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A MODIFIED LIKELIHOOD RATIO TEST FOR A MEAN VECTOR WITH MONOTONE MISSING DATA

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Abstract

In this study, we consider the likelihood ratio test (LRT) for a normal mean vector when the data have a monotone pattern of missing observations. We derive modified likelihood ratio test (MLRT) statistic by using decomposition of the likelihood ratio (LR). Further, we investigate the accuracy of the upper percentiles of this test statistic by Monte Carlo simulation.

Keywords: asymptotic expansion, maximum likelihood estimator, Monte Carlo simulation.

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1. Introduction

In statistical data analyses, testing hypotheses with missing data is an important problem. In this study, we consider the one-sample test for a normal mean vector with monotone missing data. For the one-sample problem with k-step monotone missing data, the closed-form expressions for the MLEs of the mean vector and covariance matrix were given by Jinadasa and Tracy [9]. Kanda and Fujikoshi [10] discussed the properties of the MLEs in the case of k-step monotone missing data using the conditional approach. The one-sample problem of the test for the mean vector with monotone missing data has been discussed by many authors. For discussions related to Hotelling's T^2-type statistic, see Krishnamoorthy and Pannala [13]; Chang and Richards [3]; Seko *et al.* [17]; Yagi and Seo [23]; and Kawasaki and Seo [11], among others. For a discussion of the LRT statistic, see Krishnamoorthy and Pannala [12] and Seko *et al.* [17]. For the two-sample problem, see Yu *et al.* [25]; Seko *et al.* [16]; Yagi and Seo [24]. In particular, Yagi and Seo [24] gave the approximate upper percentiles of the Hotelling's T^2-type statistics. They also discussed multivariate multiple comparisons for mean vectors. For a general missing data pattern, Srivastava [20] discussed the LRT for mean vectors, and Seo and Srivastava [18] gave a test of equality of means and the simultaneous confidence intervals. In addition, for the simultaneous testing of the mean vector and the covariance matrix with monotone missing data, see Hao and Krishnamoorthy [6]; Tsukada [22]; Hosoya and Seo [7, 8], among others. On the other hand, for non-missing and multivariate normality, the asymptotic expansion for LR-criterion was discussed by Muirhead [15]; Siotani *et al.* [19]; and Anderson [1], among others.

In this paper, for the one-sample test of the mean vector, we give the LRT statistic for k-step monotone missing data and derive MLRT statistic by using the decomposition (see Bhargava [2] and Krishnamoorthy and Pannala [12]). In the process deriving the MLRT statistic, we give asymptotic expansions of the LR of the test for a mean vector and those of sub-mean vector. For the test for a subvector under multivariate normality, see e.g., Siotani *et al.* [19]. Recently, under nonnormality, Gupta *et al.* [5] discussed the asymptotic expansion of the distribution of Rao's U-statistic, which is proposed as test for a subvector or additional information. This paper is organized in the following way. In Section 2, we present the assumptions and notation. In Section 3, we derive the LRT statistic, MLRT and modified test (MT) statistics, which converge to the χ^2 distribution faster than the LRT statistic as the sample size tends to infinity. In Section 4, some simulation results for three- and five-step monotone missing data cases are presented to investigate the accuracy of the upper percentiles of the null distributions of MT and MLRT statistics.

2. Assumptions and notation

We consider the one-sample problem of testing for a mean vector with a k-step monotone missing data pattern. Let $\boldsymbol{x}_i$ be a $p_i \times 1$ normal random vector with the mean vector $\boldsymbol{\mu}_i$ and covariance matrix $\boldsymbol{\Sigma}_i$, where $\boldsymbol{\mu}_i = (\boldsymbol{\mu})_i = (\mu_1, \mu_2, \ldots, \mu_{p_i})'$, and $\boldsymbol{\Sigma}_i$ is the $p_i \times p_i$ principal submatrix of $\boldsymbol{\Sigma}(= \boldsymbol{\Sigma}_1)$ with $p = p_1 > p_2 > \cdots > p_k > 0$. Further, let $\boldsymbol{x}_i$, $i = 1, 2, \ldots, k$ be mutually independent. Suppose that $\boldsymbol{x}_{i1}, \boldsymbol{x}_{i2}, \ldots, \boldsymbol{x}_{in_i}$ are independent and identically distributed samples from $\boldsymbol{x}_i$, $i = 1, 2, \ldots, k$, where $n_1 > p$. Note that k denotes the number of steps. The above data set is called k-step monotone missing data:

$$\left(\begin{array}{cccc} & \boldsymbol{x}'_{11} & & \\ & \vdots & & \\ & \boldsymbol{x}'_{1n_1} & & \\ \boldsymbol{x}'_{21} & & & * \cdots * \\ \vdots & & & \vdots \quad \vdots \\ \boldsymbol{x}'_{2n_2} & & & * \cdots * \\ & \ddots & & \\ \boldsymbol{x}'_{k1} & * \cdots * & & * \cdots * \\ \vdots & \vdots \quad \vdots & & \vdots \quad \vdots \\ \boldsymbol{x}'_{kn_k} & * \cdots * & & * \cdots * \end{array}\right),$$

where an asterisk indicates a missing observation. For a k-step monotone sample or k-step monotone missing data pattern, see Bhargava [2], Little and Rubin [14], and Srivastava [21], among others. In this paper, we assume that the data are missing completely at random (MCAR), and we adopt the notation from Jinadasa and Tracy [9]. As for the partitions of $\boldsymbol{\Sigma}$, for $1 \le i < j \le k$, let $(\boldsymbol{\Sigma}_i)_j$ be the principal submatrix of $\boldsymbol{\Sigma}_i$ of order $p_j \times p_j$; we define

$$\boldsymbol{\Sigma}_i = (\boldsymbol{\Sigma}_1)_i, \quad \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_i & \boldsymbol{\Sigma}_{i2} \\ \boldsymbol{\Sigma}'_{i2} & \boldsymbol{\Sigma}_{i3} \end{pmatrix}, \quad \boldsymbol{\Sigma}_{i-1} = \begin{pmatrix} \boldsymbol{\Sigma}_i & \boldsymbol{\Sigma}_{(i-1,2)} \\ \boldsymbol{\Sigma}'_{(i-1,2)} & \boldsymbol{\Sigma}_{(i-1,3)} \end{pmatrix},$$

and

$$\boldsymbol{\Sigma}_{(i-1,3)\cdot i} = \boldsymbol{\Sigma}_{(i-1,3)} - \boldsymbol{\Sigma}'_{(i-1,2)} \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\Sigma}_{(i-1,2)}, \quad i = 2, 3, \ldots, k.$$

