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FURTHER CHARACTERIZATIONS OF FUNCTIONS OF A PAIR OF ORTHOGONAL PROJECTORS

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Abstract

The paper provides several original conditions involving ranks and traces of functions of a pair of orthogonal projectors (i.e., Hermitian idempotent matrices) under which the functions themselves are orthogonal projectors. The results are established by means of a joint decomposition of the two projectors.

Keywords: Hermitian idempotent matrix, partial isometry, rank, trace, Moore-Penrose inverse.

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1. Introduction

Let $\mathbb{C}_{m,n}$ denote the set of $m \times n$ complex matrices. The symbols $\mathbf{K}^*$, $\mathcal{R}(\mathbf{K})$, and $\mathrm{rk}(\mathbf{K})$ will stand for the conjugate transpose, column space (range), and rank of $\mathbf{K} \in \mathbb{C}_{m,n}$, respectively. Further, $\mathbf{I}_n$ will be the identity matrix of order n, and for a given $\mathbf{K} \in \mathbb{C}_{n,n}$ we define $\overline{\mathbf{K}} = \mathbf{I}_n - \mathbf{K}$. Another function of a square matrix $\mathbf{K} \in \mathbb{C}_{n,n}$, which will be referred to in what follows, is its trace $\mathrm{tr}(\mathbf{K})$.

A crucial role in the considerations of the present paper is played by the class of orthogonal projectors in $\mathbb{C}_{n,1}$ (Hermitian idempotent matrices of order n), whose set will be denoted by $\mathbb{C}_n^{\mathsf{OP}}$, i.e.,

$$\mathbb{C}_n^{\mathsf{OP}} = \{\mathbf{P} \in \mathbb{C}_{n,n} \colon\ \mathbf{P}^2 = \mathbf{P} = \mathbf{P}^*\}.$$

It is known that $\mathbf{P} \in \mathbb{C}_n^{\mathsf{OP}}$ if and only if the matrix is expressible as $\mathbf{K}\mathbf{K}^\dagger$ for some $\mathbf{K} \in \mathbb{C}_{n,m}$, where $\mathbf{K}^\dagger \in \mathbb{C}_{m,n}$ is the Moore-Penrose inverse of $\mathbf{K}$, i.e., the unique matrix satisfying the equations

$$(1)\qquad \mathbf{K}\mathbf{K}^\dagger\mathbf{K} = \mathbf{K},\ \mathbf{K}^\dagger\mathbf{K}\mathbf{K}^\dagger = \mathbf{K}^\dagger,\ (\mathbf{K}\mathbf{K}^\dagger)^* = \mathbf{K}\mathbf{K}^\dagger,\ (\mathbf{K}^\dagger\mathbf{K})^* = \mathbf{K}^\dagger\mathbf{K}.$$

Then $\mathbf{P}_\mathbf{K} = \mathbf{K}\mathbf{K}^\dagger$ is the orthogonal projector onto $\mathcal{R}(\mathbf{K})$ and, consequently, $\overline{\mathbf{P}}_\mathbf{K} = \mathbf{I}_n - \mathbf{K}\mathbf{K}^\dagger$ is the orthogonal projector onto the orthogonal complement of $\mathcal{R}(\mathbf{K})$.

Let $\mathbf{P} \in \mathbb{C}_n^{\mathsf{OP}}$ be of rank r. Then there exists unitary $\mathbf{U} \in \mathbb{C}_{n,n}$ such that

$$(2)\qquad \mathbf{P} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*.$$

Clearly, any other orthogonal projector of order n, say $\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$, can be represented as

$$(3)\qquad \mathbf{Q} = \mathbf{U}\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{D} \end{pmatrix}\mathbf{U}^*,$$

with $\mathbf{A} \in \mathbb{C}_{r,r}$ and $\mathbf{D} \in \mathbb{C}_{n-r,n-r}$ being Hermitian. Two particular versions of the representation (3) are obtained when $r = 0$, in which case the matrices $\mathbf{A}$ and $\mathbf{B}$ are absent, and when $r = n$, in which case the matrices $\mathbf{D}$ and $\mathbf{B}$ are absent.

In what follows we recall selected properties of the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$ involved in the representation (3); for their formal derivations see e.g., [6, Lemmas 1 and 4].

Lemma 1. *Let $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ be of the forms* (2) *and* (3), *respectively. Then:*

(i) $\mathbf{A} = \mathbf{A}^2 + \mathbf{B}\mathbf{B}^*$,
(ii) $\overline{\mathbf{A}} = \overline{\mathbf{A}}^2 + \mathbf{B}\mathbf{B}^*$,
(iii) $\mathbf{D} = \mathbf{D}^2 + \mathbf{B}^*\mathbf{B}$,
(iv) $\overline{\mathbf{D}} = \overline{\mathbf{D}}^2 + \mathbf{B}^*\mathbf{B}$,
(v) $\mathrm{rk}(\overline{\mathbf{A}}) = r - \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{B})$,
(vi) $\mathrm{rk}(\overline{\mathbf{D}}) = n - r + \mathrm{rk}(\mathbf{B}) - \mathrm{rk}(\mathbf{D})$.

The lemma below is an extraction of the rank identities given in [6, Lemma 5].

Lemma 2. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *be of the forms* (2) *and* (3)*, respectively. Then:*

(i) $\mathrm{rk}(\mathbf{PQ}) = \mathrm{rk}(\mathbf{A})$,
(ii) $\mathrm{rk}(\mathbf{P}+\mathbf{Q}) = r + \mathrm{rk}(\mathbf{D})$,
(iii) $\mathrm{rk}(\mathbf{P}-\mathbf{Q}) = r - \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{B}) + \mathrm{rk}(\mathbf{D})$,
(iv) $\mathrm{rk}(\mathbf{I}_n - \mathbf{PQ}) = n - \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{B})$,
(v) $\mathrm{rk}(\mathbf{PQ}+\mathbf{QP}) = \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{B})$,
(vi) $\mathrm{rk}(\mathbf{PQ}-\mathbf{QP}) = 2\,\mathrm{rk}(\mathbf{B})$,
(vii) $\mathrm{rk}(\mathbf{P}+\mathbf{Q}-\mathbf{PQ}) = r + \mathrm{rk}(\mathbf{D})$,
(viii) $\mathrm{rk}(\mathbf{I}_n - \mathbf{P} - \mathbf{Q}) = n - r + \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{B}) - \mathrm{rk}(\mathbf{D})$.

