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ALL REGULAR-SOLID VARIETIES OF IDEMPOTENT SEMIRINGS

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Abstract

The lattice of all regular-solid varieties of semirings splits in two complete sublattices: the sublattice of all idempotent regular-solid varieties of semirings and the sublattice of all normal regular-solid varieties of semirings. In this paper, we discuss the idempotent part.

Keywords: semiring, hypersubstitution, regular hypersubstitution, regular hyperidentity, solid variety, regular-solid variety.

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1. Introduction

Varieties of semirings are varieties of algebras of type $(2, 2)$, where both binary operations are associative and satisfy the two usual distributive laws. Single semirings as well as classes of semirings form important structures in Automata and Formal Languages Theories [5]. To get more insight into the complete lattice of all varieties of semirings, all solid and all pre-solid varieties of semirings were determined [1, 2]. Now, we are interested in the complete lattice of all regular-solid varieties of semirings by characterizing all regular-solid varieties of idempotent semirings. To achieve our aim, we recall some basic concepts.

Let F and G be the both binary operation symbols and let $W_{(2,2)}(X_2)$ be the set of all binary terms of type $(2, 2)$ built up by variables from the alphabet $X_2 = \{x, y\}$. *Hypersubstitutions of type* $(2, 2)$ are mappings

$$\sigma : \{F, G\} \to W_{(2,2)}(X_2).$$

The set of all hypersubstitutions of type $(2, 2)$ will be denoted by Hyp. A hypersubstitution $\sigma \in Hyp$ can be extended on the set $W_{(2,2)}(X)$ of all terms of type $(2, 2)$, where X is an arbitrary countably infinite alphabet of variables, by the following steps:

(i) $\hat{\sigma}[t] := t$, if $t \in X$,

(ii) $\hat{\sigma}[t] := \sigma(f)(\hat{\sigma}[t_1], \hat{\sigma}[t_2])$, if $t = f(t_1, t_2) \in W_{(2,2)}(X)$ with $f \in \{F, G\}$, where $\sigma(f)$ can be interpreted as the term operation $\sigma(f)^{\mathcal{F}_{(2,2)}(X)}$ induced by the term $\sigma(f)$ on the free algebra $\mathcal{F}_{(2,2)}(X) := (W_{(2,2)}(X); (\overline{F}, \overline{G}))$ with $\overline{f} : (W_{(2,2)}(X))^2 \to W_{(2,2)}(X)$, $(t_1, t_2) \mapsto f(t_1, t_2)$.

It is easy to prove that the algebra $(Hyp; \circ_h, \sigma_{id})$, is a monoid with $\circ_h$ (where $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ and $\circ$ is the usual mapping composition) as binary operation and σ_{id}, defined by $\sigma_{id}(f) := f(x, y)$ for all $f \in \{F, G\}$, as an identity element. Hypersubstitutions can be applied to algebras as follows: given an algebra $\mathcal{A} = (\mathcal{A}; (\mathcal{F}^{\mathcal{A}}, \mathcal{G}^{\mathcal{A}}))$ of type $(2, 2)$ and a hypersubstitution $\sigma \in Hyp$, one defines the algebra $\sigma(\mathcal{A}) := (\mathcal{A}; (\sigma(\mathcal{F})^{\mathcal{A}}, \sigma(\mathcal{G})^{\mathcal{A}}))$. This algebra of type $(2, 2)$ is called the derived algebra by $\mathcal{A}$ and σ.

The hypersubstitution $\sigma \in Hyp$ such that $\sigma(F) = t$ and $\sigma(G) = s$ will be denoted by $\sigma_{t,s}$. For all variables u and v, the term $F(u, v)$ and $G(u, v)$ will be denoted by $u + v$ and uv, respectively.

A hypersubstitution $\sigma \in Hyp$ is called a *regular hypersubstitution* if σ maps both F and G to binary terms containing both variables x and y. It is easy to verify that the set Reg of all regular hypersubstitutions of type $(2, 2)$ forms a submonoid of the monoid Hyp. An identity $s \approx t$ in a variety V of semirings is called a *regular hyperidentity* if for every $\sigma \in Reg$, the equation $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ belongs to the set IdV of all identities satisfied in V. A variety V of semirings is called regular-solid if all identities in V are satisfied as regular hyperidentities. For more information about hypersubstitutions and varieties of algebras see in [3, 7].