For example, when $k = 3$, we can express $\boldsymbol{\Sigma}_1$ as

$$\boldsymbol{\Sigma}_1 = \left(\begin{array}{cc|c} \overbrace{\boldsymbol{\Sigma}_3}^{p_3} & \overbrace{\boldsymbol{\Sigma}_{(2,2)}}^{p_2-p_3} & \overbrace{\phantom{\boldsymbol{\Sigma}_{(1,2)}}}^{p_1-p_2} \\ \boldsymbol{\Sigma}'_{(2,2)} & \boldsymbol{\Sigma}_{(2,3)} & \boldsymbol{\Sigma}_{(1,2)} \\ \hline \multicolumn{2}{c|}{\boldsymbol{\Sigma}'_{(1,2)}} & \boldsymbol{\Sigma}_{(1,3)} \end{array}\right) \begin{array}{l} \}p_3 \\ \}p_2 - p_3 \\ \}p_1 - p_2 \end{array}.$$

3. LRT and MLRT statistics

Consider the null hypothesis

$$H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$$

against the alternative $H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$, where $\boldsymbol{\mu}_0$ is known. Without loss of generality, we can assume that $\boldsymbol{\mu}_0 = \boldsymbol{0}$. Then, the LR is given by

$$\lambda = \prod_{i=1}^{k} \left(\frac{|\widehat{\boldsymbol{\Sigma}}_i|}{|\widetilde{\boldsymbol{\Sigma}}_i|} \right)^{\frac{1}{2}n_i}, \tag{1}$$

where $\widehat{\boldsymbol{\Sigma}}_i$ is the MLE of $\boldsymbol{\Sigma}_i$ under H_1, and $\widetilde{\boldsymbol{\Sigma}}_i$ is the MLE of $\boldsymbol{\Sigma}_i$ under H_0. Let

$$\boldsymbol{E}_i = \sum_{j=1}^{n_i} (\boldsymbol{x}_{ij} - \overline{\boldsymbol{x}}_i)(\boldsymbol{x}_{ij} - \overline{\boldsymbol{x}}_i)', \quad \overline{\boldsymbol{x}}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} \boldsymbol{x}_{ij}, \quad i = 1, 2, \ldots, k,$$

$$\boldsymbol{d}_1 = \overline{\boldsymbol{x}}_1, \quad \boldsymbol{d}_i = \frac{n_i}{N_{i+1}} \Big[\overline{\boldsymbol{x}}_i - \frac{1}{N_i} \sum_{j=1}^{i-1} n_j (\overline{\boldsymbol{x}}_j)_i \Big], \quad i = 2, 3, \ldots, k,$$

$$N_1 = 0, \quad N_{i+1} = N_i + n_i \ \Big(= \sum_{j=1}^{i} n_j \Big), \quad i = 1, 2, \ldots, k.$$

Then, we can express $\widehat{\boldsymbol{\Sigma}}$ concretely (see Jinadasa and Tracy [9]) as

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{n_1} \boldsymbol{H}_1 + \sum_{i=2}^{k} \frac{1}{N_{i+1}} \boldsymbol{F}_i \left[\boldsymbol{H}_i - \frac{n_i}{N_i} \boldsymbol{L}_{i-1,1} \right] \boldsymbol{F}_i',$$

where

$$\boldsymbol{H}_1 = \boldsymbol{E}_1, \ \boldsymbol{H}_i = \boldsymbol{E}_i + \frac{N_i N_{i+1}}{n_i} \boldsymbol{d}_i \boldsymbol{d}_i', \ i = 2, 3, \ldots, k,$$

$$\boldsymbol{L}_1 = \boldsymbol{H}_1, \ \boldsymbol{L}_i = (\boldsymbol{L}_{i-1})_i + \boldsymbol{H}_i, \ i = 2, 3, \ldots, k,$$

$$\boldsymbol{L}_{i1} = (\boldsymbol{L}_i)_{i+1}, \ \boldsymbol{L}_i = \begin{pmatrix} \boldsymbol{L}_{i1} & \boldsymbol{L}_{i2} \\ \boldsymbol{L}_{i2}' & \boldsymbol{L}_{i3} \end{pmatrix}, \ i = 1, 2, \ldots, k-1,$$

$$\boldsymbol{G}_1 = \boldsymbol{I}_{p_1}, \ \boldsymbol{G}_{i+1} = \begin{pmatrix} \boldsymbol{I}_{p_{i+1}} \\ \boldsymbol{L}_{i2}' \boldsymbol{L}_{i1}^{-1} \end{pmatrix}, \ i = 1, 2, \ldots, k-1,$$

$$\boldsymbol{F}_1 = \boldsymbol{G}_1, \ \boldsymbol{F}_i = \boldsymbol{F}_{i-1} \boldsymbol{G}_i, \ i = 2, 3, \ldots, k.$$

When $k = 3$, we can see that the above result of $\widehat{\boldsymbol{\Sigma}}$ coincides with the result in the three-step case (see Yagi and Seo [23]). By the same derivation in Jinadasa and Tracy [9], it holds that the MLE of $\boldsymbol{\Sigma}$ under H_0, $\widetilde{\boldsymbol{\Sigma}}$ is equal to $\widehat{\boldsymbol{\Sigma}}$ with $\overline{\boldsymbol{x}}_i = \boldsymbol{0}$, $i = 1, 2, \ldots, k$. That is, $\widetilde{\boldsymbol{\Sigma}}$ can be obtained as $\widehat{\boldsymbol{\Sigma}}$ in the case that

$$\boldsymbol{H}_i = \sum_{j=1}^{n_i} \boldsymbol{x}_{ij}\boldsymbol{x}'_{ij}, \quad i = 1, 2, \ldots, k.$$

We note that the null distribution of the LRT statistic $Q(= -2\log\lambda)$ is asymptotically a χ^2 distribution with p degrees of freedom. However, it may be noted that the upper percentiles of the χ^2 distribution are not a good approximation to those of the LRT statistic when the sample size is not large. For example, Table 1 gives the simulated values of the upper 100α percentiles of Q, $q(\alpha)$ and

Table 1. The upper percentiles of Q and the actual Type I error rates.