On account of the fact that $\mathbf{A}$ is a contraction, the matrix was in [4, p. 2253] represented as

$$\mathbf{A} = \mathbf{V}\mathrm{diag}(\underbrace{1,\ldots,1}_{k\,\text{times}}, \alpha_1, \ldots, \alpha_l, \underbrace{0,\ldots,0}_{m\,\text{times}})\mathbf{V}^*, \tag{4}$$

where $\mathbf{V} \in \mathbb{C}_{r,r}$ is unitary, $r = k + l + m$, and α_i, $i = 1, \ldots, l$, satisfying $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_l$, are the eigenvalues of $\mathbf{A}$ belonging to the set $(0,1)$, i.e., $l = \mathrm{rk}(\mathbf{B})$. Similarly, the spectral decompositions of $\mathbf{D}$ provided in [4, p. 2251] reads

$$\mathbf{D} = \mathbf{W}\mathrm{diag}(\underbrace{1,\ldots,1}_{s\,\text{times}}, \delta_1, \ldots, \delta_t, \underbrace{0,\ldots,0}_{u\,\text{times}})\mathbf{W}^*, \tag{5}$$

with unitary $\mathbf{W} \in \mathbb{C}_{n-r,n-r}$, $n - r = s + t + u$, and δ_j, $j = 1, \ldots, t$, such that $\delta_1 \geqslant \delta_2 \geqslant \cdots \geqslant \delta_t$, denoting the eigenvalues of $\mathbf{D}$ belonging to the set $(0,1)$, i.e., $t = \mathrm{rk}(\mathbf{B})$.

The next section of the paper provides a collection of (to the best of our knowledge) original characterizations of requirements that a given function of a pair of orthogonal projectors is an orthogonal projector itself. A particular attention is paid to the characterizations which exploit the notions of rank and trace of the function under consideration.

2. Main results

In the literature, one can find several conditions equivalent to the requirement that $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$, or, equivalently, to the commutativity property $\mathbf{PQ} = \mathbf{QP}$; see e.g., [1, 2, 3, 8]. The first theorem of the paper establishes additional four conditions of the kind, each of which involves rank and/or trace of a function of the product $\mathbf{PQ}$.

Theorem 3. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$*. Then the following conditions are equivalent:*

(i) $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$,

(ii) $\mathrm{rk}(\mathbf{PQ}) = \mathrm{tr}(\mathbf{PQ})$,
(iii) $\mathrm{rk}(\mathbf{PQ}) = \mathrm{tr}[\mathbf{PQP}]$,
(iv) $\mathrm{rk}(\mathbf{PQ} + \mathbf{QP}) = \frac{1}{2}\mathrm{tr}(\mathbf{PQ} + \mathbf{QP})$,
(v) $\mathrm{tr}(\mathbf{PQ} + \mathbf{QP}) = \frac{1}{2}\mathrm{tr}[(\mathbf{PQ} + \mathbf{QP})^2]$.

Proof. We will exploit the representations (2) and (3) to show that all conditions listed in the theorem are equivalent to $\mathbf{B} = \mathbf{0}$. First observe that from (2) and (3) it follows that

$$\mathbf{PQ} = \mathbf{U}\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*, \tag{6}$$

whence we conclude that $\mathbf{PQ} = \mathbf{QP}$ or, in other words, $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$ holds if and only if $\mathbf{B} = \mathbf{0}$.

Since $\mathrm{rk}(\mathbf{PQ}) = \mathrm{rk}(\mathbf{A})$ and $\mathrm{tr}(\mathbf{PQ}) = \mathrm{tr}(\mathbf{A})$, it follows that $\mathrm{rk}(\mathbf{PQ}) = \mathrm{tr}(\mathbf{PQ})$ if and only if $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A})$. By the decomposition (4), it is seen that rank and trace of $\mathbf{A}$ coincide merely when $\mathbf{A}$ has no eigenvalues belonging to the set $(0, 1)$, which is equivalent to the requirement that $\mathbf{B} = \mathbf{0}$ (parenthetically note that $\mathbf{B} = \mathbf{0} \Leftrightarrow \mathbf{A}^2 = \mathbf{A}$).

In the light of

$$\mathbf{PQP} = \mathbf{U}\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*,$$

it is clear that also the condition in point (iii) of the theorem is equivalent to $\mathbf{B} = \mathbf{0}$.

Similarly, in view of

$$\mathbf{PQ} + \mathbf{QP} = \mathbf{U}\begin{pmatrix}2\mathbf{A} & \mathbf{B}\\ \mathbf{B}^* & \mathbf{0}\end{pmatrix}\mathbf{U}^*,$$

and the formula for rank of $\mathbf{PQ} + \mathbf{QP}$ given in Lemma 2, we arrive at the conclusion that

$$\mathrm{rk}(\mathbf{PQ} + \mathbf{QP}) = \frac{1}{2}\mathrm{tr}(\mathbf{PQ} + \mathbf{QP}) \;\Leftrightarrow\; \mathbf{B} = \mathbf{0}.$$

To complete the proof, observe that direct calculations show that the trace identity given in point (v) of the theorem is satisfied if and only if

$$\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^2) + \frac{1}{2}\mathrm{tr}(\mathbf{BB}^*), \tag{7}$$

where the fact that $\mathrm{tr}(\mathbf{BB}^*) = \mathrm{tr}(\mathbf{B}^*\mathbf{B})$ was utilized. Hence, by point (i) of Lemma 1, the condition (7) can be expressed as $\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^2)$, which, in the light of (4) is equivalent to $\mathbf{B} = \mathbf{0}$. ■

For further conditions equivalent to $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$, each of which is based on rank of a function of the product $\mathbf{PQ}$, see [3, Theorem 6]. Among the conditions given therein one finds:

(i) $\mathrm{rk}(\mathbf{I}_n - \mathbf{PQ}) + \mathrm{rk}(\mathbf{PQ}) = n$,
(ii) $\mathrm{rk}(\mathbf{P} - \mathbf{Q}) = \mathrm{rk}(\mathbf{P} + \mathbf{Q}) - \mathrm{rk}(\mathbf{PQ})$,
(iii) $\mathrm{rk}(\mathbf{PQ}) = \mathrm{rk}(\mathbf{PQ} + \mathbf{QP})$.

