) \quad << dy, d\psi >> = & << DG(t, x^o, u^o; y)dW_H \\
& + G(t, x^o, u - u^o)dW_H, (-)DG^*(t, x^o, u^o; \psi)dW_H >> .
\end{aligned}
$$

Integrating the quadratic variation term and recalling that $(\gamma(H, E_{-\theta_2}))^* \cong (\gamma(H, E^*_{-\theta_2})$ we obtain

$$
\begin{aligned}
& \mathbf{E}\int_0^T << dy, d\psi >> \\
(33) \quad & = \mathbf{E}\int_0^T << DG(t, x^o, u^o; y), (-)DG^*(t, x^o, u^o; \psi) >>_{\gamma(H,E_{-\theta_2}),\gamma(H,E^*_{-\theta_2})} dt \\
& + \mathbf{E}\int_0^T << G(t, x^o, u - u^o), (-)DG^*(t, x^o, u^o; \psi) >>_{\gamma(H,E_{-\theta_2}),\gamma(H,E^*_{-\theta_2})} dt.
\end{aligned}
$$

The first term on the right hand side of the above expression is a bilinear form on $E_\eta \times E^*_\eta$ and can be rewritten as

$$
\begin{aligned}
(34) \quad & \mathbf{E}\int_0^T << DG(t, x^o, u^o; y), (-)DG^*(t, x^o, u^o; \psi) >>_{\gamma(H,E_{-\theta_2}),\gamma(H,E^*_{-\theta_2})} dt \\
& \equiv \mathbf{E}\int_0^T (Q(t, x^o, u^o)y, \psi)_{E_{-\theta_2},E^*_{-\theta_2}} dt \equiv \mathbf{E}\int_0^T (y, Q^*(t, x^o, u^o)\psi)_{E_\eta,E^*_\eta} dt.
\end{aligned}
$$

Recall that $E^*_{-\theta_1} \hookrightarrow E^*_{-\theta_2} \hookrightarrow E^*_\eta$ and $\psi(\cdot)$ is $E^*_{-\theta_1}$-valued random process. Thus the above duality pairings are consistent. For simplicity of notation we set $Q(t, x^o(t), u^o_t) \equiv Q_o(t)$, $t \in I$, and note that, for almost all $t \in I$, it belongs to $\mathcal{L}(E_\eta, E_{-\theta_2})$ P-a.s. The duality pairing in the first term on the righthand side of the above expression is well defined because of the fact that the injection $E^*_{-\theta_1} \hookrightarrow E^*_{-\theta_2}$ is continuous and that $\psi(t) \in E^*_{-\theta_1} \subset E^*_{-\theta_2}$ P-a.s. Similarly, the second duality pairing is well defined because of the fact that $Q_o(t) \in \mathcal{L}(E_\eta, E_{-\theta_2})$ and hence $Q_o^*(t) \in \mathcal{L}(E^*_{-\theta_2}, E^*_\eta)$ and that $\psi(t) \in E^*_{-\theta_1} \hookrightarrow E^*_{-\theta_2} \hookrightarrow E^*_\eta$ for each $t \in I$, P-a.s. Integrating equation (31) and using the expressions (33)–(34) along with the initial condition $y(0) = 0$ (see equation (20)) we obtain

$$\begin{aligned}
&\mathbf{E}(y(T), \psi(T)) \\
&= \mathbf{E}\int_0^T < y(t), d\psi + A^*\psi dt + Q^*(t, x^o, u^o)\psi dt + DF^*(t, x^o, u^o; \psi)dt \\
&+ DG^*(t, x^o, u^o; \psi)dW_H > \\
&+ \mathbf{E}\int_0^T < F(t, x^o, u - u^o), \psi > dt + \mathbf{E}\int_0^T < \psi, G(t, x^o, u - u^o)dW_H > \\
&+ \mathbf{E}\int_0^T << -G(t, x^o, u - u^o), DG^*(t, x^o, u^o; \psi) >>_{\gamma(H, E_{-\theta_2}), \gamma(H, E^*_{-\theta_2})} dt.
\end{aligned} \tag{35}$$