	$(p_1, p_2, p_3) = (8, 4, 2)$		$(p_1, p_2, p_3, p_4, p_5) = (15, 12, 9, 6, 3)$	
n_1	$q(\alpha)$	α_Q	$q(\alpha)$	α_Q
		$\alpha = 0.05$		
20	20.74	16.71	46.86	47.84
30	18.53	11.24	33.86	23.03
40	17.70	9.25	30.91	16.13
50	17.18	8.17	29.52	13.05
100	16.29	6.42	27.15	8.37
200	15.88	5.65	26.03	6.53
400	15.68	5.31	25.53	5.74
∞	15.51	5.00	25.00	5.00

Note. $n_2 = n_3 = n_4 = n_5 = 10$, $\chi^2_8(0.05) = 15.51$, $\chi^2_{15}(0.05) = 25.00$.

the actual Type I error rates, $\alpha_Q = 100\,\Pr\{Q > \chi^2_p(\alpha)\}$ for the three- and five-step monotone missing data cases, where $\chi^2_p(\alpha)$ is the upper 100α percentile of the χ^2 distribution with p degrees of freedom. It may be seen that the upper percentiles of the χ^2 distribution are useful as an approximation to the upper percentiles of Q for cases in which the sample size is considerably large. Therefore, we consider the MLRT statistic whose null distribution is closer to the χ^2 distribution than that of the LRT statistic even when the sample size is small. In particular, we derive an asymptotic expansion for the distribution function of the LRT statistic in a situation when $n_1 \to \infty$ with $q_i = n_i/n_1 \to \delta_i \in [0, \infty)$, $i = 2, 3, \ldots, k$.

Using the notation in Section 2, we can write λ in (1) as

$$\lambda = \prod_{i=1}^{k} \lambda_i,$$

where

$$\lambda_1 = \left(\frac{|\widehat{\boldsymbol{\Sigma}}_k|}{|\widetilde{\boldsymbol{\Sigma}}_k|} \right)^{\frac{N_{k+1}}{2}}, \quad \lambda_i = \left(\frac{|\widehat{\boldsymbol{\Sigma}}_{(k-i+1,3)\cdot k-i+2}|}{|\widetilde{\boldsymbol{\Sigma}}_{(k-i+1,3)\cdot k-i+2}|} \right)^{\frac{N_{k-i+2}}{2}}, \quad i = 2, 3, \ldots, k, \tag{2}$$

$$N_{k+1} = \sum_{j=1}^{k} n_j, \quad N_{k-i+2} = \sum_{j=1}^{k-i+1} n_j,$$

$\widehat{\boldsymbol{\Sigma}}_{(k-i+1,3)\cdot k-i+2}$ and $\widetilde{\boldsymbol{\Sigma}}_{(k-i+1,3)\cdot k-i+2}$ are given by

$$\widehat{\boldsymbol{\Sigma}}_{(k-i+1,3)\cdot k-i+2} = \widehat{\boldsymbol{\Sigma}}_{(k-i+1,3)} - \widehat{\boldsymbol{\Sigma}}'_{(k-i+1,2)} \widehat{\boldsymbol{\Sigma}}^{-1}_{k-i+2} \widehat{\boldsymbol{\Sigma}}_{(k-i+1,2)}$$

and

$$\widetilde{\boldsymbol{\Sigma}}_{(k-i+1,3)\cdot k-i+2} = \widetilde{\boldsymbol{\Sigma}}_{(k-i+1,3)} - \widetilde{\boldsymbol{\Sigma}}'_{(k-i+1,2)} \widetilde{\boldsymbol{\Sigma}}^{-1}_{k-i+2} \widetilde{\boldsymbol{\Sigma}}_{(k-i+1,2)},$$

respectively. We note that the values of λ_i, $i = 1, 2, \ldots, k$ are mutually independent.

Further, we consider the following hypotheses:

$$H_{01} : \boldsymbol{\mu}_k = \mathbf{0} \text{ vs. } H_{11} : \boldsymbol{\mu}_k \neq \mathbf{0},$$

$$H_{0i} : \boldsymbol{A}_{k-i+1}\boldsymbol{\mu}_{k-i+1} = \mathbf{0} \text{ given } \boldsymbol{\mu}_{k-i+2} = \mathbf{0}$$
$$\text{vs. } H_{1i} : \boldsymbol{A}_{k-i+1}\boldsymbol{\mu}_{k-i+1} \neq \mathbf{0} \text{ given } \boldsymbol{\mu}_{k-i+2} = \mathbf{0}, \quad i = 2, 3, \ldots, k,$$

where $\boldsymbol{A}_{k-i+1} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{I}_{p_{k-i+1}-p_{k-i+2}} \end{pmatrix}$ is a $(p_{k-i+1} - p_{k-i+2}) \times p_{k-i+1}$ matrix.

Let the parameter spaces of Ω_0, Ω_i, $i = 1, 2, \ldots, k-1$ and Ω_k be

$$\Omega_0 = \{(\boldsymbol{\mu}, \boldsymbol{\Sigma}) : -\infty < \mu_j < \infty, \ j = 1, 2, \ldots, p, \ \boldsymbol{\Sigma} > \boldsymbol{O}\},$$

$$\Omega_i = \{(\boldsymbol{\mu}, \boldsymbol{\Sigma}) : \boldsymbol{\mu}_{k-i+1} = \mathbf{0}, \ -\infty < \mu_j < \infty, \ j = p_{k-i+1}+1, p_{k-i+1}+2, \ldots, p,$$
$$\boldsymbol{\Sigma} > \boldsymbol{O}\}, \ i = 1, 2, \ldots, k-1,$$

$$\Omega_k = \{(\boldsymbol{\mu}, \boldsymbol{\Sigma}) : \boldsymbol{\mu} = \mathbf{0}, \ \boldsymbol{\Sigma} > \boldsymbol{O}\},$$

respectively. Then, the LR for the hypothesis H_{0i} is given by

$$\lambda_{0i} = \frac{\max\limits_{(\boldsymbol{\mu},\boldsymbol{\Sigma})\in\Omega_i} L(\boldsymbol{\mu},\boldsymbol{\Sigma})}{\max\limits_{(\boldsymbol{\mu},\boldsymbol{\Sigma})\in\Omega_{i-1}} L(\boldsymbol{\mu},\boldsymbol{\Sigma})}, \quad i = 1, 2, \ldots, k. \tag{3}$$

Therefore, since λ_i in (2) is equal to λ_{0i} in (3), we have

$$\lambda = \prod_{i=1}^{k} \lambda_{0i}.$$

That is, we note $-2\log\lambda_1$ is the usual LRT statistic of the test for p_k dimensional mean vector, and $-2\log\lambda_i$, $i = 2, 3, \ldots, k$ is LRT statistic of the test for a subvector. The above result is obtained by Krishnamoorthy and Pannala [12]. For the test for a subvector under the complete data set and the multivariate normality, see e.g., Siotani *et al.* [19]. Therefore, we have the following theorem.