In the next section, we will provide some necessary conditions for a variety of semirings to be a regular-solid one. This leads to a description of the lattice of all regular-solid varieties of semirings. The last section will be devoted to the determination of the lattice of all regular-solid varieties of idempotent semirings.

2. Some Properties

A variety V of semirings is medial if $x + y + z + u \approx x + z + y + u \in IdV$ and $xyzu \approx xzyu \in IdV$, idempotent if $x + x \approx x \approx x^2 \in IdV$, distributive if $xy + z \approx (x + z)(y + z) \in IdV$ and $x + yz \approx (x + y)(x + z) \in IdV$.

An equation $s \approx t$ is called normal if either both terms s and t are equal to the same variable or none of them is a variable, that is, if $s = t$ or the complexity (number of occurrences of operation symbols) of both terms s and t is greater or equal to 1. A variety in which all identities are normal is called a normal variety.

Now, we can derive some necessary conditions for varieties of semirings to be regular-solid.

Proposition 1. *Let V be a regular-solid variety of semirings. The following properties are:*

(1) *V is medial, distributive and satisfies the identities:*

 (i) $x^2yz \approx xy^2z \approx xyz^2 \approx xyz,$

 (ii) $2x + y + z \approx x + 2y + z \approx x + y + 2z \approx x + y + z.$

(2) *V is either idempotent or normal.*

Proof. (1) It is clear that the usual distributive laws are satisfied in V. The application of the regualar hypersubstitutions $\sigma_{xy,x+y}$ to them gives the other distributive laws since V is a regular-solid variety of semirings. Moreover, applying the regular hypersubstitutions $\sigma_{xy,xy}$ and $\sigma_{yx,yx}$ to the distributive law

$$x(y + z) \approx xy + xz,$$

of V, we get the identities

$$xyz \approx xyxz \text{ and } zyx \approx zxyx, \text{ respectively, in } V.$$

It is folklore that the identities $xyz \approx xyxz \approx xzyz$ imply the medial law $xyzu \approx xzyu$ and the identities $xyz \approx x^2yz \approx xy^2z \approx xyz^2$. The application of the regular hypersubstitution $\sigma_{xy,x+y}$ to these identities gives the remaining identities.

(2) Suppose that $t \approx x$ is an identity in V which is not normal. This provides $x^k \approx x \in IdV$ for some $k \geq 2$ (by using the regular hypersubstitution $\sigma_{xy,xy}$ and identifying all variables with x). From the identity $x^2yz \approx xyz \in IdV$, we get $x^4 \approx x^3 \in IdV$ and together with $x^k \approx x \in IdV$, we obtain the idempotent law $x^2 \approx x \in IdV$. Therefore, V is idempotent by using the regular hypersubstitution $\sigma_{xy,x+y}$. ■

Proposition 1 (2), leads to a description of the complete lattice $Reg(Sr)$ of all regular-solid varieties of semirings. Denoting by $\mathcal{L}(2,2)$ the lattice of all varieties of type $(2,2)$, we have:

Corollary 2. *The lattice $Reg(Sr)$ splits into two complete sublattices of $\mathcal{L}(2,2)$, the sublattice $Reg_{Idem}(Sr)$ of all idempotent regular-solid varieties of semirings and the sublattice $Reg_N(Sr)$ of all normal regular-solid varieties of semirings.*

Proof. The lattice $\mathcal{L}_N(2,2)$ of all normal varieties of type $(2,2)$ and the lattice $\mathcal{L}_{Idem}(2,2)$ of all idempotent varieties of type $(2,2)$ are complete sublattices of $\mathcal{L}(2,2)$ (see [4, 7]). Therefore, since $Reg_N(Sr) = Reg(Sr) \cap \mathcal{L}_N(2,2)$ (the intersection of two complete sublattices) and since $Reg_{Idem}(Sr) = Reg(Sr) \cap \mathcal{L}_{Idem}(2,2)$ (the intersection of two complete sublattices), it arises that both lattices $Reg_{Idem}(Sr)$ and $Reg_N(Sr)$ are complete sublattices. By Proposition 1 (2) the lattices $Reg_{Idem}(Sr)$ and $Reg_N(Sr)$ are disjoint and their union is $Reg(Sr)$. ∎