Now setting

$$\begin{aligned}
-d\psi &= A^*\psi dt + Q_o^*(t)\psi dt + DF^*(t, x^o, u^o; \psi)dt \\
&\quad + DG^*(t, x^o, u^o; \psi)dW_H + \ell_x(t, x^o, u^o)dt
\end{aligned} \tag{36}$$

and $\psi(T) \equiv \Phi_x(x^o(T))$ in the expression (35), we obtain the following expression (37)

$$\begin{aligned}
&\mathbf{E}(y(T), \psi(T)) + \mathbf{E}\int_0^T < y, \ell_x(t, x^o, u^o) > dt \\
&= \mathbf{E}\int_0^T < F(t, x^o, u - u^o), \psi > dt \\
&+ \mathbf{E}\int_0^T << G(t, x^o, u - u^o), (-)DG^*(t, x^o, u^o; \psi) >>_{\gamma(H, E_{-\theta_2}), \gamma(H, E^*_{-\theta_2})} dt \\
&+ \mathbf{E}\int_0^T < \psi, G(t, x^o, u - u^o)dW_H > .
\end{aligned}$$

Here we notice that the expression on the left-hand side of the above equation coincides with the linear functional $L(y)$ given by equation (24). And the last term of (37) involving the stochastic integral vanishes. This follows from stopping time argument. Denote $G_o(t) \equiv G(t, x^o(t), u_t - u_t^o)$ for $t \in I$. Since $E^*_{-\theta_1} \subset E^*_{-\theta_2}$ and $\psi(t) \in E^*_{-\theta_1}$ P-a.s, we have

$$\mathbf{E}\int_0^T < \psi(t), G_o(t)dW_H(t) >_{E^*_{-\theta_1}, E_{-\theta_2}} = \mathbf{E}\int_0^T < \psi(t), G_o(t)dW_H(t) >_{E^*_{-\theta_2}, E_{-\theta_2}}.$$

For each $n \in N$, define the stopping time

$$\tau_n \equiv \inf\left\{t \in [0, T] : \parallel \psi(t) \parallel_{E^*_{-\theta_2}} \parallel G_o(t) \parallel_{\gamma(H, E_{-\theta_2})} > n\right\}.$$

If the set is empty, set $\tau_n = T$. It is clear that $\{\tau_n\}$ is a monotone increasing (nondecreasing) sequence of stopping times. Now considering the above stochastic integral, it is obvious that for every $n \in N$, we have

$$\begin{aligned} &\mathbf{E}\int_0^{T\wedge\tau_n} < \psi(t), G_o(t)dW_H >_{E^*_{-\theta_1},E_{-\theta_2}} \\ &= \mathbf{E}\int_0^{T\wedge\tau_n} < \psi(t), G_o(t)dW_H >_{E^*_{-\theta_2},E_{-\theta_2}} = 0. \end{aligned}$$

It follows from assumption **(A3)** that $G_o(t) \in \gamma(H, E_{-\theta_2})$ satisfying

$$\sup\Big\{ \parallel G_o(t) \parallel_{\gamma(H,E_{-\theta_2})},\ t \in I\Big\} < \infty.$$

Thus $\lim_{n\to\infty} \tau_n = T$, P-a.s, and hence $\mathbf{E}\int_0^T < \psi(t), G_o(t)dW_H >_{E^*_{-\theta_2},E_{-\theta_2}} = 0$. Hence we conclude that the expression (37) reduces to the following identity,

$$\begin{aligned} L(y) = {}& \mathbf{E}\int_0^T < F(t, x^o, u - u^o), \psi >_{E_{-\theta_1},E^*_{-\theta_1}} dt \\ & + \mathbf{E}\int_0^T << G(t, x^o, u - u^o), (-)DG^*(t, x^o, u^o; \psi) >>_{\Gamma,\Gamma^*} dt, \end{aligned} \tag{38}$$