Theorem 1. *Suppose that $\boldsymbol{x}_{ij}$ is distributed as $N_{p_i}(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n_i$, where $p = p_1 > p_2 > \cdots > p_k > p_{k+1} = 0$ and $n_1 > p$. Then, when the null hypothesis H_{0i} is true, the cumulative distribution function of Q_i^* $(= -2\rho_i \log\lambda_i)$ can be expressed for large N_{k-i+2} as*

$$\Pr(Q_i^* \le x) = G_{p_{k-i+1}-p_{k-i+2}}(x) + O(N_{k-i+2}^{-2}), \quad i = 1, 2, \ldots, k,$$

where λ_i is given in (2) and

$$\rho_i = 1 - \frac{1}{2N_{k-i+2}}(p_{k-i+1} + p_{k-i+2} + 2), \quad i = 1, 2, \ldots, k,$$

$G_p(x)$ is the distribution function of a χ^2-variate with p degrees of freedom.

Proof. First we derive an asymptotic expansion of the characteristic function of Q_1 $(= -2\log\lambda_1)$. We use the following notation to simplify setting. Let $\boldsymbol{y}_1$, $\boldsymbol{y}_2, \ldots, \boldsymbol{y}_{N_{k+1}}$ be distributed as p_k dimensional multivariate normal distribution. Then λ_1 can be written as

$$\lambda_1 = \left(\frac{|\boldsymbol{U}_k|}{|\boldsymbol{U}_k + N_{k+1}\overline{\boldsymbol{y}}_k\overline{\boldsymbol{y}}_k'|}\right)^{\frac{N_{k+1}}{2}},$$

where

$$\overline{\boldsymbol{y}}_k = \frac{1}{N_{k+1}}\sum_{j=1}^{N_{k+1}} \boldsymbol{y}_j, \quad \boldsymbol{U}_k = \sum_{j=1}^{N_{k+1}} (\boldsymbol{y}_j - \overline{\boldsymbol{y}}_k)(\boldsymbol{y}_j - \overline{\boldsymbol{y}}_k)'.$$

Therefore, expanding $Q_1(=-2\log\lambda_1)$ by the perturbation method and calculating the characteristic function, we obtain

$$\mathrm{E}[\exp(itQ_1)] = (1-2it)^{-\frac{p_k}{2}}\left[1+\frac{\beta_1}{N_{k+1}}\left\{1-(1-2it)^{-1}\right\}\right]+O\left(N_{k+1}^{-2}\right),$$

where

$$\beta_1 = -\frac{1}{4}p_k(p_k+2).$$

Inverting the characteristic function, we have

$$\Pr(Q_1\le x) = G_{p_k}(x)+\frac{\beta_1}{N_{k+1}}\left[G_{p_k}(x)-G_{p_k+2}(x)\right]+O\left(N_{k+1}^{-2}\right).$$

Therefore, if $\rho_1 = 1-(p_k+2)/(2N_{k+1})$, then the cumulative distribution function of Q_1^* $(=-2\rho_1\log\lambda_1)$ is given by

$$\Pr(Q_1^*\le x) = G_{p_k}(x)+O\left(N_{k+1}^{-2}\right).$$

Similar to the case of Q_1, we consider the cumulative distribution function of Q_i $(=-2\log\lambda_i)$, $i=2,3,\ldots,k$. Let $\boldsymbol{y}_1,\boldsymbol{y}_2,\ldots,\boldsymbol{y}_{N_{k-i+2}}$ be distributed as $N_{p_{k-i+1}}(\boldsymbol{\eta},\boldsymbol{\Delta})$, where $\boldsymbol{\eta}$ is a $p_{k-i+1}\times 1$ mean vector and $\boldsymbol{\Delta}$ is a $p_{k-i+1}\times p_{k-i+1}$ covariance matrix. In order to be the notation shorter, we omit the index i of $\boldsymbol{\eta}$ and $\boldsymbol{\Delta}$. Further let $\boldsymbol{\eta}$ and $\boldsymbol{\Delta}$ be partitioned as

$$\boldsymbol{\eta} = \begin{pmatrix}\boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{pmatrix}\begin{matrix}\}r_i\\ \}s_i\end{matrix},\quad \boldsymbol{\Delta} = \left(\begin{array}{c|c}\overbrace{\boldsymbol{\Delta}_{11}}^{r_i} & \overbrace{\boldsymbol{\Delta}_{12}}^{s_i}\\ \hline \boldsymbol{\Delta}_{21} & \boldsymbol{\Delta}_{22}\end{array}\right)\begin{matrix}\}r_i\\ \}s_i\end{matrix},$$

where $r_i = p_{k-i+2}$, $s_i = p_{k-i+1}-p_{k-i+2}$. Then, since λ_i is the LR for testing

$$H_i:\boldsymbol{\eta}_2=\boldsymbol{0}\text{ given }\boldsymbol{\eta}_1=\boldsymbol{0}\text{ vs. }K_i:\boldsymbol{\eta}_2\ne\boldsymbol{0}\text{ given }\boldsymbol{\eta}_1=\boldsymbol{0},\quad i=2,3,\ldots,k,$$

we can write

$$\lambda_i = \left(\frac{1+N_{k-i+2}\overline{\boldsymbol{y}}_1'\boldsymbol{U}_{11}^{-1}\overline{\boldsymbol{y}}_1}{1+N_{k-i+2}\overline{\boldsymbol{y}}'\boldsymbol{U}^{-1}\overline{\boldsymbol{y}}}\right)^{\frac{N_{k-i+2}}{2}},$$

where

$$\overline{\boldsymbol{y}} = \frac{1}{N_{k-i+2}}\sum_{j=1}^{N_{k-i+2}}\boldsymbol{y}_j,\quad \boldsymbol{U} = \sum_{j=1}^{N_{k-i+2}}(\boldsymbol{y}_j-\overline{\boldsymbol{y}})(\boldsymbol{y}_j-\overline{\boldsymbol{y}})',$$

and

$$\overline{\boldsymbol{y}} = \begin{pmatrix}\overline{\boldsymbol{y}}_1\\ \overline{\boldsymbol{y}}_2\end{pmatrix}\begin{matrix}\}r_i\\ \}s_i\end{matrix},\quad \boldsymbol{U} = \left(\begin{array}{c|c}\overbrace{\boldsymbol{U}_{11}}^{r_i} & \overbrace{\boldsymbol{U}_{12}}^{s_i}\\ \hline \boldsymbol{U}_{21} & \boldsymbol{U}_{22}\end{array}\right)\begin{matrix}\}r_i\\ \}s_i\end{matrix}.$$