3. All Regular-Solid Varieties of Idempotent Semirings

In this section, the lattice of all regular-solid varieties of idempotent semirings will be determined. An equation $s \approx t$ is outermost if the terms s and t start with the same variable (we write $leftmost(s) = leftmost(t)$) and end also with the same variable (we write $rightmost(s) = rightmost(t)$). A variety V is called outermost if all equations in IdV are outermost. A variety V of semirings is commutative if $x + y \approx y + x \in IdV$ and $xy \approx yx \in IdV$. The following result gives a description of idempotent regular-solid varieties of semirings.

Proposition 3. *Each idempotent regular-solid variety of semirings is either outermost or commutative.*

Proof. Let V be an idempotent regular-solid variety of semirings. Assume that V is not outermost. We will show that V is commutative. Since V is not outermost, without loss of generality, we can assume that there exists an equation $s \approx t$ in IdV such that $leftmost(s) = x \neq y = leftmost(t)$. Applying the regular hypersubstitution $\sigma_{xy,xy}$ to the identity $s \approx t \in IdV$, we get the following identity $s_1 \approx t_1$ in V (where $leftmost(s_1) = x \neq y = leftmost(t_1)$). Let us consider the function $h : X \to W_{(2,2)}(X),\ w \mapsto \begin{cases} x & \text{if } w = x \\ y & \text{otherwise.} \end{cases}$
It is well known that this function can be uniquely extended to an endomorphism $\overline{h}$ on $\mathcal{F}_{(\in,\in)}(\mathcal{X})$. Then, $\overline{h}(s_1) \approx \overline{h}(t_1) \in IdV$ and $\overline{h}(s_1)yx \approx \overline{h}(t_1)yx \in IdV$, so $xyx \approx yx \in IdV$ because of the idempotent law. Applying the regular hypersubstitution $\sigma_{yx,yx}$ to the latter identity, the following equations $xy \approx xyx \approx yx$ hold in V as identities. The application of $\sigma_{xy,x+y}$ to $xy \approx yx$ shows that V is commutative. ∎

Now, we determine the commutative part of $Reg_{Idem}(Sr)$. Proposition 1 (1) shows that every regular-solid variety of idempotent semirings is a subvariety of the variety V_{MID} of all medial idempotent and distributive semirings. But the subvariety lattice of V_{MID} is fully described by Pastjin in [6] as follows:

Let us consider the two-element algebras (using the same notations as in [6]):

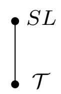
$\mathcal{A} \ = (\{0,1\};\ e_1^2,\ e_1^2)$, e_1^2 is the binary projection $\{0,1\}^2 \to \{0,1\}$ on the first input;
$\mathcal{A}^\circ = (\{0,1\};\ e_2^2,\ e_2^2)$, e_2^2 is the binary projection $\{0,1\}^2 \to \{0,1\}$ on the second input;
$\mathcal{B} \ = (\{0,1\};\ e_1^2, \wedge)$, where $\wedge$ denotes the conjunction;
$\mathcal{B}^\circ = (\{0,1\};\ e_2^2, \wedge)$;
$\mathcal{B}^\bullet = (\{0,1\};\ \wedge, e_1^2)$;
$\mathcal{B}^{\bullet\circ} = (\{0,1\};\ \wedge, e_2^2)$;
$\mathcal{F} \ = (\{0,1\};\ e_1^2,\ {}^2_2)$;
$\mathcal{F}^\circ = (\{0,1\};\ e_2^2,\ e_1^2)$;
$\mathcal{J} \ = (\{0,1\};\ \wedge, \vee)$, where $\vee$ denotes the disjunction;
$\mathcal{L} \ = (\{0,1\};\ \wedge, \wedge)$.