where, for simplicity of notation, we used $\Gamma \equiv \gamma(H, E_{-\theta_2})$ and $\Gamma^* \equiv \gamma(H, E^*_{-\theta_2})$. Substituting the above expression for $L(y)$ in the inequality (25) we obtain the necessary condition (14). It follows from equation (36) with the boundary condition as stated there, that ψ must satisfy the adjoint equation (16) in the mild sense. This verifies the necessary condition (16). The operator valued process $Q_o(t) \equiv Q(t, x^o(t), u^o_t)$, $t \in I$, is as identified in the expression (34). Equation (15) gives the dynamics of the given system corresponding to the optimal control u^o with x^o being the corresponding solution trajectory. So there is nothing to prove. Comparing (29) with (38) we can identify $\Sigma(t) \equiv (-)DG^*(t, x^o(t), u^o_t; \psi(t))$ which is Γ^*-valued. Clearly, for each $t \in I$, $\Sigma(t)$ is an operator valued $\mathcal{F}_t$-adapted random process taking values in the space of bounded linear operators from H to $E^*_{-\theta_2}$ and belongs to Γ^*. Using successive approximation as in Hu and Peng [12, Theorem 3.1, p. 455], it is not difficult to verify that the adjoint evolution equation (16) has a unique $\mathcal{F}_t$-adapted mild solution $\psi \in L_q(\Omega, B_0(I, E^*_{-\theta_1})) \subset L_q(\Omega, B_0(I, E^*_\eta))$. This proves all the necessary conditions (14)–(16). ■

5. AN ILLUSTRATION:(LQR PROBLEM)

For a simple illustration, we consider the following linear quadratic regulator problem. The system is given by

$$dx = Axdt + B(t)udt + \sigma(t)dW_H,\ x(0) = x_0 \tag{39}$$

where A is the infinitesimal generator of an analytic semigroup on a UMD Banach space E as in the preceding sections, B is an operator valued function with values in $\mathcal{L}(U, E_{-\theta_1})$ and σ is also an operator valued function with values in $\gamma(H, E_{-\theta_2})$. Note that it follows from the analyticity of the semigroup $\{S(t), t \geq 0\}$ that $E_{-\theta_1}$ is invariant under the semigroup $S(t)$. Thus under the given assumptions, for every $x_0 \in L_2(\Omega, E_{-\theta_1})$, equation (39) has a unique solution $x \in L_2(\Omega, C(I, E_{-\theta_1}))$. Recall that the dual of the Banach space $E_{-\theta_1}$ is denoted by $E^*_{-\theta_1}$. The cost functional is given by

$$(40) \quad J(u) \equiv \mathbf{E}\left\{\int_0^T (1/2)\{(Q(t)x, x) + (R(t)u, u)\}dt + (1/2)(Mx(T), x(T))\right\},$$

where Q is a symmetric positive operator valued function with values $Q(t) \in \mathcal{L}(E_{-\theta_1}, E^*_{-\theta_1})$ and $M \in \mathcal{L}(E_{-\theta_1}, E^*_{-\theta_1})$ is also symmetric and positive. The duality pairings (scalar products) in the cost functional (40) are defined accordingly. We assume that U is a real separable Hilbert space (so Polish space) and R is a positive symmetric operator valued function with values in $\mathcal{L}(U)$ and there exists a real number $\rho > 0$ so that $(R(t)u, u) \geq \rho \parallel u \parallel^2$ for all $u \in U$ and $t \in I$. This implies that $R^{-1}(t) \in \mathcal{L}(U)$, $t \in I$. Since the problem is convex we do not need relaxed controls. In fact the ordinary controls are special relaxed controls given by Dirac measures along the path $\{u(t),\, t \in I\}$ with $u \in L^a_2(I, U)$. The objective is to find a control from the space $L^a_2(I, U)$ ($\mathcal{F}_t$-adapted U valued norm square integrable processes) that minimizes the functional (40) subject to the dynamic constraint (39). Positive semi-definiteness of Q and M, and positive definiteness of R with $\rho > 0$, imply that J is radially unbounded on the Hilbert space $L^a_2(I, U)$ satisfying $J \geq 0$, and hence it attains its minimum on $L^a_2(I, U)$ implying existence of an optimal control. Using the necessary conditions (14)-(16) of Theorem 4.2, it is easy to verify that the optimal control, state, and the corresponding adjoint state satisfy the following 2-point boundary value problem (in the mild sense):

$$(41) \qquad dx^o = Ax^o dt - BR^{-1}B^*\psi dt + \sigma dW_H, x^o(0) = x_0$$

$$(42) \qquad -d\psi = A^*\psi dt + Qx^o dt, \quad \psi(T) = Mx^o(T).$$

The optimal control is given by $u^o = -R^{-1}B^*\psi$. Using the above necessary optimality conditions and defining $\psi(t) \equiv K(t)x^o(t) + r(t), t \geq 0$, one can easily derive the operator Riccati equation for K and the backward SDE for r respectively as follows:

