ONE MORE TURÁN NUMBER AND RAMSEY NUMBER FOR THE LOOSE 3-UNIFORM PATH OF LENGTH THREE

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Abstract

Let $P$ denote a 3-uniform hypergraph consisting of 7 vertices $a, b, c, d, e, f, g$ and 3 edges $\{a, b, c\}, \{c, d, e\}, \{e, f, g\}$. It is known that the $r$-color Ramsey number for $P$ is $R(P; r) = r + 6$ for $r \leq 9$. The proof of this result relies on a careful analysis of the Turán numbers for $P$. In this paper, we refine this analysis further and compute the fifth order Turán number for $P$, for all $n$. Using this number for $n = 16$, we confirm the formula $R(P; 10) = 16$.

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1. Introduction

For the sake of brevity, 3-uniform hypergraphs will be called here 3-graphs. Given a family of 3-graphs $\mathcal{F}$, we say that a 3-graph $H$ is $\mathcal{F}$-free if for all $F \in \mathcal{F}$ we have $H \nsubseteq F$.

For a family of 3-graphs $\mathcal{F}$ and an integer $n \geq 1$, the Turán number of the 1st order, that is, the ordinary Turán number, is defined as

$$\text{ex}(n; \mathcal{F}) = \text{ex}^{(1)}(n; \mathcal{F}) = \max \{|E(H)| : |V(H)| = n \text{ and } H \text{ is } \mathcal{F}\text{-free}\}.$$ 

Every $n$-vertex $\mathcal{F}$-free 3-graph with $\text{ex}^{(1)}(n; \mathcal{F})$ edges is called 1-extremal for $\mathcal{F}$. We denote by $\text{Ex}^{(1)}(n; \mathcal{F})$ the family of all, pairwise non-isomorphic, $n$-vertex 3-graphs which are 1-extremal for $\mathcal{F}$. Further, for an integer $s \geq 1$, the Turán number of the $(s + 1)$-st order is defined as

$$\text{ex}^{(s+1)}(n; \mathcal{F}) = \max \{|E(H)| : |V(H)| = n, H \text{ is } \mathcal{F}\text{-free, and } \forall H' \in \text{Ex}^{(1)}(n; \mathcal{F}) \cup \cdots \cup \text{Ex}^{(s)}(n; \mathcal{F}), H \nsubseteq H'\},$$
if such a 3-graph $H$ exists. Note that if $\text{ex}^{(s+1)}(n; \mathcal{F})$ exists then, by definition,

$$\text{ex}^{(s+1)}(n; \mathcal{F}) < \text{ex}^{(s)}(n; \mathcal{F}).$$

An $n$-vertex $\mathcal{F}$-free 3-graph $H$ is called $(s + 1)$-extremal for $\mathcal{F}$ if $|E(H)| = \text{ex}^{(s+1)}(n; \mathcal{F})$ and for every $H' \in \text{Ex}^{(1)}(n; \mathcal{F}) \cup \cdots \cup \text{Ex}^{(s)}(n; \mathcal{F}), H \not\subseteq H'$; we denote by $\text{Ex}^{(s+1)}(n; \mathcal{F})$ the family of $n$-vertex 3-graphs which are $(s + 1)$-extremal for $\mathcal{F}$. In the case when $\mathcal{F} = \{F\}$, we will write $F$ instead of $\{F\}$.

A loose 3-uniform path of length 3 is a 3-graph $P$ consisting of 7 vertices, say, $a, b, c, d, e, f, g$, and 3 edges $\{a, b, c\}, \{c, d, e\}$, and $\{e, f, g\}$. The Ramsey number $R(P; r)$ is the least integer $n$ such that every $r$-coloring of the edges of the complete 3-graph $K_n$ results in a monochromatic copy of $P$. Gyárfás and Räisä [6] proved, among many other results, that $R(P; 2) = 8$. (This result was later extended to loose paths of arbitrary lengths, but still $r = 2$ in [13]). Then Jackowska [9] showed that $R(P; 3) = 9$ and $r + 6 \leq R(P; r)$ for all $r \geq 3$. In turn, in [10, 11] and [15], the Turán numbers of the first four orders, $\text{ex}^{(i)}(n; P)$, $i = 1, 2, 3, 4$, have been determined for all feasible values of $n$. Using these numbers, in [11] and [15], we were able to compute the Ramsey numbers $R(P; r)$ for $4 \leq r \leq 9$.

**Theorem 1** [6, 9, 11, 15]. For all $r \leq 9$, $R(P; r) = r + 6$.

In this paper we determine, for all $n \geq 7$, the Turán numbers for $P$ of the fifth order, $\text{ex}^{(5)}(n; P)$. This allows us to compute one more Ramsey number.

**Theorem 2.** $R(P; 10) = 16$.

It seems that in order to make a further progress in computing the Ramsey numbers $R(P; r)$, $r \geq 11$, one would need to determine still higher order Turán numbers $\text{ex}^{(s)}(n; P)$, at least for some small values of $n$.

Throughout, we denote by $S_n$ the 3-graph on $n$ vertices and with $\binom{n-1}{2}$ edges, in which one vertex, referred to as the center, forms edges with all pairs of the remaining vertices. Every sub-3-graph of $S_n$ without isolated vertices is called a star, while $S_n$ itself is called the full star. We denote by $C$ the triangle, that is, a 3-graph with six vertices $a, b, c, d, e, f$ and three edges $\{a, b, c\}, \{c, d, e\}$, and $\{e, f, a\}$. Finally, $M$ stands for a pair of disjoint edges. For a given 3-graph $H$ and a vertex $v \in V(G)$ we denote by $\deg_H(v)$ the number of edges in $H$ containing $v$.