Without loss of generality, we can assume that $\boldsymbol{\eta} = \mathbf{0}$, and $\boldsymbol{\Delta} = \boldsymbol{I}$. Let

$$\overline{\boldsymbol{y}} = \frac{1}{\sqrt{N_{k-i+2}}}\boldsymbol{z}, \quad \frac{1}{N_{k-i+2}-1}\boldsymbol{U} = \boldsymbol{I} + \frac{1}{\sqrt{N_{k-i+2}}}\boldsymbol{V}.$$

We use partitions of $\boldsymbol{z}$ and $\boldsymbol{V}$ as

$$\boldsymbol{z} = \begin{pmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \end{pmatrix} \begin{matrix} \}r_i \\ \}s_i \end{matrix}, \quad \boldsymbol{V} = \begin{pmatrix} \overbrace{\boldsymbol{V}_{11}}^{r_i} & \overbrace{\boldsymbol{V}_{12}}^{s_i} \\ \boldsymbol{V}_{21} & \boldsymbol{V}_{22} \end{pmatrix} \begin{matrix} \}r_i \\ \}s_i \end{matrix}.$$

Then, we can expand Q_i as

$$\begin{aligned} Q_i &= \boldsymbol{z}'\boldsymbol{z} - \boldsymbol{z}_1'\boldsymbol{z}_1 - \frac{1}{\sqrt{N_{k-i+2}}}(\boldsymbol{z}'\boldsymbol{V}\boldsymbol{z} - \boldsymbol{z}_1'\boldsymbol{V}_{11}\boldsymbol{z}_1) \\ &+ \frac{1}{N_{k-i+2}}\Big\{\boldsymbol{z}'\boldsymbol{V}^2\boldsymbol{z} - \boldsymbol{z}_1'\boldsymbol{V}_{11}^2\boldsymbol{z}_1 - \frac{1}{2}(\boldsymbol{z}'\boldsymbol{z} - \boldsymbol{z}_1'\boldsymbol{z}_1)(\boldsymbol{z}'\boldsymbol{z} + \boldsymbol{z}_1'\boldsymbol{z}_1 - 2)\Big\} + O_p\Big(N_{k-i+2}^{-\frac{3}{2}}\Big). \end{aligned}$$

Hence, for $j = 2, 3, \ldots, k$,

$$\begin{aligned} E[\exp\{it(Q_j)\}] &= E\left[\exp\{it(\boldsymbol{z}_2'\boldsymbol{z}_2)\}\right] + E\Big[\Big(X + \frac{1}{2}X^2\Big)\exp\{it(\boldsymbol{z}_2'\boldsymbol{z}_2)\}\Big] \\ &\quad + O\Big(N_{k-j+2}^{-\frac{3}{2}}\Big), \end{aligned}$$

where $i = \sqrt{-1}$ and

$$\begin{aligned} X = it\Big[&-\frac{1}{\sqrt{N_{k-j+2}}}(\boldsymbol{z}'\boldsymbol{V}\boldsymbol{z} - \boldsymbol{z}_1'\boldsymbol{V}_{11}\boldsymbol{z}_1) \\ &+ \frac{1}{N_{k-j+2}}\Big\{\boldsymbol{z}_1'\boldsymbol{V}_{12}\boldsymbol{V}_{21}\boldsymbol{z}_1 + 2\boldsymbol{z}_1'(\boldsymbol{V}_{11}\boldsymbol{V}_{12} + \boldsymbol{V}_{12}\boldsymbol{V}_{22})\boldsymbol{z}_2 \\ &+ \boldsymbol{z}_2'(\boldsymbol{V}_{21}\boldsymbol{V}_{12} + \boldsymbol{V}_{22}^2)\boldsymbol{z}_2 - \frac{1}{2}\boldsymbol{z}_2'\boldsymbol{z}_2(2\boldsymbol{z}_1'\boldsymbol{z}_1 + \boldsymbol{z}_2'\boldsymbol{z}_2 - 2)\Big\}\Big]. \end{aligned}$$

Therefore, after calculating the expectation, we obtain

$$\mathrm{E}[\exp(itQ_j)] = (1-2it)^{-\frac{s_j}{2}}\left[1 + \frac{\beta_j}{N_{k-j+2}}\left\{1 - (1-2it)^{-1}\right\}\right] + O\left(N_{k-j+2}^{-2}\right),$$

where

$$\beta_j = -\frac{1}{4}s_j(2r_j + s_j + 2),$$

and hence

$$\Pr(Q_j \le x) = G_{s_j}(x) + \frac{\beta_j}{N_{k-j+2}} \left[G_{s_j}(x) - G_{s_j+2}(x)\right] + O\left(N_{k-j+2}^{-2}\right).$$

Therefore, if $\rho_i = 1-(2r_i+s_i+2)/(2N_{k-i+2})$, then the cumulative distribution function of $Q_i^*(= -2\rho_i \log \lambda_i)$ is given by

$$\Pr(Q_i^* \le x) = G_{s_i}(x) + O\left(N_{k-i+2}^{-2}\right),$$

and the proof is complete. ■

Using Theorem 1, we can give the MT statistic $Q^*(= \sum_{i=1}^{k} Q_i^*)$ with an improved chi-squared approximation. However, this transformation statistic Q^* is not always monotone. For the monotone transformation, see Fujikoshi [4]. On the other hand, by gathering up the expanded results for the characteristic functions of Q_i, $i = 1, 2, \dots, k$, we obtain the following theorem.

Theorem 2. *Under H_0, the cumulative distribution function of $Q^\dagger(= -2\rho \log \lambda)$ can be expressed for large n_1 as*

$$\Pr(Q^\dagger \le x) = G_p(x) + O(n_1^{-2}),$$

where

$$\rho = 1 - \frac{1}{2n_1 p} \sum_{i=1}^{k} \frac{1}{m_{k-i+1}} (p_{k-i+1} - p_{k-i+2})(p_{k-i+1} + p_{k-i+2} + 2),$$

$m_{k-i+1} = 1 + \sum_{j=2}^{k-i+1} q_j$, $q_j(= n_j/n_1)$ *is a nonnegative constant, and* $p_{k+1} = 0$.