The next theorem provides a number of conditions equivalent to $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$ involving the notion of a partial isometry. Recall that the class of partial isometries is defined by

$$\mathbb{C}_{n,m}^{\mathsf{PI}} = \{\mathbf{K} \in \mathbb{C}_{n,m} \colon \ \mathbf{K}^* = \mathbf{K}^\dagger\};$$

for alternative characterizations of the class see [9, Theorem 5, §6.4]. In what follows, whenever $m = n$, instead of $\mathbb{C}_{n,n}^{\mathsf{PI}}$ we will write $\mathbb{C}_n^{\mathsf{PI}}$.

Theorem 4. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then the following conditions are equivalent:*

(i) $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$,
(ii) $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{PI}}$,
(iii) $\mathbf{P} - \mathbf{Q} \in \mathbb{C}_n^{\mathsf{PI}}$,
(iv) $\mathbf{PQP} \in \mathbb{C}_n^{\mathsf{PI}}$,
(v) $\mathbf{I}_n - \mathbf{P} - \mathbf{Q} \in \mathbb{C}_n^{\mathsf{PI}}$,
(vi) $\mathbf{P} + \mathbf{Q} - \mathbf{PQ} \in \mathbb{C}_n^{\mathsf{PI}}$.

Proof. In the light of (6) and the representation of the Moore–Penrose inverse of $\mathbf{PQ}$ given in the Appendix, we conclude that $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{PI}}$ if and only if $\mathbf{P}_{\mathbf{A}} = \mathbf{A}$ and $\mathbf{B}^* = \mathbf{B}^*\mathbf{A}^\dagger$. By point (i) of Lemma 1 it is seen that the former of these conditions is equivalent to $\mathbf{B} = \mathbf{0}$ or, in other words, to $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$.

A proof referring to the remaining points of the theorem will be established in a similar fashion. Exploiting the representation of $(\mathbf{P} - \mathbf{Q})^\dagger$, we conclude that $\mathbf{P} - \mathbf{Q} \in \mathbb{C}_n^{\mathsf{PI}}$ is equivalent to the following conjunction of three conditions

$$\mathbf{P}_{\overline{\mathbf{A}}} = \overline{\mathbf{A}}, \ \mathbf{B} = \mathbf{BD}^\dagger, \ \mathbf{P}_{\mathbf{D}} = \mathbf{D}.$$

Combining now, $\mathbf{P}_{\overline{\mathbf{A}}} = \overline{\mathbf{A}}$ with point (ii) of Lemma 1 proves that $\mathbf{P}_{\overline{\mathbf{A}}} = \overline{\mathbf{A}} \Leftrightarrow \mathbf{B} = \mathbf{0}$, whereas combining $\mathbf{P}_{\mathbf{D}} = \mathbf{D}$ with point (iii) of Lemma 1 gives $\mathbf{P}_{\mathbf{D}} = \mathbf{D} \Leftrightarrow \mathbf{B} = \mathbf{0}$. The proof of point (iii) is, thus, complete.

To show that $\mathbf{PQP} \in \mathbb{C}_n^{\mathsf{PI}}$ is also equivalent to $\mathbf{B} = \mathbf{0}$, note that on account of the representation of $(\mathbf{PQP})^\dagger$, given in the Appendix, we obtain

$$\mathbf{PQP} \in \mathbb{C}_n^{\mathsf{PI}} \ \Leftrightarrow \ \mathbf{A} = \mathbf{A}^\dagger.$$

By (4) it is clear that $\mathbf{A} = \mathbf{A}^\dagger$ holds if and only if $\mathbf{B} = \mathbf{0}$.

Utilizing the representation of the Moore–Penrose inverse of $\mathbf{I}_n - \mathbf{P} - \mathbf{Q}$ yields

$$\mathbf{I}_n - \mathbf{P} - \mathbf{Q} \in \mathbb{C}_n^{\mathsf{PI}} \;\Leftrightarrow\; \mathbf{P_A} = \mathbf{A},\; \mathbf{B} = \mathbf{A}^\dagger\mathbf{B},\; \mathbf{P}_{\overline{\mathbf{D}}} = \overline{\mathbf{D}}.$$

In the light of points (i) and (iv) of Lemma 1 it is seen that $\mathbf{P_A} = \mathbf{A}$ and $\mathbf{P}_{\overline{\mathbf{D}}} = \overline{\mathbf{D}}$ are both equivalent to $\mathbf{B} = \mathbf{0}$, what establishes the part (i) $\Leftrightarrow$ (v) of the theorem.

To prove that also the condition (vi) holds if and only if $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$, we again exploit the corresponding representation of the Moore–Penrose inverse. Hence, we conclude that

$$\mathbf{P} + \mathbf{Q} - \mathbf{PQ} \in \mathbb{C}_n^{\mathsf{PI}} \;\Leftrightarrow\; \mathbf{B} = \mathbf{0}, \mathbf{D} = \mathbf{D}^\dagger.$$

Now, in view of (5), we see that $\mathbf{B} = \mathbf{0} \Leftrightarrow \mathbf{D} = \mathbf{D}^\dagger$, what completes the proof. ∎

Since any $\mathbf{K} \in \mathbb{C}_{n,n}$ satisfies $\mathbf{K} \in \mathbb{C}_n^{\mathsf{PI}} \Leftrightarrow \mathbf{K}^\dagger \in \mathbb{C}_n^{\mathsf{PI}} \Leftrightarrow \mathbf{K}^* \in \mathbb{C}_n^{\mathsf{PI}}$ (see [9, Theorem 5, §6.4]), further conditions equivalent to $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$ ore obtained from Theorem 2 by replacing the functions of $\mathbf{P}$ and $\mathbf{Q}$ on the right-hand sides of the conditions (ii)–(vi) either by their Moore–Penrose inverses or conjugate transposes.