In the next section we state some known and new results on Turán numbers for $P$, including Theorem 11 which provides a complete formula for $\text{ex}^{(5)}(n; P)$. We also define conditional Turán numbers and quote from [11] and [14] some useful lemmas about the conditional Turán numbers with respect to $P, C, M$. Then, in Section 3, we prove Theorem 2, while the remaining sections are devoted to proving Theorem 11.
2. Turán Numbers

We restrict ourselves exclusively to the case $k = 3$ only. A celebrated result of Erdős, Ko, and Rado [2] asserts, in the case of $k = 3$, that for $n \geq 6$, $\text{ex}^{(1)}(n; M) = \binom{n-1}{2}$. Moreover, for $n \geq 7$, $\text{Ex}^{(1)}(n; M) = \{S_n\}$. We will need the higher order versions of this Turán number, together with its extremal families. The second of these numbers has been found by Hilton and Milner, [8] (see [4] and [14] for a simple proof). For a given set of vertices $V$, with $|V| = n \geq 7$, let us define two special 3-graphs. Let $x, y, z, v \in V$ be four different vertices of $V$. We set

$$G_1(n) = \{\{x, y, z\}\} \cup \left\{ h \in \binom{V}{3} : v \in h, h \cap \{x, y, z\} \neq \emptyset \right\},$$

$$G_2(n) = \{\{x, y, z\}\} \cup \left\{ h \in \binom{V}{3} : |h \cap \{x, y, z\}| = 2 \right\}.$$

Note that for $i \in \{1, 2\}$, $M \not\subseteq G_i(n)$ and $|G_i(n)| = 3n - 8$.

**Theorem 3** [8]. For $n \geq 7$, $\text{ex}^{(2)}(n; M) = 3n - 8$ and $\text{Ex}^{(2)}(n; M) = \{G_1(n), G_2(n)\}$.

Later, we will use the fact that $C \subset G_i(n) \not\subset P$, $i = 1, 2$.

Recently, the third order Turán number for $M$ has been established for general $k$ by Han and Kohayakawa in [7]. Let $G_3(n)$ be the 3-graph on $n$ vertices, with distinguished vertices $x, y_1, y_2, z_1, z_2$ whose edge set consists of all edges spanned by $x, y_1, y_2, z_1, z_2$ except for $\{y_1, y_2, z_i\}, i = 1, 2$, and all edges of the form $\{x, z_i, v\}, i = 1, 2$, where $v \notin \{x, y_1, y_2, z_1, z_2\}$.

**Theorem 4** [7]. For $n \geq 7$, $\text{ex}^{(3)}(n; M) = 2n - 2$ and $\text{Ex}^{(3)}(n; M) = \{G_3(n)\}$.

For $k = 3$ we were able to take the next step and determine the next Turán number for $M$.

**Theorem 5** [14]. For $n \geq 7$, $\text{ex}^{(4)}(n; M) = n + 4$.

The number $\binom{n-1}{2}$ serves as the Turán number for two other 3-graphs, $C$ and $P$. The Turán number $\text{ex}^{(1)}(n; C)$ has been determined in [3] for $n \geq 75$ and later for all $n$ in [1].

**Theorem 6** [1]. For $n \geq 6$, $\text{ex}^{(1)}(n; C) = \binom{n-1}{2}$. Moreover, for $n \geq 8$, it holds $\text{Ex}^{(1)}(n; C) = \{S_n\}$.

In [10], we filled an omission of [5] and [12] and calculated $\text{ex}^{(1)}(n; P)$ for all $n$. Given two 3-graphs $F_1$ and $F_2$, by $F_1 \cup F_2$ denote a vertex-disjoint union of $F_1$ and $F_2$. If $F_1 = F_2 = F$ we will sometimes write $2F$ instead of $F \cup F$. 


Theorem 7 [10]. \( \text{ex}^{(1)}(n; P) = \)

\[
\begin{cases}
\binom{n}{2} & \text{and } \text{Ex}^{(1)}(n; P) = \{K_n\} \quad \text{for } n \leq 6, \\
20 & \text{and } \text{Ex}^{(1)}(n; P) = \{K_6 \cup K_1\} \quad \text{for } n = 7, \\
\binom{n-1}{2} & \text{and } \text{Ex}^{(1)}(n; P) = \{S_n\} \quad \text{for } n \geq 8.
\end{cases}
\]

In [11] we have completely determined the second order Turán number \( \text{ex}^{(2)}(n; P) \), together with the corresponding 2-extremal 3-graphs. A comet \( \text{Co}(n) \) is an \( n \)-vertex 3-graph consisting of the complete 3-graph \( K_4 \) and the full star \( S_{n-3} \), sharing exactly one vertex which is the center of the star (see Figure 1). This vertex is called the center of the comet, while the set of the remaining three vertices of the \( K_4 \) is called its head.