We note that the value of ρ coincides with that of Krishnamoorthy and Pannala [12] when $k = 2$.

4. Simulation studies

In this section, we study the numerical accuracy of the upper percentiles of the MT and MLRT statistics using the actual Type I error rates. In order to investigate the accuracy of the approximation, we compute the upper percentiles of Q, Q^* and $Q^\dagger$ with monotone missing data by Monte Carlo simulation. For each parameter, the simulation was executed 10^6 times using normal random vectors generated from $N_{p_i}(\mathbf{0}, \boldsymbol{I}_{p_i})$, $i = 1, 2, \dots, k$.

In Tables 2–5, we provide the simulated upper 100α percentiles of Q, Q^* and $Q^\dagger$ for the three-step and five-step cases. Further, we provide the actual Type I error rates, α_Q, α_{Q^*} and $\alpha_{Q^\dagger}$ given by

$$\alpha_Q = 100\,\mathrm{Pr}\{Q > \chi_p^2(\alpha)\}, \quad \alpha_{Q^*} = 100\,\mathrm{Pr}\{Q^* > \chi_p^2(\alpha)\},$$

and

$$\alpha_{Q^\dagger} = 100\,\mathrm{Pr}\{Q^\dagger > \chi_p^2(\alpha)\},$$

respectively, where $\chi_p^2(\alpha)$ is the upper 100α percentile of the χ^2 distribution with p degrees of freedom.

It may be noted from Tables 2–5 that each value of $q(\alpha)$, $q^*(\alpha)$ and $q^\dagger(\alpha)$ is closer to the upper percentiles of the χ^2 distribution with p degrees of freedom, $\chi_p^2(\alpha)$, when n_1 becomes large. It is seen from Tables 2–4 that $q^*(\alpha)$ for $(p_1, p_2, p_3) = (8, 4, 2)$ and $(15, 12, 9)$ is a considerably good approximate value when n_1 is greater than 20. Similarly, it is seen from Table 5 that $q^*(\alpha)$ for $(p_1, p_2, p_3.p_4, p_5) = (15, 12, 9, 6, 3)$ is a considerably good approximate value when n_1 is greater than 20 without regard to the sample size of n_i, $i \geq 2$. As for $q^\dagger(\alpha)$, it is seen from Tables 2 and 3 that $q^\dagger(\alpha)$ for $(p_1, p_2, p_3) = (8, 4, 2)$ is a good approximate value when n_1 is greater than 30. Similarly, it is seen from Table 4 that $q^\dagger(\alpha)$ for $(p_1, p_2, p_3) = (15, 12, 9)$ is a good approximate value when n_1 is greater than 50. For the case of five-step monotone missing data in Table 5, $q^\dagger(\alpha)$ for $(p_1, p_2, p_3, p_4, p_5) = (15, 12, 9, 6, 3)$ is a good approximation to $\chi_p^2(\alpha)$ when n_1 is greater than 50. It may be noted from the simulation results that the MT statistic Q^* converges to the χ^2 distribution faster than the MLRT statistic $Q^\dagger$ in almost all cases, including the case of unbalanced sample sizes.

5. Conclusions

We have developed the MLRT statistic $Q^\dagger$ and the MT statistic Q^* with general monotone missing data in one-sample problem, where Q^* is not always monotone. Further, we presented that the LR for the one-sample test of the mean vector with monotone missing data can be expressed as the products of the LR of the test for a mean vector and those of subvector, and derived the asymptotic expansion by the perturbation method. The null distribution of MLRT or MT statistic is considerably closer to the χ^2 distribution than that of the LRT statistic, even for small samples.

Table 2. The upper percentiles of Q, Q^*, $Q^\dagger$ and the actual Type I error rates when $(p_1, p_2, p_3) = (8, 4, 2)$ and $\alpha = 0.05$.

n_1	n_2	n_3	$q(\alpha)$	$q^*(\alpha)$	$q^\dagger(\alpha)$	α_Q	α_{Q^*}	$\alpha_{Q^\dagger}$
10	10	10	41.98	16.77	24.49	56.66	**7.20**	22.06
20	10	10	20.74	15.58	16.16	16.71	**5.12**	6.15
30	10	10	18.53	15.52	15.72	11.24	**5.02**	5.37
40	10	10	17.70	15.55	15.65	9.25	**5.07**	5.24
50	10	10	17.18	15.52	15.57	8.17	**5.02**	5.11
100	10	10	16.29	15.50	15.51	6.42	4.98	**5.00**
200	10	10	15.88	15.48	15.49	5.65	4.96	**4.97**
400	10	10	15.68	15.49	15.49	5.31	**4.97**	**4.97**
10	20	20	41.70	16.77	25.30	55.54	**7.21**	23.63
20	20	20	20.55	15.57	16.27	16.00	**5.11**	6.29
30	20	20	18.43	15.55	15.78	10.88	**5.07**	5.45
40	20	20	17.54	15.50	15.60	8.97	**4.99**	5.16
50	20	20	17.12	15.52	15.58	8.03	**5.01**	5.12
100	20	20	16.29	15.52	15.53	6.37	**5.01**	5.03
200	20	20	15.89	15.51	15.51	5.67	**5.00**	**5.00**
400	20	20	15.69	15.49	15.50	5.30	4.97	**4.98**
10	50	50	41.40	16.77	26.03	54.61	**7.19**	25.01
20	50	50	20.29	15.56	16.36	15.40	**5.09**	6.46
30	50	50	18.21	15.51	15.79	10.45	**5.00**	5.47
40	50	50	17.44	15.53	15.66	8.74	**5.04**	5.26
50	50	50	17.01	15.52	15.60	7.83	**5.02**	5.15
100	50	50	16.24	15.51	15.53	6.30	**5.01**	5.03
200	50	50	15.91	15.53	15.54	5.68	**5.04**	5.05
400	50	50	15.69	15.50	15.50	5.30	**4.99**	**4.99**
30	30	30	18.30	15.52	15.76	10.65	**5.01**	5.43
40	40	40	17.47	15.53	15.65	8.78	**5.03**	5.24
100	100	100	16.18	15.49	15.51	6.22	4.98	**5.00**
200	200	200	15.83	15.50	15.50	5.57	4.98	**4.99**
400	400	400	15.68	15.51	15.51	5.28	**5.01**	**5.01**

Note. $\chi^2_8(0.05) = 15.51$. The closest to $\alpha (= 0.05)$ in the values α_Q, α_{Q^*} and $\alpha_{Q^\dagger}$ of each low is boldface.