Further conditions equivalent to $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$, each of which involves the Moore–Penrose inverse of a function of $\mathbf{P}$ and $\mathbf{Q}$ are given in the following corollary. Its proof is omitted, for it is a direct consequence of Theorem 4.

Corollary 5. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then the following conditions are equivalent:*

(i) $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$,
(ii) $\mathbf{PQ} = (\mathbf{PQ})^\dagger$,
(iii) $\mathbf{P} - \mathbf{Q} = (\mathbf{P} - \mathbf{Q})^\dagger$,
(iv) $\mathbf{PQP} = (\mathbf{PQP})^\dagger$,
(v) $\mathbf{I}_n - \mathbf{P} - \mathbf{Q} = (\mathbf{I}_n - \mathbf{P} - \mathbf{Q})^\dagger$,
(vi) $\mathbf{P} + \mathbf{Q} - \mathbf{PQ} = (\mathbf{P} + \mathbf{Q} - \mathbf{PQ})^\dagger$.

An interesting observation is that $\mathbf{B}$ does not need to be the zero matrix to ensure that $\mathbf{PQ} - \mathbf{QP} \in \mathbb{C}_n^{\mathsf{PI}}$. To be more precise, by exploiting the representation of $(\mathbf{PQ} - \mathbf{QP})^\dagger$, provided in the Appendix, we conclude that $\mathbf{PQ} - \mathbf{QP} \in \mathbb{C}_n^{\mathsf{PI}} \Leftrightarrow \mathbf{B} \in \mathbb{C}_{r,n-r}^{\mathsf{PI}}$. Hence, $\mathbf{B} = \mathbf{0}$ clearly implies $\mathbf{PQ} - \mathbf{QP} \in \mathbb{C}_n^{\mathsf{PI}}$, but this implication is not reversible.

Another fact related to Theorem 4 is that yet another condition equivalent to $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$, namely $\mathbf{I}_n - \mathbf{PQ} \in \mathbb{C}_n^{\mathsf{PI}}$, was identified in [5, Theorem 10(iii)]).

In what follows we focus our attention on the condition $\mathbf{P} + \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. Recall that $\mathbf{P} + \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}} \Leftrightarrow \mathbf{PQ} = \mathbf{0} \Leftrightarrow \mathbf{A} = \mathbf{0}$, with the last equality obtained by combining the representation (6) with point (i) of Lemma 1.

Theorem 6. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$*. Then the following conditions are equivalent:*
(i) $\mathbf{P} + \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$,
(ii) $\mathbf{P} + \mathbf{Q} \in \mathbb{C}_n^{\mathsf{PI}}$,
(iii) $\mathbf{PQ} + \mathbf{QP} \in \mathbb{C}_n^{\mathsf{PI}}$,
(iv) $\mathrm{tr}(\mathbf{P} + \mathbf{Q}) = \mathrm{tr}[(\mathbf{P} + \mathbf{Q})^2]$.

Proof. The expression for the Moore–Penrose inverse of $\mathbf{P}+\mathbf{Q}$ given in Appendix yields

$$(8) \qquad \mathbf{P} + \mathbf{Q} \in \mathbb{C}_n^{\mathsf{PI}} \;\Leftrightarrow\; \mathbf{A} = -\frac{1}{2}\overline{\mathbf{P}}_{\overline{\mathbf{A}}},\ \mathbf{B} = -\mathbf{B}\mathbf{D}^{\dagger}, \text{and } \mathbf{D} = 2\mathbf{D}^{\dagger} - \mathbf{P}_{\mathbf{D}}.$$

Direct calculations show that if $\mathbf{A}$ is of the form (4), then

$$\overline{\mathbf{P}}_{\overline{\mathbf{A}}} = \mathbf{V}\mathrm{diag}(\underbrace{1,\dots,1}_{k\,\text{times}}, \underbrace{0,\dots,0}_{l\,\text{times}}, \underbrace{0,\dots,0}_{m\,\text{times}})\mathbf{V}^*.$$

Hence, it is seen that $\mathbf{A} = -\frac{1}{2}\overline{\mathbf{P}}_{\overline{\mathbf{A}}}$ if and only if $\mathbf{A}$ has no nonzero eigenvalues, which means that $\mathbf{A} = \mathbf{0}$. Analogous derivations lead to the conclusion that if $\mathbf{D}$ is of the form (5), then $\mathbf{D} = 2\mathbf{D}^{\dagger} - \mathbf{P}_{\mathbf{D}}$ if and only if

$$\delta_i^2 + \delta_i - 2 = 0 \quad \text{for all } i = 1, \dots, t.$$

Since this equation does not have a solution in the set $(0, 1)$, we see that the last condition on the right-hand side of the equivalence (8) holds if and only if $\mathbf{B} = \mathbf{0}$, which completes the proof dealing with point (ii) of the theorem.

Let us now consider the condition $\mathbf{PQ} + \mathbf{QP} \in \mathbb{C}_n^{\mathsf{PI}}$. Utilizing the expression for the Moore–Penrose inverse of $\mathbf{PQ} + \mathbf{QP}$ proves that the sum of two products is a partial isometry if and only if $\mathbf{B} = \mathbf{0}$ and $4\mathbf{A} = \mathbf{A}^{\dagger}$. Exploiting again the representation (4) leads to the conclusion that these two conditions are equivalent to $\mathbf{A} = \mathbf{0}$. Hence, $\mathbf{PQ} + \mathbf{QP} \in \mathbb{C}_n^{\mathsf{PI}} \Leftrightarrow \mathbf{A} = \mathbf{0}$.