![Figure 1. The comet Co(n).](image)

Theorem 8 [11]. \( \text{ex}^{(2)}(n; P) = \)

\[
\begin{cases}
15 & \text{and } \text{Ex}^{(2)}(n; P) = \{S_7\} \quad \text{for } n = 7, \\
20 + \binom{n-6}{3} & \text{and } \text{Ex}^{(2)}(n; P) = \{K_6 \cup K_{n-6}\} \quad \text{for } 8 \leq n \leq 12, \\
40 & \text{and } \text{Ex}^{(2)}(n; P) = \{2K_6 \cup K_1, \text{Co}(13)\} \quad \text{for } n = 13, \\
4 + \binom{n-4}{2} & \text{and } \text{Ex}^{(2)}(n; P) = \{\text{Co}(n)\} \quad \text{for } n \geq 14.
\end{cases}
\]

In [11] \((n = 12)\) and in [15] (for all \( n \)), we calculated the third order Turán number for \( P \).

Theorem 9 [11, 15]. \( \text{ex}^{(3)}(n; P) = \)

\[
\begin{cases}
3n - 8 & \text{and } \text{Ex}^{(3)}(n; P) = \{G_1(n), G_2(n)\} \quad \text{for } 7 \leq n \leq 10, \\
25 & \text{and } \text{Ex}^{(3)}(n; P) = \{G_1(n), G_2(n), \text{Co}(n)\} \quad \text{for } n = 11, \\
32 & \text{and } \text{Ex}^{(3)}(n; P) = \{\text{Co}(n)\} \quad \text{for } n = 12, \\
20 + \binom{n-7}{3} & \text{and } \text{Ex}^{(3)}(n; P) = \{K_6 \cup S_{n-6}\} \quad \text{for } 13 \leq n \leq 14, \\
4 + \binom{n-5}{2} & \text{and } \text{Ex}^{(3)}(n; P) = \{K_4 \cup S_{n-4}\} \quad \text{for } n \geq 15.
\end{cases}
\]
Surprisingly, as an immediate consequence we obtained also an exact formula for the 4th Turán number for $P$. We define a rocket $Ro(n)$ to be the 3-graph obtained from the star $S_{n-4}$ with center $x$ by adding to it 4 more vertices, say, $a, b, c, d$, and three edges: $\{x, a, b\}, \{a, b, c\}, \{a, b, d\}$. Let $K_5^{+t}$ be the 3-graph obtained from $K_5$ by fixing two of its vertices, say $a, b$, and adding $t$ more vertices $v_1, v_2, \ldots, v_t$ and $t$ edges $\{a, v_i\}, i = 1, 2, \ldots, t$.

**Theorem 10** [15]. $\text{ex}^{(4)}(n; P) =$

\[
\begin{align*}
12 &\quad \text{and} \quad \text{ex}^{(4)}(n; P) = \{G_3(n), K_5^{+2}\} \quad \text{for } n = 7, \\
2n - 2 &\quad \text{and} \quad \text{ex}^{(4)}(n; P) = \{G_3(n)\} \quad \text{for } 8 \leq n \leq 9, \\
20 &\quad \text{and} \quad \text{ex}^{(4)}(n; P) = \{K_5 \cup K_5\} \quad \text{for } n = 10, \\
20 &\quad \text{and} \quad \text{ex}^{(4)}(n; P) = \{G_3(n)\} \quad \text{for } n = 11, \\
28 &\quad \text{and} \quad \text{ex}^{(4)}(n; P) = \{G_1(n), G_2(n)\} \quad \text{for } n = 12, \\
33 &\quad \text{and} \quad \text{ex}^{(4)}(n; P) = \{K_6 \cup G_1(n), K_5 \cup G_2(n)\} \quad \text{for } n = 13, \\
40 &\quad \text{and} \quad \text{ex}^{(4)}(n; P) = \{2K_6 \cup 2K_1, K_4 \cup S_{10}\} \quad \text{for } n = 14, \\
48 &\quad \text{and} \quad \text{ex}^{(4)}(n; P) = \{Ro(n), K_6 \cup S_9\} \quad \text{for } n = 15, \\
3 + \binom{n-5}{2} &\quad \text{and} \quad \text{ex}^{(4)}(n; P) = \{Ro(n)\} \quad \text{for } n \geq 16.
\end{align*}
\]

The main Turán-type result of this paper provides a complete formula for the fifth order Turán number for $P$.