$$(43) \qquad -\dot{K} = KA + A^*K - K(BR^{-1}B^*)K + Q, K(T) = M,$$

$$(44) \qquad -dr = A^*r dt - K(BR^{-1}B^*)r dt + K\sigma dW_H, r(T) = 0.$$

For convenience of notation set $\Gamma(t) \equiv (B(t)R^{-1}(t)B^*(t))$, $t \geq 0$, and note that it is an operator valued function with values in $\mathcal{L}(E^*_{-\theta_1}, E_{-\theta_1})$.

Given K, we define the operator valued function $\mathcal{A}(t) \equiv (A-(1/2)\Gamma(t)K(t))$. It is easy to verify that equation (43) is equivalent to the following operator equation,

$$-\dot{K} = \mathcal{A}^*K + K\mathcal{A} + Q,\ t \in I,\ K(T) = M, \tag{45}$$

on the Banach space of bounded linear operators $\mathcal{L}(E_{-\theta_1}, E^*_{-\theta_1})$. Let $\mathcal{A}$ generate the evolution operator $\mathcal{S}(t,s), 0 \leq s < t < \infty$, on the Banach space $\mathcal{L}(E_{-\theta_1})$. Then K is given by

$$K(t) = \mathcal{S}^*(T,t)M\mathcal{S}(T,t) + \int_t^T \mathcal{S}^*(s,t)Q(s)\mathcal{S}(s,t)ds,\ t \in I. \tag{46}$$

Clearly, this is only an implicit formula for K because the evolution operator $\mathcal{S}$ is itself dependent on K. However, this tells us that if equation (43) has any solution K, it must be symmetric and positive with values in $\mathcal{L}(E_{-\theta_1}, E^*_{-\theta_1})$. Using this fact, we can prove existence of solution of equation (43). For more on similar questions relating the properties of $\mathcal{S}$ and K see [5, Proposition 5.2 and Corollary 5.1]. For convenience of notation set $\mathcal{L}(E_{-\theta_1}, E^*_{-\theta_1}) \equiv \mathcal{Z}$. Under the assumption that the operator valued functions $\{B, R\}$ are uniformly measurable and bounded we have $\Gamma \in B_0(I, \mathcal{L}(E^*_{-\theta_1}, E_{-\theta_1}))$ and hence there exists a finite positive number b such that $\sup\Big\{ \parallel \Gamma(t) \parallel_{\mathcal{L}(E^*_{-\theta_1}, E_{-\theta_1})},\ t \in I\Big\} \leq b$.

Theorem 5.1. *Consider the operator Riccati differential equation* (43) *and suppose A is the generator of an analytic semigroup on E, $B \in B_0(I, \mathcal{L}(U, E_{-\theta_1}))$, $Q \in C(I, \mathcal{L}(E_{-\theta_1}, E^*_{-\theta_1}))$ and $M \in \mathcal{L}(E_{-\theta_1}, E^*_{-\theta_1})$ and both Q and M are positive and symmetric. Let U be a real separable Hilbert space and suppose the operator valued function $R \in C(I, \mathcal{L}(U))$ is symmetric and there exists a $\rho > 0$ such that $(R(t)u, u) \geq \rho \parallel u \parallel_U^2$ for all $u \in U$ and $t \in I$. Then equation* (43) *has a unique solution $K \in C(I, \mathcal{L}(E_{-\theta_1}, E^*_{-\theta_1}))$.*

Proof. As discussed above, if equation (43) has a solution, it most be positive and symmetric. Using variation of constants formula and equation (43) one can verify that K satisfies the following integral equation

$$\begin{aligned} K(t) &= S^*(T-t)MS(T-t) \\ &\quad - \int_t^T S^*(s-t)K(s)(BR^{-1}B^*)(s)K(s)S(s-t)ds \\ &\quad + \int_t^T S^*(s-t)Q(s)S(s-t)ds,\ t \in I. \end{aligned} \tag{47}$$