So, point (iv) of the theorem is left to be considered. As can be directly verified, the trace identity given therein holds if and only if

$$\mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{A}^2) + \mathrm{tr}(\mathbf{B}\mathbf{B}^*) + \mathrm{tr}(\mathbf{B}^*\mathbf{B}) + \mathrm{tr}(\mathbf{D}^2) = \mathrm{tr}(\mathbf{D}).$$

On account of the relationships in points (i) and (iii) of Lemma 1, this equality reduces to $\mathrm{tr}(\mathbf{A}) = 0$, i.e., $\mathbf{A} = \mathbf{0}$. ∎

The corollary below is a direct consequence of points (ii) and (iii) of Theorem 6.

Corollary 7. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$*. Then the following conditions are equivalent:*
(i) $\mathbf{P} + \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$,

(ii) $\mathbf{P}+\mathbf{Q}=(\mathbf{P}+\mathbf{Q})^\dagger$,
(iii) $\mathbf{PQ}+\mathbf{QP}=(\mathbf{PQ}+\mathbf{QP})^\dagger$.

It is clear that $\mathbf{P}+\mathbf{Q}\in\mathbb{C}_n^{\mathsf{OP}}$ implies $\mathbf{PQ}\in\mathbb{C}_n^{\mathsf{OP}}$, but not vice versa. The lemma below lists three conditions each of which combined with the requirement that the product $\mathbf{PQ}$ is an orthogonal projector ensures that the sum $\mathbf{P}+\mathbf{Q}$ is an orthogonal projector.

Lemma 8. *Let $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathsf{OP}}$. Then $\mathbf{P}+\mathbf{Q}\in\mathbb{C}_n^{\mathsf{OP}}$ if and only if $\mathbf{PQ}\in\mathbb{C}_n^{\mathsf{OP}}$ and any of the following conditions is satisfied:*
(i) $\mathrm{rk}(\mathbf{P}+\mathbf{Q})=\mathrm{tr}(\mathbf{P}+\mathbf{Q})$,
(ii) $\mathrm{rk}(\mathbf{P}+\mathbf{Q})=\mathrm{tr}[(\mathbf{P}+\mathbf{Q})^2]$,
(iii) $\mathrm{rk}(\mathbf{P}+\mathbf{Q})=\mathrm{tr}[(\mathbf{P}+\mathbf{Q})^\dagger]$.

Proof. Combining the expression for rank of $\mathbf{P}+\mathbf{Q}$ given in Lemma 2 with $\mathrm{tr}(\mathbf{P}+\mathbf{Q})=r+\mathrm{tr}(\mathbf{A})+\mathrm{tr}(\mathbf{D})$ shows that

$$\mathrm{rk}(\mathbf{P}+\mathbf{Q})=\mathrm{tr}(\mathbf{P}+\mathbf{Q})\ \Leftrightarrow\ \mathrm{rk}(\mathbf{D})=\mathrm{tr}(\mathbf{A})+\mathrm{tr}(\mathbf{D}).$$

Now, if $\mathbf{PQ}\in\mathbb{C}_n^{\mathsf{OP}}$, i.e., if $\mathbf{B}=\mathbf{0}$, then $\mathrm{rk}(\mathbf{D})=\mathrm{tr}(\mathbf{D})$, whence we arrive at $\mathbf{A}=\mathbf{0}$, which completes the proof dealing with point (i) of the theorem.

Similarly, on account of $\mathbf{B}=\mathbf{0}$, we obtain

$$\mathrm{rk}(\mathbf{P}+\mathbf{Q})=\mathrm{tr}[(\mathbf{P}+\mathbf{Q})^2]\ \Leftrightarrow\ \mathrm{rk}(\mathbf{D})=2\mathrm{tr}(\mathbf{A})+\mathrm{tr}(\mathbf{A}^2)+\mathrm{tr}(\mathbf{D}^2).$$

Hence, by $\mathrm{rk}(\mathbf{D})=\mathrm{tr}(\mathbf{D}^2)$ and $\mathrm{tr}(\mathbf{A})=\mathrm{tr}(\mathbf{A}^2)$ we in turn conclude that $\mathbf{A}$ is necessarily the zero matrix.

The proof of the last point is established analogously. First observe that the condition $\mathbf{B}=\mathbf{0}$ entails $\mathrm{tr}(\mathbf{D})=\mathrm{tr}(\mathbf{D}^\dagger)=\mathrm{tr}(\mathbf{P_D})$. In consequence, direct calculations show that

$$\mathrm{rk}(\mathbf{P}+\mathbf{Q})=\mathrm{tr}[(\mathbf{P}+\mathbf{Q})^\dagger]\ \Leftrightarrow\ \overline{\mathbf{P}}_{\overline{\mathbf{A}}}=\mathbf{0}.$$

This identity is satisfied exclusively when $\mathbf{A}$ has no unit eigenvalues, which (in the light of $\mathbf{B}=\mathbf{0}$) can happen only when $\mathbf{A}=\mathbf{0}$. ∎

Subsequently we focus our attention on the difference $\mathbf{P}-\mathbf{Q}$. It is known that $\mathbf{P}-\mathbf{Q}\in\mathbb{C}_n^{\mathsf{OP}}\Leftrightarrow\mathbf{PQ}=\mathbf{Q}\Leftrightarrow\mathbf{D}=\mathbf{0}$; see [3, Lemma 2]. The next theorem provides two rank/trace counterparts of the condition $\mathbf{P}-\mathbf{Q}\in\mathbb{C}_n^{\mathsf{OP}}$.

Theorem 9. *Let $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathsf{OP}}$. Then the following conditions are equivalent:*
(i) $\mathbf{P}-\mathbf{Q}\in\mathbb{C}_n^{\mathsf{OP}}$,
(ii) $\mathrm{tr}(\mathbf{P}-\mathbf{Q})=\mathrm{tr}[(\mathbf{P}-\mathbf{Q})^2]$,
(iii) $\mathrm{rk}(\mathbf{P}-\mathbf{Q})=\mathrm{tr}[(\mathbf{P}-\mathbf{Q})^\dagger]$.