**Theorem 11.** $\text{ex}^{(5)}(n; P) =$

\[
\begin{align*}
11 &\quad \text{and} \quad \text{ex}^{(5)}(n; P) = \text{ex}^{(4)}(7; M) \quad \text{for } n = 7, \\
13 &\quad \text{and} \quad \text{ex}^{(5)}(n; P) = \{K_5^{+3}\} \quad \text{for } n = 8, \\
14 &\quad \text{and} \quad \text{ex}^{(5)}(n; P) = \{K_5^{+4}, K_5 \cup K_4\} \cup \text{ex}(9; \{P, C\}|M) \quad \text{for } n = 9, \\
19 &\quad \text{and} \quad \text{ex}^{(5)}(n; P) = \{Co(10)\} \quad \text{for } n = 10, \\
19 &\quad \text{and} \quad \text{ex}^{(5)}(n; P) = \{K_4 \cup S_7\} \quad \text{for } n = 11, \\
25 &\quad \text{and} \quad \text{ex}^{(5)}(n; P) = \{K_5 \cup S_7, K_4 \cup S_8\} \quad \text{for } n = 12, \\
32 &\quad \text{and} \quad \text{ex}^{(5)}(n; P) = \{K_4 \cup S_9, K_6 \cup K_5^{+2}, K_6 \cup G_3(7)\} \quad \text{for } n = 13, \\
39 &\quad \text{and} \quad \text{ex}^{(5)}(n; P) = \{Ro(14)\} \quad \text{for } n = 14, \\
46 &\quad \text{and} \quad \text{ex}^{(5)}(n; P) = \{K_5 \cup S_{10}\} \quad \text{for } n = 15, \\
56 &\quad \text{and} \quad \text{ex}^{(5)}(n; P) = \{K_6 \cup S_{10}\} \quad \text{for } n = 16, \\
65 &\quad \text{and} \quad \text{ex}^{(5)}(n; P) = \{K_5 \cup S_{12}, K_5 \cup S_{11}\} \quad \text{for } n = 17, \\
10 + \binom{n-6}{2} &\quad \text{and} \quad \text{ex}^{(5)}(n; P) = \{K_5 \cup S_{n-5}\} \quad \text{for } n \geq 18.
\end{align*}
\]

To determine Turán numbers, it is sometimes useful to rely on Theorem 3 and divide all 3-graphs into those which contain $M$ and those which do not. To this end, it is convenient to define conditional Turán numbers (see [10, 11]). For a family of 3-graphs $\mathcal{F}$, an $\mathcal{F}$-free 3-graph $G$, and an integer $n \geq |V(G)|$, the conditional Turán number is defined as

\[
\text{ex}(n; \mathcal{F}|G) = \max\{|E(H)| : |V(H)| = n, \ H \text{ is } \mathcal{F}\text{-free, and } H \supseteq G\}.
\]
Every $n$-vertex $F$-free 3-graph with $\text{ex}(n; F|G)$ edges and such that $H \supseteq G$ is called $G$-extremal for $F$. We denote by $\text{Ex}(n; F|G)$ the family of all $n$-vertex 3-graphs which are $G$-extremal for $F$. (If $F = \{F\}$, we simply write $F$ instead of $\{F\}$.)

To illustrate the above mentioned technique, observe that for $n \geq 7$

$$\text{ex}^{(2)}(n; P) = \max\{\text{ex}(n; P|M), \text{ex}^{(2)}(n; M)\} \stackrel{\text{Thm. 3}}{=} \max\{\text{ex}(n; P|M), 3n - 8\} = \text{ex}(n; P|M),$$

the last equality holds for sufficiently large $n$ (see [11] for details).

In the proof of Theorem 11 we will use the following five lemmas, all proved in [11] and [14]. For the first two we need one more piece of notation. If, in the above definition, we restrict ourselves to connected 3-graphs only (connected in the weakest, obvious sense), then the corresponding conditional Turán number and the extremal family are denoted by $\text{ex}_{\text{conn}}(n; F|G)$ and $\text{Ex}_{\text{conn}}(n; F|G)$, respectively.

**Lemma 12** [11]. For $n \geq 7$,

$$\text{ex}_{\text{conn}}(n; P|C) = 3n - 8 \quad \text{and} \quad \text{Ex}_{\text{conn}}(n; P|C) = \{G_1(n), G_2(n)\}.$$ 

Lemma 12 as stated in [11] does not provide the family $\text{Ex}_{\text{conn}}(n; P|C)$. However, it is clear from its proof that the extremal 3-graphs are the same as in Theorem 3. We will also need another lemma, which is not stated explicitly in [11], but it immediate results from the proof of the previous one.

**Lemma 13** [11]. For $n \geq 7$,

$$\text{ex}_{\text{conn}}(n; P|\{C, M\}) = n + 5 \quad \text{and} \quad \text{Ex}_{\text{conn}}(n; P|\{C, M\}) = \left\{K_5^{+(n-5)}\right\}.$$ 

Moreover, if $H$ is an $n$-vertex connected $P$-free 3-graph such that $C \subset H$ and $M \subset H$, then $H \subseteq K_5^{+(n-5)}$.

**Lemma 14** [11].

$$\text{ex}(n; \{P, C\}|M) = \begin{cases} 2n - 4 & \text{for } 6 \leq n \leq 9, \\ 20 & \text{for } n = 10, \\ 4 + \binom{n-4}{2} & \text{and} \ \text{Ex}(n; \{P, C\}|M) = \{\text{Co}(n)\} \quad \text{for } n \geq 11. \end{cases}$$

**Lemma 15** [11]. For $n \geq 6$,

$$\text{ex}(n; \{P, C, P_2 \cup K_3\}|M) = 2n - 4,$$

where $P_2$ is a pair of edges sharing one vertex.

**Lemma 16** [14]. For $n \geq 6$,

$$\text{ex}^{(2)}(n; \{M, C\}) = \max\{10, n\}.$$
3. Proof of Theorem 2

As mentioned in the Introduction, Jackowska has shown in [9] that $R(P; r) \geq r+6$ for all $r \geq 1$. We are going to show that $R(P; 10) \leq 16$.

We will show that every 10-coloring of $K_{16}$ yields a monochromatic copy of $P$. The idea of the proof is to gradually reduce the number of vertices and colors (by one in each step), until we reach a coloring which yields a monochromatic copy of $P$.