Since the operator valued function K (if one exists) must be symmetric and positive, it follows from the above expression that K must satisfy the following inequality,

$$< K(t)e, e >_{E^*_{-\theta_1}, E_{-\theta_1}} \leq \; < MS(T-t)e, S(T-t)e >_{E^*_{-\theta_1}, E_{-\theta_1}}$$

$$+ \int_t^T < Q(s)S(s-t)e, S(s-t)e) >_{E^*_{-\theta_1}, E_{-\theta_1}} ds, \tag{48}$$

for all $t \in I$ and for all $e \in E_{-\theta_1}$. Thus K satisfies an apriori bound and it follows from the above expression that there exists a number $r > 0$ ($r = r(M, Q, M_0, T)$) so that

$$\sup \Big\{ \parallel K(t) \parallel_{\mathcal{L}(E_{-\theta_1}, E^*_{-\theta_1})}, \; t \in I \Big\} \leq r < \infty,$$

where $M_0 \equiv \sup\{\parallel S(t) \parallel, t \in I\}$. Define the operator Φ on the Banach space $\mathcal{Z}$ as follows

$$\begin{aligned} \Phi(K)(t) \equiv\; & S^*(T-t)MS(T-t) \\ & - \int_t^T S^*(s-t)K(s)(BR^{-1}B^*)(s)K(s)S(s-t)ds \\ & + \int_t^T S^*(s-t)Q(s)S(s-t)ds, \; t \in I. \end{aligned} \tag{49}$$

Clearly, it follows from the above analysis that Φ maps $\mathcal{Z}$ into itself. We show that Φ has a fixed point in $C(I, \mathcal{Z})$. First we note that Φ is locally Lipschitz. Indeed, for $K, L \in \mathcal{B}_r \subset C(I, \mathcal{Z})$, where $\mathcal{B}_r$ denotes the closed ball of radius r in the Banach space $C(I, \mathcal{Z})$, we have

$$\parallel \Phi(K)(t) - \Phi(L)(t) \parallel_{\mathcal{Z}} \; \leq \; 2M_0^2 rb \int_t^T \parallel K(s) - L(s) \parallel_{\mathcal{Z}} ds, \; t \in I, \tag{50}$$

where b is the upper bound of the norm of the operator valued function Γ as stated above the statement of Theorem 5.1. Now we choose an integer n_0 sufficiently large so that $T/n_0 \equiv \lambda \leq 1/(4M_0^2 rb)$ and partition the interval $[0, T)$ into n_0 disjoint intervals $\{[T - m\lambda, T - (m-1)\lambda), m = 1, 2, \ldots, n_0\}$ each of length λ. First we consider the last interval $[T - \lambda, T)$ and find that

$$\begin{aligned} & \sup_{t \in [T-\lambda, T]} \{\parallel \Phi(K)(t) - \Phi(L)(t) \parallel_{\mathcal{Z}}\} \\ & \leq (2M_0^2 rb) \sup_{t \in [T-\lambda, T]} \int_t^T \parallel K(s) - L(s) \parallel_{\mathcal{Z}} ds \\ & \leq (1/2) \sup_{t \in [T-\lambda, T]} \{\parallel K(s) - L(s) \parallel_{\mathcal{Z}}\}. \end{aligned} \tag{51}$$

Thus Φ is a contraction on the Banach space $C([T - \lambda, T], \mathcal{Z})$ and hence it has a unique fixed point, say, $K_1 \in C([T - \lambda, T], \mathcal{Z})$ satisfying $K_1(T) = M$. Next, starting with $K_1(T-\lambda)$ and carrying out similar steps for the interval $[T-2\lambda, T-\lambda)$, we obtain a similar inequality

$$\begin{aligned}
&\sup_{t\in[T-2\lambda,T-\lambda]} \{\| \Phi(K)(t) - \Phi(L)(t) \|_{\mathcal{Z}}\} \\
&\leq (2M_0^2 rb) \sup_{t\in[T-2\lambda,T-\lambda]} \int_t^{T-\lambda} \| K(s) - L(s) \|_{\mathcal{Z}}\, ds \\
&\leq (1/2) \sup_{s\in[T-2\lambda,T-\lambda]} \{\| K(s) - L(s) \|_{\mathcal{Z}}\}.
\end{aligned} \tag{52}$$