Proof. In view of points (ii) and (iii) of Lemma 1, we obtain $\mathrm{tr}[(\mathbf{P}-\mathbf{Q})^2] = r - \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{D})$. Hence, the equivalence between condition (ii) of the theorem and identity $\mathbf{D} = \mathbf{0}$ follows by direct calculations.

From the representation of the Moore–Penrose inverse of $\mathbf{P}-\mathbf{Q}$, we obtain

$$\mathrm{tr}[(\mathbf{P}-\mathbf{Q})^\dagger] = \mathrm{tr}(\mathbf{P}_{\overline{\mathbf{A}}}) - \mathrm{tr}(\mathbf{P}_{\mathbf{D}}) = \mathrm{rk}(\mathbf{P}_{\overline{\mathbf{A}}}) - \mathrm{rk}(\mathbf{P}_{\mathbf{D}}) = \mathrm{rk}(\overline{\mathbf{A}}) - \mathrm{rk}(\mathbf{D}).$$

Thus, on account of the formulae for ranks of $\overline{\mathbf{A}}$ and $\mathbf{P}-\mathbf{Q}$, we arrive at

$$\mathrm{rk}(\mathbf{P}-\mathbf{Q}) = \mathrm{tr}[(\mathbf{P}-\mathbf{Q})^\dagger] \;\Leftrightarrow\; \mathbf{D} = \mathbf{0},$$

which completes the proof. ■

The lemma below identifies a condition which combined with the requirement that $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$ constitutes a conjunction equivalent to $\mathbf{P}-\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$.

Lemma 10. *Let $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. Then $\mathbf{P}-\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ if and only if $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$ holds along with $\mathrm{rk}(\mathbf{P}-\mathbf{Q}) = \mathrm{tr}(\mathbf{P}-\mathbf{Q})$.*

Proof. Since $\mathbf{B} = \mathbf{0}$ yields, both, $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A})$ and $\mathrm{rk}(\mathbf{D}) = \mathrm{tr}(\mathbf{D})$, on account of the formula for rank of $\mathbf{P}-\mathbf{Q}$ provided in Lemma 2, we obtain $\mathrm{rk}(\mathbf{P}-\mathbf{Q}) = \mathrm{tr}(\mathbf{P}-\mathbf{Q}) \Leftrightarrow \mathbf{D} = \mathbf{0}$. ■

In what follows our attention focuses on the condition $\mathbf{PQP} \in \mathbb{C}_n^{\mathsf{OP}}$. In fact, it is known (and can be easily verified within the setting utilized in the present paper) that $\mathbf{PQP} \in \mathbb{C}_n^{\mathsf{OP}} \Leftrightarrow \mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$; this is a particular case of a more general result established in [8, Theorem].

Theorem 11. *Let $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. Then the following conditions are equivalent:*
(i) $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$,
(ii) $\mathbf{PQP} \in \mathbb{C}_n^{\mathsf{OP}}$,
(iii) $\mathrm{rk}(\mathbf{PQP}) = \mathrm{tr}(\mathbf{PQP})$,
(iv) $\mathrm{rk}(\mathbf{PQP}) = \mathrm{tr}[(\mathbf{PQP})^2]$,
(v) $\mathrm{tr}(\mathbf{PQP}) = \mathrm{tr}[(\mathbf{PQP})^2]$,
(vi) $\mathrm{rk}(\mathbf{PQP}) = \mathrm{tr}[(\mathbf{PQP})^\dagger]$,
(vii) $\mathrm{tr}(\mathbf{PQP}) = \mathrm{tr}[(\mathbf{PQP})^\dagger]$.

Proof. The proof is limited to an observation that the conditions provided in points (iii)–(vii) of the theorem are equivalent to $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A})$, $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^2)$, $\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^2)$, $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^\dagger)$, and $\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^\dagger)$, respectively, each of which is equivalent to $\mathbf{B} = \mathbf{0}$. ■

Yet another set of conditions equivalent to $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$ is given in what follows. The theorem below puts emphasis on the function $\mathbf{I}_n - \mathbf{PQ}$, which, clearly, satisfies the equivalence $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}} \Leftrightarrow \mathbf{I}_n - \mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$.

Theorem 12. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$*. Then the following conditions are equivalent:*

(i) $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$,

(ii) $\mathbf{I}_n - \mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$,

(iii) $\mathrm{rk}(\mathbf{I}_n - \mathbf{PQ}) = \mathrm{tr}(\mathbf{I}_n - \mathbf{PQ})$,

(iv) $\mathrm{rk}(\mathbf{I}_n - \mathbf{PQ}) = \mathrm{tr}[(\mathbf{I}_n - \mathbf{PQ})(\mathbf{I}_n - \mathbf{PQ})^*]$,

(v) $\mathrm{tr}(\mathbf{I}_n - \mathbf{PQ}) = \mathrm{tr}[(\mathbf{I}_n - \mathbf{PQ})^2]$,

(vi) $\mathrm{rk}(\mathbf{I}_n - \mathbf{PQ}) = \mathrm{tr}[(\mathbf{I}_n - \mathbf{PQ})^\dagger]$.

Proof. By exploiting the formula for rank of $\mathbf{I}_n - \mathbf{PQ}$ given in Lemma 2, we arrive at the conclusion that the condition given in point (iii) of the theorem is satisfied if and only if

$$\mathrm{rk}(\mathbf{A}) - \mathrm{tr}(\mathbf{A}) = \mathrm{rk}(\mathbf{B}). \tag{9}$$

In the light of the symbols used in (4), we have $\mathrm{rk}(\mathbf{A}) = k+l$, $\mathrm{tr}(\mathbf{A}) = k+\sum_{i=1}^{l} \alpha_i$, and $\mathrm{rk}(\mathbf{B}) = l$. Hence, the relationship (9) reduces to $\sum_{i=1}^{l} \alpha_i = 0$, which is equivalent to $\mathbf{B} = \mathbf{0}$, i.e., the part (i) $\Leftrightarrow$ (iii) is established.