Let us consider an arbitrary 10-coloring of $K_{16}$, $K_{16} = \bigcup_{i=1}^{10} G_i$, and assume that for each $i \in [10]$, $P \not\subseteq G_i$. Since $|K_{16}| = 560$, the average number of edges per color is 56, and therefore, by Theorems 7–11, either for each $i \in [10]$, $G_i = K_6 \cup S_{10}$, or there exists a color, say $G_{10}$, contained in one of the 3-graphs: $S_{16}, Co(16), K_4 \cup S_{12}, Ro(16)$. We will show that the latter case must occur. Indeed, for each vertex $v \in V(K_{16})$ we have $\deg_{K_{16}}(v) = \binom{15}{2} = 105$ whereas for $v \in V(K_6 \cup S_{10})$, $\deg_{K_6 \cup S_{10}}(v) \in \{10, 36, 8\}$ depending on whether $v$ is a vertex of $K_6$, the center of the star $S_{10}$ or another vertex of the star. Since we are not able to obtain an odd number as a sum of even numbers, we can not decompose $K_{16}$ into edge-disjoint copies of $K_6 \cup S_{10}$. Let us turn back to $G_{10}$. No matter in which of the four 3-graph $G_{10}$ is contained, we remove the center of the star (or comet, or rocket) together with up to four more edges of $G_{10}$, so that we get rid of color 10 completely (note that some other colors can also be affected by this deletion).

As a result, we obtain a 3-graph $H_{15}$ on 15 vertices, colored with 9 colors, $H_{15} = \bigcup_{i=1}^{9} G_i$, with $|H(15)| \geq 451$ (with some abuse of notation we will keep denoting the subgraphs of $G_i$ obtained in each step again by $G_i$). The average number of edges per color is at least 50.1, and therefore there exists a color, say $G_9$, with $|G_9| \geq 51$. This time we use Theorems 7–9 to conclude that either $G_9 \subset S_{15}$ or $G_9 \subset Co(15)$. In either case we remove the center and, in case of the comet, one more edge being its head.

We get a 3-graph $H(14)$ on 14 vertices with $|H(14)| \geq 359$, colored by 8 colors, $H(14) = \bigcup_{i=1}^{8} G_i$. The average number of edges per color is at least 44.9, and hence there exists a color, say $G_8$, with $|G_8| \geq 45$. Similarly as in the previous step we reduce the picture to a 3-graph $H(13)$ on 13 vertices with $|H(13)| \geq 280$, colored by 7 colors, $H(13) = \bigcup_{i=1}^{7} G_i$.

This time the average number of edges per color is at least 40, and therefore, by Theorems 7 and 8, either each color is a copy of $Co(13)$ or $K_6 \cup K_6 \cup K_1$, or there exists a color, say $G_7$, contained in the full star $S_{13}$. We will show in the similar way as before that $H(13)$ can not by decomposed into edge-disjoint copies of $Co(13)$ and $K_6 \cup K_6 \cup K_1$, and therefore the latter case must occur. Indeed, let us assume that no color is contained in the full star $S_{13}$. First notice that there is not enough space for two edge-disjoint copies of $K_6 \cup K_6 \cup K_1$ in $K_{13}$ and
therefore also in $H(13)$. Fix one copy of $K_6 \cup K_6 \cup K_1$ in $K_{13}$. By the pigeon-hole principle, any other copy of $K_6$ must share at least three vertices with one of the fixed copies of $K_6$ and therefore they are not edge-disjoint. Now observe that since during our procedure we have lost at most 6 edges of $K_{13}$, for each vertex $v \in V(H(13))$ we have $\deg_{H(13)}(v) \geq \binom{12}{2} - 6 = 60$ and also for each vertex of a comet $Co(13)$ which is not its center we have $\deg_{Co(13)}(v) \leq 8$. If $H(13)$ is decomposed into seven copies of $Co(13)$ or six copies of $Co(13)$ and one copy of $2K_6 \cup K_1$, then there must exist a vertex $v \in V(H(13))$ which is not a center of any of these comets and therefore $\deg_{H(13)}(v) \leq 10 + 6 \cdot 8 = 58 < 60$, a contradiction. Consequently, we have $G_7 \subseteq S_{13}$ and, by removing the center of this star, we obtain a 6-coloring of a 3-graph $H(12)$ on 12 vertices with $|H(12)| \geq 214$.

To proceed, let us assume for a while, that none of the colors $G_i$, $i \in [6]$, is a star. Then, by Theorems 7–9, each color with more than 32 edges is a subset of $K_6 \cup K_6$. The average number of edges per color is at least 35.6, and hence there exists a color, say $G_6$, with $G_6 \subset K_6 \cup K_6$. We remove all edges of this copy of $K_6 \cup K_6$, getting a bipartite 3-graph $H'(12)$ with a bipartition $V(H'(12)) = W \cup U$, $|W| = |U| = 6$, and with $|H'(12)| \geq 174$ edges colored by 5 colors, $H'(12) = \bigcup_{i=1}^5 G_i$. Note that every subgraph of $K_6 \cup K_6$ contained in $H'(12)$ (and consequently each color class of $H'(12)$) has at most 36 edges. Since $2 \cdot 36 + 3 \cdot 33 = 171 < 174$, at least 3 color classes have at least 34 edges, and thus each of them must be subsets of $K_6 \cup K_6$. Now observe that if two color classes, say $G_1$ and $G_2$, have at least 34 edges each, then they are disjoint unions of two copies of $K_6$, one on the vertex set $U'_i \cup W'_i$, the other one on $U''_i \cup W''_i$, with four missing edges $U'_i, U''_i, W'_i, W''_i$, where $U = U'_1 \cup U''_1$, $W = W'_1 \cup W''_1$, $i = 1, 2$, and $\{U'_1, U''_1\} = \{U'_2, U''_2\}$ (see Figure 2).