This shows that Φ is also a contraction $C([T-2\lambda, T-\lambda], \mathcal{Z})$ and hence it has a unique fixed point $K_2 \in C([T-2\lambda, T-\lambda], \mathcal{Z})$ satisfying $K_2(T-\lambda) = K_1(T-\lambda)$. Continuing this process step by step backward in time till the interval $[0, \lambda)$ is covered, we obtain the sequence $\{K_i,\ i = 1, 2, \ldots, n_0\}$. By concatenating these pieces in the reverse order we obtain $K^o \in C(I, Z)$ and conclude that Φ has a unique fixed point K^o in the Banach space $C(I, \mathcal{Z})$ and hence the operator Riccati differential equation (43) has a unique mild solution $K^o \in C(I, \mathcal{Z})$. This completes the proof. ∎

The proof given here is much simpler compared to the proof based on the so called fractional step method [4, Ahmed & Teo, Theorem 5.1.10, p. 310]. For a different technique See also [5, Ahmed, Proposition 5.2, Corollary 5.1, Theorem 5.1].

In order to develop a complete state feedback control law one must prove that the Backward SDE (44) has an appropriate $\mathcal{F}_t$-adapted solution taking values in $E^*_{-\theta_1}$. For the existence of mild solution, we must show that the following integral equation,

$$\begin{aligned}
r(t) = &\int_t^T S^*(s-t)K(s)(BR^{-1}B^*)(s)r(s)ds \\
&+ \int_t^T S^*(s-t)K(s)\sigma(s)dW_H(s),\ t \in I,
\end{aligned} \tag{53}$$

considered on the Banach space $E^*_{-\theta_1}$, has a unique solution $r \in L_2^a(\Omega, C(I, E^*_{-\theta_1}))$. Hu and Peng [12, Theorem 3.1, p. 455] used classical Picard approximation technique to solve such problems in Hilbert space setting. In fact the same approach also applies to Banach space setting as stated in Theorem 4.2.

An alternative set of assumptions. We can prove results similar to that of Theorem 5.1 under a different set of assumptions thereby extending the scope of applications. These are stated in Lemma 5.2 and Theorem 5.3 below.

Lemma 5.2. *Consider the system* (39) *on the Banach space* E_η. *Suppose* $B \in B_0(I, \mathcal{L}(U, E_\eta))$, $\sigma \in B_0(I, \gamma(H, E_{-\theta_2}))$, $\eta, \theta_2 \geq 0$, *and* $0 \leq \eta + \theta_2 < (1/2)$. *Then,* *for every $\mathcal{F}_0$ measurable random variable $x_0 \in L_2(\Omega, E_\eta)$, equation* (39) *has a unique $\mathcal{F}_t$-adapted mild solution $x \in L_2^a(\Omega, C(I, E_\eta))$.*

Using the above result and following similar steps as in Theorem 5.1 we can prove the following alternative result.

Theorem 5.3. *Consider the operator Riccati evolution equation* (43) *and suppose A is the generator of an analytic semigroup on the UMD Banach space E, and let U be a real separable Hilbert space. Suppose $B \in C(I, \mathcal{L}(U, E_\eta))$, $Q \in C(I, \mathcal{L}(E_\eta, E_\eta^*))$ and $M \in \mathcal{L}(E_\eta, E_\eta^*)$ and both Q and M are positive and symmetric. Suppose the operator valued function $R \in C(I, \mathcal{L}(U))$ is symmetric and there exists a constant $\rho > 0$ such that $(R(t)u, u) \geq \rho \parallel u \parallel_U^2$ for all $u \in U$ and $t \in I$. Then equation* (43) *has a unique solution $K \in C(I, \mathcal{L}(E_\eta, E_\eta^*))$.*

From Optimality to Stability. In this section we show that the optimality of the system (41)–(42) implies stability of the following feedback system

$$dx = Axdt - BR^{-1}B^*(Kx + r)dt + \sigma dW_H \tag{54}$$

with K and r satisfying equations (43) and (44) respectively. Here we introduce the product space $\mathcal{K} \equiv E_{-\theta_1} \times E_{-\theta_1}^*$. Clearly it is a Banach space with respect to the standard norm topology given by $\parallel \vartheta \parallel_{\mathcal{K}} = \parallel \vartheta_1 \parallel_{E_{-\theta_1}} + \parallel \vartheta_2 \parallel_{E_{-\theta_1}^*}$, where the vector $\vartheta \equiv (\vartheta_1, \vartheta_2)' \equiv (x, r)' \in \mathcal{K}$.