The algebra $\mathcal{J}$ generates the variety DL of all distributive lattices and $\mathcal{L}$ generates the variety SL of bi-semilattices. Then we have

Lemma 4 [6]**.** *The subvariety lattice of the variety V_{MID} of all medial idempotent and distributive semirings is a Boolean lattice with* 10 *atoms and* 10 *dual atoms, i.e., with* 2^{10} *elements. The atoms are exactly the varieties $V(\mathcal{A})$, $V(\mathcal{A}^\circ)$, $V(\mathcal{B})$, $V(\mathcal{B}^\circ), V(\mathcal{B}^\bullet)$, $V(\mathcal{B}^{\bullet\circ})$, $V(\mathcal{F})$, $V(\mathcal{F}^\circ), DL$ and SL, where $V(K)$ is the variety generated by a given algebra K of type* $(2,2)$. ∎

Therefore, each subvariety of V_{MID} is a join of some of these 10 atoms.
An equation $s \approx t$ is said to be regular if both terms s and t use the same variables and a variety of semirings is regular if all identities in that variety are regular. The lattice of all regular-solid varieties of commutative and idempotent semirings is determined as follows:

Theorem 5. *The two-element lattice*



is the lattice of all regular-solid varieties of commutative and idempotent semirings, where $\mathcal{T} = Mod\{x \approx y\}$ is the trivial variety of type $(2,2)$.

Proof. Let V be a regular-solid variety of commutative and idempotent semirings. By Proposition 1 (1), the variety V is a commutative subvariety of V_{MID}. So V is either trivial or a join of some commutative atoms listed in Lemma 4. This means that either V is trivial or $V \in \{SL, DL, SL \vee DL\}$. But the varieties DL and $SL \vee DL$ are not regular-solid. Indeed, the application of $\sigma_{x+xy,x+xy}$ to the commutative identity $xy \approx yx$ gives the identity $x+xy \approx y+yx$ which cannot be satisfied in DL because of the absorption laws. $IdSL$ is the set of all regular identities of type $(2,2)$. It is clear that applying regular hypersubstitution to any regular identity, one gets a regular identity. So SL is regular-solid. ■

We are now interested in the outermost part of $Reg_{Idem}(Sr)$. Some definitions and facts will be referred.

Definition. A variety V of semirings is s-outermost if for any identity $s \approx t \in IdV$, the equations $s \approx t$ as well as $\hat{\sigma}_{x+y,yx}[s] \approx \hat{\sigma}_{x+y,yx}[t]$ are outermost.

This definition coincides with that one given in [1] and it is clear that every outermost regular-solid variety of semirings is s-outermost since the hypersubstitution $\sigma_{x+y,yx}$ is regular.

A variety V of semirings is said to be a solid variety if for all $s \approx t \in IdV$ and for all $\sigma \in Hyp$, we get $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$. It is well known that the variety $RA_{(2,2)}$ generated by all projection algebras of type $(2,2)$ is a variety of semirings and it is defined by $RA_{(2,2)} = Mod\{(xy)z \approx x(yz) \approx xz, (x+y)+z \approx x+(y+z) \approx x+z, (x+y)(z+u) \approx xz+yu, x^2 \approx x \approx x+x\}$ [1]. It is already proved:

Lemma 6 [1]**.** *The lattice of all solid varieties of semirings is the four-element chain represented by* $\mathcal{T} \subset RA_{(2,2)} \subset V_{BE} \subset V_{MID}$, *where*

$$RA_{(2,2)} = V(\mathcal{A}) \vee V(\mathcal{A}^{\circ}) \vee V(\mathcal{F}) \vee V(\mathcal{F}^{\circ}) \quad \textit{and}$$
$$V_{BE} = RA_{(2,2)} \vee SL \vee V(\mathcal{B}) \vee V(\mathcal{B}^{\circ}) \vee V(\mathcal{B}^{\bullet}) \vee V(\mathcal{B}^{\bullet\circ}).$$

Moreover, it holds

Lemma 7 [1]**.** *The variety* $RA_{(2,2)}$ *is the least s-outermost variety of semirings.*

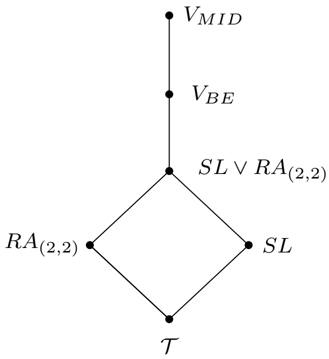
Now, we can prove:

Lemma 8. *Let* V *be an outermost regular-solid variety of idempotent semirings. If* V *is different from* $RA_{(2,2)}$ *then* V *is regular i.e all equations in* IdV *are regular.*

Proof. We will prove that if V is not regular then $V = RA_{(2,2)}$. Since V is outermost regular-solid variety of semirings, V is s-outermost and we have $RA_{(2,2)} \subseteq V$ (Lemma 7). It left to prove that $V \subseteq RA_{(2,2)}$ i.e $Id(RA_{(2,2)}) \subseteq IdV$. Since V is not regular, there exists an identity $s \approx t$ in IdV such that, without loss of generality, a variable x_i occurs in s but not in t. Applying $\sigma_{xy,xy}$ to $s \approx t$ and identifying all variables different from x_i with x, we get $xx_ix \approx x \in IdV$ because V is outermost and idempotent. Therefore, $xyz \approx xz \in IdV$. The application of $\sigma_{xy,x+y}$ to this identity gives $x+y+z \approx x+z \in IdV$. Moreover, using the previous identity, the distributivity and the idempotency, the basis identities of $RA_{(2,2)}$ are also identities in V. This finishes the proof of $Id(RA_{(2,2)}) \subseteq IdV$. ■

Now, we have all tools to prove our main result:

Theorem 9. *The lattice of all regular-solid varieties of idempotent semirings is the lattice*



Proof. Let V be a regular-solid variety of idempotent semirings. Then V is either commutative or outermost (Proposition 3).

If V is commutative then $V \in \{\mathcal{T}, SL\}$ (Theorem 5). Otherwise, V is outermost. Then $V = RA_{(2,2)}$ or V is regular (Lemma 8). Therefore, $V = RA_{(2,2)}$ or $SL \subseteq V$ since $Id(SL)$ is the set of all regular identities of type $(2,2)$. Moreover, V is s-outermost and thus $RA_{(2,2)} \subseteq V$ (Lemma 7). Altogether, we have $V = RA_{(2,2)}$ or $RA_{(2,2)} \vee SL \subseteq V$.

Let σ_i, $i = 1,2,3,4$, be hypersubstitutions defined by

$$\begin{array}{llllllll} \sigma_1\colon & F \mapsto F(y,x) & \sigma_2\colon & F \mapsto G(x,y) & \sigma_3: & F \mapsto G(x,y) & \sigma_4: & F \mapsto G(y,x) \\ & G \mapsto G(x,y) & & G \mapsto F(x,y) & & G \mapsto F(y,x) & & G \mapsto F(x,y). \end{array}$$

Then $\mathcal{B}^\circ = \sigma_1(\mathcal{B})$, $\mathcal{B} = \sigma_1(\mathcal{B}^\circ)$, $\mathcal{B}^\bullet = \sigma_2(\mathcal{B})$, $\mathcal{B} = \sigma_2(\mathcal{B}^\bullet)$, $\mathcal{B}^{\bullet\circ} = \sigma_3(\mathcal{B})$ and $\mathcal{B} = \sigma_4(\mathcal{B}^{\bullet\circ})$. Since a regular-solid variety has to contain all its derived algebras by using regular hypersubstitutions, all of the varieties $V(\mathcal{B})$, $V(\mathcal{B}^\circ), V(\mathcal{B}^\bullet)$ and $V(\mathcal{B}^{\bullet\circ})$ are contained in the variety V if it contains one of them. It follows that V_{BE} is the only one dual atom of V_{MID} which is a regular-solid variety of semirings, since V_{BE} is solid and $DL \not\subseteq V_{BE}$ (Lemma 4 and Lemma 6).

Therefore, $V \in \{RA_{(2,2)}, RA_{(2,2)} \vee SL, V_{BE}, V_{MID}\}$. Each element of the previous set is a regular-solid variety of semirings (by using Theorem 5, Lemma 6 and the fact that $RA_{(2,2)} \vee SL$ is a join of two regular-solid varieties). ■

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