![Figure 2](image)

Figure 2. The partition of the set of vertices of $H'(12)$, $G_1$ and $G_2$.

Indeed, otherwise, if $1 \leq |U'_1 \cap U'_2| \leq 2$, then $G_1$ and $G_2$ would have at least six edges, and thus $|G_1| + |G_2| \leq 36 + 36 - 6 < 2 \cdot 34$. This simply means that...
of the partitions, of $U$ or of $W$, must be swapped. But this is impossible for three color classes. Consequently, at least one color, say $G_6$, is a star. We remove the center of this star to get a 5-coloring of a 3-graph $H(11)$ on 11 vertices with $|H(11)| \geq 159$.

By repeating this argument three more times, we finally arrive at a 2-coloring of a 3-graph $H(8) = G_1 \cup G_2$, with $|H(8)| \geq 50$ which, by Theorem 7, should contain a copy of $P$, a contradiction.

4. Proof of Theorem 11

Let us define $H_n = \text{Ex}^{(1)}(n; P) \cup \text{Ex}^{(2)}(n; P) \cup \text{Ex}^{(3)}(n; P) \cup \text{Ex}^{(4)}(n; P)$. To prove Theorem 11 we need to find, for each $n \geq 7$, a $P$-free $n$-vertex 3-graph $H$ with the biggest possible number of edges such that, whenever $G \in H_n$, then $H \not\supset G$. Moreover, we will show that $|H| = h_n$, where $h_n$ is the number of edges, given by the formula to be proved.

First note that for each $n \geq 7$, all candidates for being 5-extremal 3-graphs do qualify, that is, are $P$-free, are not contained in any of the 3-graphs from $H_n$, and have $h_n$ edges. To finish the proof, we will show that each $P$-free $n$-vertex 3-graph $H$, not contained in any of 3-graph from $H_n$, satisfy $|H| < h_n$ unless it is one of the candidates for being 5-extremal 3-graph itself.

For the latter task, we distinguish two cases: when $H$ is connected and disconnected. The entire proof is inductive, in the sense that here and there we apply the very Theorem 11 for smaller instances of $n$, once they have been confirmed.

For all $n \geq 7$, let $H$ be $P$-free $n$-vertex 3-graph such that for each $G \in H_n$, $H \not\supset G$. Moreover, let $H$ be different from all candidates for being 5-extremal 3-graphs with the same number of vertices. We will show that $|H| < h_n$.

4.1. Connected case

We start with the connected case. First let us assume that $M \not\supset H$ and consider consecutive intersecting families. Recall that for all $n \geq 7$, $H \not\supset S_n$, for $7 \leq n \leq 12$, $H \not\supset G_1(n)$ and $H \not\supset G_2(n)$, for $7 \leq n \leq 9$ and $n = 11$, $H \not\supset G_3(n)$, and finally, for $n = 7$, $H$ is not equal to any of 4-extremal 3-graphs for $M$. Therefore, by Theorems 3, 4 and 5, we get that for all $n \geq 7$,

$$|H| < h_n.$$  

Consequently, we will be assuming by the end of the proof that $M \subset H$. If additionally $C \subset H$, then by Lemma 13, $H \leq K_5^{+(n-5)}$ and hence $|H| \leq |K_5^{+(n-5)}| = n + 5$. Therefore, for $n \geq 10$, $|H| < h_n$. If $n = 7$, as $K_5^{+2} \in H_7$, we have $H \not\supset K_5^{+2}$ and thus we may exclude this case. Lastly, for $8 \leq n \leq 9$, by the definition of $H$,
$H \neq K_5^{(n-5)}$ and hence $|H| < h_n$. Therefore, in the rest of the proof we will be assuming that $C \nsubseteq H$.

Finally, let $H$ be connected $\{P, C\}$-free 3-graph containing $M$. Then by Lemma 14, for $7 \leq n \leq 8$, $|H| \leq 2n - 4 < h_n$ and for $n = 9$, since $H \notin \operatorname{Ex}(9, \{P, C\}|M)$, we have $|H| < 14 = h_9$.

For $10 \leq n \leq 11$ we need two more facts, which we state here without the proof. Namely, $\operatorname{ex}_{\text{conn}}(10; \{P, C\}|M) = 19$ and $\operatorname{Ex}_{\text{conn}}(10; \{P, C\}|M) = \{\operatorname{Co}(10)\}$. Since, by the definition of $H$, $H \neq \operatorname{Co}(10)$, this implies that $|H| < 19 = h_{10}$. Whereas for $n = 11$ we have $\operatorname{ex}_{\text{conn}}(11; \{P, C\}|M) = 18$, and therefore, as $H \nsubseteq \operatorname{Co}(11)$, we get $|H| \leq \operatorname{ex}_{\text{conn}}(11; \{P, C\}|M) = 18 < 19 = h_{11}$.