Direct calculations show that also the condition in point (iv) of the theorem is satisfied if and only if identity (9) holds.

The proof concerned with point (v) also follows straightforwardly leading to the conclusion that the identity therein is equivalent to $\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^2)$, which, as already remarked in the paper, can alternatively be expressed as $\mathbf{B} = \mathbf{0}$.

To complete the proof we need to consider its condition (vi). It can be easily verified that it is satisfied if and only if

$$\mathrm{rk}(\mathbf{A}) + \mathrm{tr}(\overline{\mathbf{A}}^\dagger) = r + \mathrm{rk}(\mathbf{B}). \tag{10}$$

In the light of the relationships provided right below identity (9), as well as the fact that $\mathrm{tr}(\overline{\mathbf{A}}^\dagger) = r - k - l + \sum_{i=1}^{l} \frac{1}{1-\alpha_i}$, we conclude that (10) is equivalent to $\sum_{i=1}^{l} \frac{1}{1-\alpha_i} = l$. However, this identity cannot be satisfied if α_i, $i = 1, \ldots, l$, belong to the set $(0, 1)$. Thus, it is seen that all nonzero eigenvalues of $\mathbf{A}$ are equal to 1, i.e., $\mathbf{B} = \mathbf{0}$. ∎

Observe that the equivalences between the requirement that $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$ and the conditions (iii) and (v) of Theorem 12 were established (by exploiting different approach) in [5, Theorem 10], where also other properties of the function $\mathbf{I}_n - \mathbf{PQ}$ were listed.

It is an easy exercise to show that

$$\mathbf{PQ} - \mathbf{QP} \in \mathbb{C}_n^{\mathsf{OP}} \;\Leftrightarrow\; \mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}.$$

Three rank/trace identities equivalent to $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$, each of which involves the difference $\mathbf{PQ} - \mathbf{QP}$, are given in the following theorem.

Theorem 13. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then the following conditions are equivalent:*

(i) $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$,
(ii) $\mathbf{PQ} - \mathbf{QP} \in \mathbb{C}_n^{\mathsf{OP}}$,
(iii) $\mathrm{rk}(\mathbf{PQ} - \mathbf{QP}) = \mathrm{tr}(\mathbf{PQ} - \mathbf{QP})$,
(iv) $\mathrm{rk}(\mathbf{PQ} - \mathbf{QP}) = \mathrm{tr}[(\mathbf{PQ} - \mathbf{QP})^\dagger]$,
(v) $\mathrm{tr}(\mathbf{PQ} - \mathbf{QP}) = \mathrm{tr}[(\mathbf{PQ} - \mathbf{QP})(\mathbf{PQ} - \mathbf{QP})^*]$.

Proof. Recall that the formula for rank of $\mathbf{PQ} - \mathbf{QP}$ was provided in Lemma 2. Now, it can be easily verified that $\mathrm{tr}(\mathbf{PQ} - \mathbf{QP}) = 0$, $\mathrm{tr}[(\mathbf{PQ} - \mathbf{QP})^\dagger] = 0$, and $\mathrm{tr}[(\mathbf{PQ} - \mathbf{QP})(\mathbf{PQ} - \mathbf{QP})^*] = 2\mathrm{tr}(\mathbf{BB}^*)$. Hence, the equivalences between each of the conditions (iii)–(v) of the theorem and $\mathbf{B} = \mathbf{0}$ is clearly seen. ∎

It can be shown without a substantial effort in the present setup that

$$\mathbf{PQ} + \mathbf{QP} \in \mathbb{C}_n^{\mathsf{OP}} \ \Leftrightarrow \ \mathbf{P} + \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}} \ \Leftrightarrow \ \mathbf{PQ} + \mathbf{QP} = \mathbf{0} \ \Leftrightarrow \ \mathbf{PQ} = \mathbf{0};$$

the result is also known in the literature; see e.g., [7, Theorems 1, 2, and 4]. Hence, it is not too interesting to characterize the condition $\mathbf{PQ} + \mathbf{QP} \in \mathbb{C}_n^{\mathsf{OP}}$ in terms of the sum $\mathbf{PQ} + \mathbf{QP}$. However, in the light of the equivalence

$$\mathbf{PQ} + \mathbf{QP} \in \mathbb{C}_n^{\mathsf{OP}} \ \Leftrightarrow \ \mathbf{I}_n - \mathbf{P} - \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}},$$

whose validity can be verified directly, it seems reasonable to use for this purpose the function $\mathbf{I}_n - \mathbf{P} - \mathbf{Q}$.

Theorem 14. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then the following conditions are equivalent:*

(i) $\mathbf{P} + \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$,
(ii) $\mathbf{I}_n - \mathbf{P} - \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$,
(iii) $\mathrm{rk}(\mathbf{I}_n - \mathbf{P} - \mathbf{Q}) = \mathrm{tr}[(\mathbf{I}_n - \mathbf{P} - \mathbf{Q})^\dagger]$.

Proof. On account of $\mathrm{tr}(\mathbf{P_A}) = \mathrm{rk}(\mathbf{P_A}) = \mathrm{rk}(\mathbf{A})$ and $\mathrm{tr}(\mathbf{P}_{\overline{\mathbf{D}}}) = \mathrm{rk}(\mathbf{P}_{\overline{\mathbf{D}}}) = \mathrm{rk}(\overline{\mathbf{D}})$, as well as the formula for $\mathrm{rk}(\overline{\mathbf{D}})$ given in Lemma 2, we conclude that the condition in point (iii) of the theorem is satisfied if and only if $\mathbf{A} = \mathbf{0}$. The proof is, thus, complete. ∎

The lemma below indicates that the list of three conditions provided in Lemma 8 can be extended by an additional one determined by the function $\mathbf{I}_n - \mathbf{P} - \mathbf{Q}$.