Corollary 5.4. *The feedback system* (54) *coupled with* (44), *is stable in the mean square sense in the product space $\mathcal{K} \equiv E_{-\theta_1} \times E_{-\theta_1}^*$. Further, if the operator valued functions $K\Gamma K + Q$ and Γ taking values in $\mathcal{L}(E_{-\theta_1}, E_{-\theta_1}^*)$ and $\mathcal{L}(E_{-\theta_1}^*, E_{-\theta_1})$ respectively, are strictly positive, the system is asymptotically stable with respect to $0 \in \mathcal{K}$.*

Proof. Define the (Lyapunov) function V as follows:

$$V(t) \equiv (1/2)(K(t)x(t), x(t))_{E_{-\theta_1}^*, E_{-\theta_1}} + (r(t), x(t))_{E_{-\theta_1}^*, E_{-\theta_1}}, \ t \geq 0. \tag{55}$$

By evaluating the stochastic differential of V in a similar way as Itô differential, and carrying out some tedious but straightforward algebraic computation, one can verify that

$$\begin{aligned} dV(t) &= -([KBR^{-1}B^*K + Q]x, x)dt - (BR^{-1}B^*r, r)dt \\ &\quad - (KBR^{-1}B^*r, x)dt - Tr(\sigma^* K \sigma)dt + < r, \sigma dW_H > . \end{aligned} \tag{56}$$

For notational simplicity, let $\Gamma(t) \equiv B(t)R^{-1}(t)B^*(t)$ and note that it is positive and symmetric and takes values from $\mathcal{L}(E_{-\theta_1}^*, E_{-\theta_1})$. Using this notation we can rewrite equation (56) as follows.

$$dV(t) = -([K\Gamma K + Q]x, x)dt - (\Gamma r, r)dt - (K\Gamma r, x)dt - Tr(\sigma^* K\sigma)dt + < r, \sigma dW_H > . \tag{57}$$

Note that the order of duality pairings of the first three components on the righthand side of the above expression are as follows: $\{E^*_{-\theta_1}, E_{-\theta_1}\}$, $\{E_{-\theta_1}, E^*_{-\theta_1}\}$, and $\{E^*_{-\theta_1}, E_{-\theta_1}\}$ respectively. Let $\mathcal{L}_2(H)$ denote the space of Hilbert-Schmidt operators on the Hilbert space H with $\mathcal{L}_2^+(H) \subset \mathcal{L}_2(H)$ denoting the class of positive self adjoint operators. In the fourth term, containing the trace, the operator valued function $(\sigma^* K\sigma)$ is $\mathcal{L}_2^+(H)$-valued. The last term has the pairing between $\{E^*_{-\theta_1}, E_{-\theta_1}\}$ where the later space contains $E_{-\theta_2}$. We introduced the Banach space $\mathcal{K} \equiv E_{-\theta_1} \times E^*_{-\theta_1}$ as the state space, for the coupled system (54) and (44), equipped with the standard norm topology. Let $\vartheta = (\vartheta_1, \vartheta_2)' \equiv (x, r)' \in \mathcal{K}$ denote the state of the system consisting of the pair of evolution equations (54) and (44). Define the matrix of operator valued functions $\{\Lambda(t), t \geq 0\}$ as follows:

$$\Lambda \equiv \begin{pmatrix} K\Gamma K + Q & (1/2)K\Gamma \\ (1/2)\Gamma K & \Gamma \end{pmatrix}$$

and note that for each $t \geq 0$, $\Lambda(t) \in \mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ is a positive operator. Using this notation, the Lyapunov derivative of V can be rewritten in the following form,

$$dV(t) = - < \Lambda\vartheta, \vartheta >_{\mathcal{K}^*,\mathcal{K}} dt - Tr(\sigma^* K\sigma)dt + (\vartheta_2, \sigma dW_H). \tag{58}$$