Table 3. The upper percentiles of Q, Q^*, $Q^\dagger$ and the actual Type I error rates when $(p_1, p_2, p_3) = (8, 4, 2)$, $\alpha = 0.05$, and $n_2 \neq n_3$.

n_1	n_2	n_3	$q(\alpha)$	$q^*(\alpha)$	$q^\dagger(\alpha)$	α_Q	α_{Q^*}	$\alpha_{Q^\dagger}$
10	10	20	42.00	16.77	24.67	56.44	**7.22**	22.36
20	10	20	20.69	15.56	16.18	16.50	**5.08**	6.15
30	10	20	18.52	15.53	15.75	11.17	**5.05**	5.39
40	10	20	17.64	15.51	15.62	9.17	**5.01**	5.17
50	10	20	17.18	15.53	15.59	8.18	**5.03**	5.14
10	10	50	41.84	16.77	24.80	56.29	**7.21**	22.73
20	10	50	20.65	15.58	16.22	16.46	**5.12**	6.22
30	10	50	18.46	15.52	15.75	11.06	**5.02**	5.41
40	10	50	17.57	15.49	15.60	9.04	**4.98**	5.15
50	10	50	17.17	15.54	15.60	8.13	**5.05**	5.15
10	20	10	41.74	16.77	25.22	55.65	**7.22**	23.39
20	20	10	20.55	15.57	16.24	16.14	**5.10**	6.25
30	20	10	18.42	15.53	15.75	10.94	**5.03**	5.41
40	20	10	17.62	15.55	15.66	9.07	**5.08**	5.25
50	20	10	17.13	15.52	15.58	8.06	**5.02**	5.13
10	20	50	41.63	16.78	25.41	55.36	**7.25**	23.83
20	20	50	20.44	15.53	16.24	15.89	**5.05**	6.27
30	20	50	18.35	15.51	15.75	10.75	**5.00**	5.41
40	20	50	17.50	15.48	15.59	8.89	**4.95**	5.14
50	20	50	17.09	15.52	15.58	8.01	**5.01**	5.13
10	50	10	41.45	16.76	25.95	54.71	**7.19**	24.80
20	50	10	20.30	15.56	16.33	15.48	**5.09**	6.43
30	50	10	18.21	15.49	15.76	10.50	**4.97**	5.42
40	50	10	17.48	15.55	15.67	8.79	**5.06**	5.28
50	50	10	16.98	15.47	15.55	7.79	**4.94**	5.06
10	50	20	41.47	16.77	26.00	54.77	**7.21**	24.94
20	50	20	20.34	15.60	16.38	15.49	**5.15**	6.49
30	50	20	18.23	15.52	15.79	10.48	**5.02**	5.46
40	50	20	17.44	15.52	15.64	8.73	**5.01**	5.23
50	50	20	17.01	15.51	15.58	7.84	**5.00**	5.11

Note. $\chi^2_8(0.05) = 15.51$. The closest to $\alpha(= 0.05)$ in the values α_Q, α_{Q^*} and $\alpha_{Q^\dagger}$ of each low is boldface.

Table 4. The upper percentiles of Q, Q^*, $Q^\dagger$ and the actual Type I error rates when $(p_1, p_2, p_3) = (15, 12, 9)$ and $\alpha = 0.05$.

n_1	n_2	n_3	$q(\alpha)$	$q^*(\alpha)$	$q^\dagger(\alpha)$	α_Q	α_{Q^*}	$\alpha_{Q^\dagger}$
20	10	10	47.04	25.17	32.73	48.54	**5.23**	17.38
30	10	10	34.05	25.08	26.56	23.47	**5.11**	7.23
40	10	10	31.07	25.06	25.68	16.51	**5.09**	5.97
50	10	10	29.58	25.00	25.34	13.31	**5.01**	5.46
100	10	10	27.17	25.00	25.06	8.43	**5.00**	5.09
200	10	10	26.09	25.02	25.03	6.60	**5.03**	5.05
400	10	10	25.51	24.98	24.98	5.74	**4.98**	**4.98**
20	20	20	45.50	25.14	33.78	43.71	**5.20**	18.81
30	20	20	32.98	25.02	26.72	20.62	**5.04**	7.48
40	20	20	30.36	25.04	25.74	14.79	**5.05**	6.04
50	20	20	29.10	25.01	25.39	12.16	**5.02**	5.54
100	20	20	26.99	24.99	25.06	8.08	**4.99**	5.08
200	20	20	26.02	25.00	25.01	6.50	**5.00**	5.02
400	20	20	25.50	24.98	24.99	5.71	4.98	**4.99**
20	50	50	43.91	25.09	34.89	39.21	**5.13**	20.48
30	50	50	31.77	25.03	26.98	17.59	**5.04**	7.88
40	50	50	29.40	25.00	25.82	12.70	**5.00**	6.15
50	50	50	28.38	25.01	25.46	10.63	**5.02**	5.64
100	50	50	26.68	24.98	25.06	7.58	**4.98**	5.09
200	50	50	25.91	24.99	25.01	6.34	**4.99**	5.02
400	50	50	25.47	24.98	24.98	5.66	**4.98**	**4.98**
30	30	30	32.40	25.02	26.84	19.12	**5.04**	7.67
40	40	40	29.60	24.99	25.79	13.17	**4.99**	6.11
100	100	100	26.43	24.97	25.06	7.13	**4.97**	5.09
200	200	200	25.71	25.03	25.05	6.01	**5.04**	5.07
400	400	400	25.34	25.00	25.01	5.46	**5.01**	5.02

Note. $\chi^2_{15}(0.05) = 25.00$. The closest to $\alpha(= 0.05)$ in the values α_Q, α_{Q^*} and $\alpha_{Q^\dagger}$ of each low is boldface.