Lemma 15. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then* $\mathbf{P} + \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *if and only if* $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$ *holds along with* $\mathrm{rk}(\mathbf{I}_n - \mathbf{P} - \mathbf{Q}) = \mathrm{tr}(\mathbf{I}_n - \mathbf{P} - \mathbf{Q})$.

Proof. Direct calculations show that $\mathrm{rk}(\mathbf{I}_n - \mathbf{P} - \mathbf{Q}) = \mathrm{tr}(\mathbf{I}_n - \mathbf{P} - \mathbf{Q})$ holds if and only if

$$\mathrm{rk}(\mathbf{A}) + \mathrm{tr}(\mathbf{A}) + \mathrm{rk}(\mathbf{B}) = \mathrm{rk}(\mathbf{D}) - \mathrm{tr}(\mathbf{D}).$$

It is clear that combining this identity with $\mathbf{B} = \mathbf{0}$ entails $\mathbf{A} = \mathbf{0}$. ∎

By exploiting the present setup, it can be proved that $\mathbf{P}+\mathbf{Q}-\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$ is satisfied if and only if $\mathbf{B}=\mathbf{0}$. In consequence,

$$\mathbf{P}+\mathbf{Q}-\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}} \;\Leftrightarrow\; \mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}.$$

The paper is concluded with a theorem recognizing two equalities involving rank and trace of functions of $\mathbf{I}_n-\mathbf{P}-\mathbf{Q}$, each of which is equivalent to the requirement that $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$.

Theorem 16. *Let* $\mathbf{P},\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then the following conditions are equivalent:*

(i) $\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$,
(ii) $\mathbf{P}+\mathbf{Q}-\mathbf{PQ} \in \mathbb{C}_n^{\mathsf{OP}}$,
(iii) $\mathrm{rk}(\mathbf{P}+\mathbf{Q}-\mathbf{PQ}) = \mathrm{tr}(\mathbf{P}+\mathbf{Q}-\mathbf{PQ})$,
(iv) $\mathrm{rk}(\mathbf{P}+\mathbf{Q}-\mathbf{PQ}) = \mathrm{tr}[(\mathbf{P}+\mathbf{Q}-\mathbf{PQ})^\dagger]$.

Proof. Direct calculations lead to the conclusions that the conditions in points (iii) and (iv) of the theorem are equivalent to $\mathrm{rk}(\mathbf{D}) = \mathrm{tr}(\mathbf{D})$ and $\mathrm{rk}(\mathbf{D}) = \mathrm{tr}(\mathbf{D}^\dagger)$, respectively. In the light of the representation (5), we see that each of the two equalities is satisfied merely when $\mathbf{B}=\mathbf{0}$. ■

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Appendix

In what follows we provide the representations of the Moore–Penrose inverses of selected functions of orthogonal projectors $\mathbf{P}$ and $\mathbf{Q}$ having the forms (2) and (3), respectively.

$$(\mathbf{PQ})^{\dagger} = \mathbf{U}\begin{pmatrix} \mathbf{P_A} & \mathbf{0} \\ \mathbf{B}^*\mathbf{A}^{\dagger} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

$$(\mathbf{P}+\mathbf{Q})^{\dagger} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r - \frac{1}{2}\overline{\mathbf{P}}_{\overline{\mathbf{A}}} & -\mathbf{BD}^{\dagger} \\ -\mathbf{D}^{\dagger}\mathbf{B}^* & 2\mathbf{D}^{\dagger} - \mathbf{P_D} \end{pmatrix}\mathbf{U}^*,$$

$$(\mathbf{P}-\mathbf{Q})^{\dagger} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{A}}} & -\mathbf{BD}^{\dagger} \\ -\mathbf{D}^{\dagger}\mathbf{B}^* & -\mathbf{P_D} \end{pmatrix}\mathbf{U}^*,$$

$$(\mathbf{PQP})^{\dagger} = \mathbf{U}\begin{pmatrix} \mathbf{A}^{\dagger} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

$$(\mathbf{I}_n - \mathbf{PQ})^{\dagger} = \mathbf{U}\begin{pmatrix} \overline{\mathbf{A}} & -\mathbf{B} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*,$$

$$(\mathbf{PQ}+\mathbf{QP})^{\dagger} = \mathbf{U}\begin{pmatrix} \frac{1}{2}\mathbf{A}^{\dagger} - \frac{1}{2}\mathbf{A}^{\dagger}\mathbf{B}(\mathbf{B}^*\mathbf{A}^{\dagger}\mathbf{B})^{\dagger}\mathbf{B}^*\mathbf{A}^{\dagger} & \mathbf{A}^{\dagger}\mathbf{B}(\mathbf{B}^*\mathbf{A}^{\dagger}\mathbf{B})^{\dagger} \\ (\mathbf{B}^*\mathbf{A}^{\dagger}\mathbf{B})^{\dagger}\mathbf{B}^*\mathbf{A}^{\dagger} & -2(\mathbf{B}^*\mathbf{A}^{\dagger}\mathbf{B})^{\dagger} \end{pmatrix}\mathbf{U}^*,$$

$$(\mathbf{PQ}-\mathbf{QP})^{\dagger} = \mathbf{U}\begin{pmatrix} \mathbf{0} & -(\mathbf{B}^*)^{\dagger} \\ \mathbf{B}^{\dagger} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

$$(\mathbf{I}_n - \mathbf{P} - \mathbf{Q})^{\dagger} = \mathbf{U}\begin{pmatrix} -\mathbf{P_A} & -\mathbf{A}^{\dagger}\mathbf{B} \\ -\mathbf{B}^*\mathbf{A}^{\dagger} & \mathbf{P}_{\overline{\mathbf{D}}} \end{pmatrix}\mathbf{U}^*,$$

$$(\mathbf{P}+\mathbf{Q}-\mathbf{PQ})^{\dagger} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ -\mathbf{D}^{\dagger}\mathbf{B}^* & \mathbf{D}^{\dagger} \end{pmatrix}\mathbf{U}^*.$$

Validity of these representations can be verified by exploiting the four Penrose conditions given in (1). Details on how most of these representations were derived can be found in articles [3, 4] and [6, 7].