Since the Riccati operator K is positive and symmetric for all $t \geq 0$, $Tr(\sigma^* K\sigma)$ is nonnegative for all $t \geq 0$. Integrating the above equation we have

$$\mathbf{E}V(t) = EV(0) - \mathbf{E}\int_0^t < \Lambda(s)\vartheta(s), \vartheta(s) >_{\mathcal{K}^*,\mathcal{K}} ds - \mathbf{E}\int_0^t Tr(\sigma^*(s)K(s)\sigma(s))ds, \ \forall\ t \geq 0. \tag{59}$$

It is clear from this expression that the feedback system, consisting of the evolution equations (54) and (44), is mean square stable. Further, if both the operator valued functions $(K\Gamma K + Q)$ and Γ, taking values in $\mathcal{L}(E_{-\theta_1}, E^*_{-\theta_1})$ and $\mathcal{L}(E^*_{-\theta_1}, E_{-\theta_1})$ respectively, are strictly positive then the operator valued function Λ is strictly positive. In this case the system is asymptotically stable in the mean square sense with respect to the origin in the state space $\mathcal{K}$. This completes the proof. ■

Under the alternative set of assumptions, as stated in Lemma 5.2 and Theorem 5.3, we can prove a similar stability result. In this case the state space is given by the product space $\mathcal{M} \equiv E_\eta \times E_\eta^*$. Equipped with the standard norm topology, it is a Banach space. Since the Banach space E_η is reflexive, the topological dual of $\mathcal{M}$ is given by $\mathcal{M}^* = E_\eta^* \times E_\eta$.

Corollary 5.5. *Under the alternative set of assumptions, as stated in Lemma* 5.2 *and Theorem* 5.3, *with* $\sigma \in B_0(I, \gamma(H, E_\eta))$, *the feedback system* (54) *coupled with* (44), *is stable in the mean square sense in the state space* $\mathcal{M}$. *Further, if the operator valued functions* $K\Gamma K + Q$ *and* Γ *taking values in* $\mathcal{L}(E_\eta, E_\eta^*)$ *and* $\mathcal{L}(E_\eta^*, E_\eta)$ *respectively, are strictly monotone positive, the system is asymptotically stable in the mean square sense with respect to the zero state* $0 \in \mathcal{M}$.

Proof. The proof is identical to that of Corollary 5.4. In this case the Lyapunov function V is given by a similar expression with appropriate changes in the Banach spaces determining the duality pairings in the following expression,

$$V(t) \equiv (1/2)(Kx, x)_{E_\eta^*, E_\eta} + (r, x)_{E_\eta^*, E_\eta}, \; t \geq 0. \tag{60}$$

Let $\vartheta \equiv (\vartheta_1, \vartheta_2)' \equiv (x, r)' \in \mathcal{M}$ denote the state of the coupled system (54) and (44). Following similar steps as in Corollary 5.4, we arrive at the Lyapunov derivative as follows:

$$dV(t) = - < \Lambda\vartheta, \vartheta >_{\mathcal{M}^*, \mathcal{M}} dt - Tr(\sigma^* K \sigma) dt + (\vartheta_2, \sigma dW_H). \tag{61}$$

In this case $\vartheta_2 = r$ satisfies the same integral equation (53), this time in the Banach space E_η^*, and the solution $\vartheta_2 \in L_2^a(I, E_\eta^*))$. Under the additional assumption on σ, we have

$$\mathbf{E} \int_0^t \parallel \sigma^*(s)\vartheta_2(s) \parallel_H^2 ds < \infty$$

for all $t < \infty$. Thus Integrating the above equation we obtain

$$\mathbf{E}V(t) = EV(0) - \mathbf{E} \int_0^t < \Lambda\vartheta, \vartheta >_{\mathcal{M}^*, \mathcal{M}} ds - \mathbf{E} \int_0^t Tr(\sigma^* K \sigma) ds, \; \forall \, t \geq 0. \tag{62}$$

From this we conclude that the system is asymptotically stable in the mean square sense with respect to the zero state in $\mathcal{M}$. This completes the proof. •

Remark 5.6. Under the assumptions of Corollary 5.4 (as well as Corollary 5.5), it is easy to verify that the process $\{V(t), t \geq 0\}$ (Lyapunov function) is a supermartingale with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$. That is $\mathbf{E}\{V(t)|_{\mathcal{F}_s}\} \leq V(s)$ for all $t \geq s \geq 0$. Then it follows from well known Doob's martingale convergence theorem that the system is asymptotically stable almost surely.

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