Table 5. The upper percentiles of Q, Q^*, $Q^\dagger$ and the actual Type I error rates when $(p_1, p_2, p_3, p_4, p_5) = (15, 12, 9, 6, 3)$ and $\alpha = 0.05$.

n_1	$n_2 = \cdots = n_5$	$q(\alpha)$	$q^*(\alpha)$	$q^\dagger(\alpha)$	α_Q	α_{Q^*}	$\alpha_{Q^\dagger}$
20	10	46.86	25.08	33.06	47.84	**5.13**	17.89
30	10	33.86	24.99	26.63	23.03	**4.99**	7.40
40	10	30.91	24.98	25.69	16.13	**4.98**	5.97
50	10	29.52	24.99	25.39	13.05	**4.99**	5.53
100	10	27.15	25.02	25.08	8.37	**5.03**	5.11
200	10	26.03	24.98	24.99	6.53	4.98	**4.99**
400	10	25.53	25.00	25.01	5.74	**5.01**	**5.01**
20	20	45.27	25.11	33.97	43.11	**5.16**	19.09
30	20	32.82	25.01	26.79	20.24	**5.01**	7.60
40	20	30.21	24.99	25.76	14.49	**4.99**	6.07
50	20	29.01	25.00	25.43	11.96	**5.00**	5.58
100	20	26.96	24.99	25.07	8.03	**4.99**	5.10
200	20	25.98	24.97	24.99	6.44	4.97	**4.99**
400	20	25.53	25.01	25.01	5.75	**5.02**	5.03
20	50	43.83	25.07	35.03	38.92	**5.10**	20.66
30	50	31.68	25.03	27.03	17.32	**5.05**	7.94
40	50	29.34	25.01	25.87	12.51	**5.02**	6.23
50	50	28.31	25.01	25.48	10.47	**5.02**	5.67
100	50	26.65	25.00	25.08	7.50	**5.00**	5.11
200	50	25.83	24.94	24.95	6.20	4.92	**4.94**
400	50	25.49	25.01	25.02	5.67	**5.02**	5.03
30	30	32.24	25.00	26.88	18.84	**5.01**	7.75
40	40	29.52	25.00	25.84	12.93	**5.00**	6.18
100	100	26.41	25.01	25.10	7.15	**5.02**	5.14
200	200	25.65	24.98	25.01	5.91	4.97	**5.01**
400	400	25.32	25.00	25.01	5.44	**5.01**	5.02

Note. $\chi^2_{15}(0.05) = 25.00$. The closest to $\alpha(= 0.05)$ in the values α_Q, α_{Q^*} and $\alpha_{Q^\dagger}$ of each low is boldface.

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References

[1] T.W. Anderson, Introduction to Multivariate Statistical Analysis 3rd ed. (New Jersey, Hoboken: John Wiley & Sons, Inc., 2003).

[2] R. Bhargava, *Multivariate tests of hypotheses with incomplete data*, California, Stanford: Technical report No. 3, Applied Mathematics and Statistics Laboratories, Stanford University (1962).

[3] W.-Y. Chang and D.St.P. Richards, *Finite-sample inference with monotone incomplete multivariate normal data,* I., Journal of Multivariate Analysis **100** (2009) 1883–1899.

[4] Y. Fujikoshi, *Transformations with improved chi-squared approximations*, Journal of Multivariate Analysis **72** (2000) 249–263.

[5] A.K. Gupta, J. Xu and Y. Fujikoshi, *An asymptotic expansion of the distribution of Rao's U-statistic under a general condition*, Journal of Multivariate Analysis **97** (2006) 492–513.

[6] J. Hao and K. Krishnamoorthy, *Inferences on a normal covariance matrix and generalized variance with monotone missing data*, Journal of Multivariate Analysis **78** (2001) 62–82.

[7] M. Hosoya and T. Seo, *Simultaneous testing of the mean vector and the covariance matrix with two-step monotone missing data*, SUT Journal of Mathematics **51** (2015) 83–98.

[8] M. Hosoya and T. Seo, *On the likelihood ratio test for the equality of multivariate normal populations with two-step monotone missing data*, Journal of Statistical Theory and Practice **10** (2016) 673–692.

[9] K.G. Jinadasa and D.S. Tracy, *Maximum likelihood estimation for multivariate normal distribution with monotone sample*, Communications in Statistics – Theory and Methods **21** (1992) 41–50.

[10] T. Kanda and Y. Fujikoshi, *Some basic properties of the MLE's for a multivariate normal distribution with monotone missing data*, American Journal of Mathematical and Management Sciences **18** (1998) 161–190.

[11] T. Kawasaki and T. Seo, *Bias correction for T^2 type statistic with two-step monotone missing data*, Statistics **50** (2016) 76–88.

[12] K. Krishnamoorthy and M.K. Pannala, *Some simple test procedures for normal mean vector with incomplete data*, Annals of the Institute of Statistical Mathematics **50** (1998) 531–542.

[13] K. Krishnamoorthy and M.K. Pannala, *Confidence estimation of a normal mean vector with incomplete data*, The Canadian Journal of Statistics **27** (1999) 395–407.

[14] R.J.A. Little and D.B. Rubin, Statistical Analysis with Missing Data, 2nd ed. (New York: John Wiley & Sons, Inc., 2002).

[15] R. J. Muirhead, Aspects of Multivariate Statistical Theory (New Jersey, Hoboken: Wiley & Sons, Inc., 1982).

[16] N. Seko, T. Kawasaki and T. Seo, *Testing equality of two mean vectors with two-step monotone missing data*, American Journal of Mathematical and Management Sciences **31** (2011) 117–135.

[17] N. Seko, A. Yamazaki and T. Seo, *Tests for mean vector with two-step monotone missing data*, SUT Journal of Mathematics **48** (2012) 13–36.

[18] T. Seo and M.S. Srivastava, *Testing equality of means and simultaneous confidence intervals in repeated measures with missing data*, Biometrical Journal **42** (2000) 981–993.

[19] M. Siotani, T. Hayakawa and Y. Fujikoshi, Modern Multivariate Statistical Analysis: A Graduate Course and Handbook (Ohio, Columbus: American Science Press, 1985).

[20] M.S. Srivastava, *Multivariate data with missing observations*, Communications in Statistics – Theory and Methods **14** (1985) 775–792.

[21] M.S. Srivastava, Methods of Multivariate Statistics (New York, John Wiley & Sons, Inc., 2002).

[22] S. Tsukada, *Equivalence testing of mean vector and covariance matrix for multi-populations under a two-step monotone incomplete sample*, Journal of Multivariate Analysis **132** (2014) 183–196.

[23] A. Yagi and T. Seo, *A test for mean vector and simultaneous confidence intervals with three-step monotone missing data*, American Journal of Mathematical and Management Sciences **33** (2014) 161–175.

[24] A. Yagi and T. Seo, *Tests for normal mean vectors with monotone incomplete data*, American Journal of Mathematical and Management Sciences **36** (2017) 1–20.

[25] J. Yu, K. Krishnamoorthy and K.M. Pannala, *Two-sample inference for normal mean vectors based on monotone missing data*, Journal of Multivariate Analysis **97** (2006) 2162–2176.

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