Recall that for all $n \geq 11$, $H \nsubseteq \operatorname{Co}(n)$. Moreover, for $12 \leq n \leq 13$, since $|\operatorname{Ro}(n)| < h_n$, we may assume that $H \nsubseteq \operatorname{Ro}(n)$. Further, for $n = 14$, by the definition of $H$ we have $H \neq \operatorname{Ro}(14)$ and thus, if $H \subseteq \operatorname{Ro}(14)$, then $|H| < |\operatorname{Ro}(14)| = h_n$. Finally, for all $n \geq 15$ we have $H \nsubseteq \operatorname{Ro}(n)$. Therefore, since for all $n \geq 12$ we have

$$h_n \geq \binom{n-6}{2} + 10,$$

to complete the proof of the connected case it is enough to prove the following Lemma.

**Lemma 17.** If $H$ is a connected, $n$-vertex, $n \geq 12$, $\{P, C\}$-free 3-graph containing $M$ such that $H \nsubseteq \operatorname{Co}(n)$ and $H \nsubseteq \operatorname{Ro}(n)$, then $|H| < \binom{n-6}{2} + 10$.

4.2. **Disconnected case**

Now let $H$ be disconnected and let $m = m(H)$ be the number of vertices in the smallest component of $H$. We have $m \neq 2$, since no component of a 3-graph may have two vertices. We now break the proof into several cases.

Let us express $H$ as a vertex disjoint union of two 3-graphs:

$$H = H' \cup H'', \quad |V(H')| = m, \quad |V(H'')| = n - m.$$  

Then, clearly, both $H'$ and $H''$ are $P$-free, and thus

$$|H| \leq \operatorname{ex}^{(1)}(m; P) + \operatorname{ex}^{(1)}(n - m; P).$$

Below, to bound $|H|$, we use the Turán numbers for $P$ of the 1\textsuperscript{st}, 2\textsuperscript{nd}, 3\textsuperscript{rd}, 4\textsuperscript{th} and 5\textsuperscript{th} order and utilize, respectively, Theorems 7, 8, 9, 10 and 11 (by induction).

Let $\nu$ be an isolated vertex ($m = 1$). Since for $n = 7$ and any 3-graph $H''$, $K_1 \cup H'' \subseteq K_1 \cup K_6 \in \mathcal{H}_7$, we may assume that $n \geq 8$. For $8 \leq n \leq 11$, the proof is similar.
as $H$ cannot be a sub-3-graph of $S_n$, $K_6 \cup K_{n-6}$, $G_1(n)$ or $G_2(n)$, $H''$ is not a sub-3-graph of $S_{n-1}$, $K_6 \cup K_{n-7}$, $G_1(n-1)$ and $G_2(n-1)$. Consequently, for $n = 8, 10$,

$$|H| = |H''| \leq \text{ex}^{(4)}(n-1; P) < h_n.$$

For $n = 9$ additionally we have $H'' \not\subseteq G_3(8)$ and therefore

$$|H| \leq \text{ex}^{(5)}(8; P) = 13 < 14 = h_9,$$

whereas for $n = 11$, $H'' \not\subseteq K_5 \cup K_5$ and $H'' \not\subseteq \text{Co}(10)$. Consequently,

$$|H| = |H''| < \text{ex}^{(5)}(10; P) = 19 = h_{11}.$$ 

For $n \geq 12$, since $H = K_1 \cup H''$ is not a sub-3-graph of any of the 3-graphs in $\mathcal{H}_n$, we have $H'' \not\subseteq S_{n-1}$ and $H'' \not\subseteq \text{Co}(n-1)$. Moreover, for $n = 12, 13$, $H'' \not\subseteq K_6 \cup K_{n-7}$, for $n = 12$, $H'' \not\subseteq G_1(n-1)$ and $H'' \not\subseteq G_2(n-1)$, for $n = 14$, $H'' \not\subseteq 2K_6 \cup K_1$, for $n = 14, 15$, $H'' \not\subseteq K_6 \cup S_{n-7}$ and finally, for $n \geq 15$, $H'' \not\subseteq K_4 \cup S_{n-5}$. Consequently,

$$|H| = |H''| \leq \text{ex}^{(4)}(n-1; P) < h_n.$$ 

For $m = 3$ and $n = 7, 8$, by (2) we get

$$|H| \leq \text{ex}^{(1)}(3; P) + \text{ex}^{(1)}(n-3; P) = 1 + \text{ex}^{(1)}(n-3; P) < h_n.$$ 

Since each disconnected 3-graph $H = H' \cup H''$ with $|V(H')| = 3$ and $|V(H'')| = 6$ is a sub-3-graph of $K_3 \cup K_6 \in \mathcal{H}_9$, we may assume that $n \neq 9$. For $n = 10$ we have $K_3 \cup K_6 \cup K_1 \subseteq K_4 \cup K_6 \in \mathcal{H}_{10}$. Consequently, $H'' \not\subseteq K_6 \cup K_1$ and thus $|H''| \leq \text{ex}^{(2)}(7; P) = 15$. Hence $|H| \leq 1 + 15 = 16 < 19 = h_{10}$.

Further, for all $n \geq 11$, since $K_3 \cup S_{n-3} \subseteq \text{Co}(n) \in \mathcal{H}_n$, we have $H'' \not\subseteq S_{n-3}$. Therefore for $n \geq 12$,

$$|H| \leq 1 + \text{ex}^{(2)}(n-3; P) < h_n,$$

whereas, for $n = 11$ additionally we have $H \not\subseteq K_3 \cup K_6 \cup K_2 \subseteq K_6 \cup K_5 \in \mathcal{H}_{11}$. Thus $H'' \not\subseteq K_6 \cup K_2$ and consequently,

$$|H| \leq 1 + \text{ex}^{(3)}(8; P) = 17 < 19 = h_{11}.$$ 

For $m = 4$ and $n = 8$ by (2) we have

$$|H| \leq \text{ex}^{(1)}(4; P) + \text{ex}^{(1)}(4; P) = 4 + 4 = 8 < h_8.$$ 

For $n = 9$, by the definition of $H$, $H \neq K_4 \cup K_5$ and therefore $|H| < |K_4 \cup K_5| = 14 = h_9$. Similarly like before, we may skip the case $n = 10$, because each
disconnected 3-graph $H = H' \cup H''$ with $|V(H')| = 4$ and $|V(H'')| = 6$ is a sub-3-graph of $K_4 \cup K_6 \in \mathcal{H}_{10}$. For $n = 11$, since $K_4 \cup K_6 \cup K_1 \subset K_5 \cup K_6 \in \mathcal{H}_{11}$, we have $H'' \not\subset K_5 \cup K_1$ and therefore $|H''| \leq \text{ex}(2)(7; P) = 15$ with the equality only for $H'' = S_7$. But, by the definition of $H$, $H \neq K_4 \cup S_7$, and hence

$$|H| < |K_4 \cup S_7| = 19 = h_{11}.$$ Further, for $n = 12, 13$, since $\text{Ex}^{(1)}(n - 4; P) = \{S_{n-4}\}$ and $H \notin H_4 \cup S_{n-4}$, we have $|H| < |H_4 \cup S_{n-4}| = h_n$. Finally, for $n \geq 14$, since $K_4 \cup S_{n-4} \in \mathcal{H}_n$ we get $H'' \not\subset S_{n-4}$ and consequently,

$$|H| \leq \text{ex}^{(1)}(4; P) + \text{ex}^{(2)}(n - 4; P) < h_n.$$

Now let $m = 5$. Notice that each disconnected 3-graph $H = H' \cup H''$ with $|V(H')| = 5$ and $5 \leq |V(H'')| \leq 6$ is a sub-3-graph of $K_5 \cup K_5 \in \mathcal{H}_{10}$ and $K_5 \cup K_6 \in \mathcal{H}_{11}$, respectively. Therefore we may consider only $n \geq 12$. For $n = 12$, since $K_5 \cup K_6 \cup K_1 \subset K_6 \cup K_6 \in \mathcal{H}_{12}$, we have $|H''| \leq \text{ex}(2)(7; P) = 15$ with the equality only for $H'' = S_7$. But, by the definition of $H$, $H \neq K_5 \cup S_7$ and hence $|H| < |K_5 \cup S_7| = 25 = h_{12}$. Finally, for $n \geq 13$, by (2),

$$|H| \leq \text{ex}^{(1)}(5; P) + \text{ex}^{(1)}(n - 5; P) = 10 + \left(\frac{n - 6}{2}\right) \leq h_n,$$

where the equality is achieved only by the candidates for 5-extremal 3-graphs with the proper number of vertices.

For $m = 6$ we have $n \geq 12$, but as each disconnected 3-graph $H' \cup H''$ with $|V(H')| = |V(H'')| = 6$ is a sub-3-graph of $K_6 \cup K_6 \in \mathcal{H}_{12}$, we may consider only $n \geq 13$. Recall that $\{2K_6 \cup K_1, K_6 \cup S_7, K_6 \cup G_1(7), K_6 \cup G_2(7)\} \subset \mathcal{H}_{13}$ and therefore, for $n = 13$, $H''$ is not contained in any of the 3-graphs $K_6 \cup K_1, S_7, G_1(7), G_2(7)$. Consequently, $|H''| \leq \text{ex}^{(4)}(7; P) = 12$ with the equality only for $H'' = G_3(7)$ and $H'' = K_5^+$. But, by the definition of $H$, $H \neq K_6 \cup K_5^+$ and $H \neq K_6 \cup G_3(7)$ and thus

$$|H| < |K_6 \cup K_5^+| = |K_6 \cup G_3(7)| = h_{13}.$$ For the same reason, if $n = 14$, then $H'' \not\subset S_8$ and $H'' \not\subset K_6 \cup K_2$. Consequently,

$$|H| = |H'| + |H''| \leq \text{ex}^{(1)}(6; P) + \text{ex}^{(3)}(8; P) = 20 + 16 < 39 = h_{14},$$

wheras for $n = 15$, we have $H'' \not\subset S_9$ and hence

$$|H| \leq \text{ex}^{(1)}(6; P) + \text{ex}^{(2)}(9; P) = 20 + 21 < 46 = h_{15}.$$ Further, for $n = 16, 17$, by the definition of $H$, $H \neq K_6 \cup S_{n-6}$. Consequently, as $\text{Ex}(n - 6; P) = \{S_{n-6}\}$, we get

$$|H| < |K_6 \cup S_{n-6}| = h_n.$$
Finally, for \( n \geq 18 \), by (2),

\[
|H| \leq \text{ex}^{(1)}(6; P) + \text{ex}^{(1)}(n - 6; P) = 20 + \binom{n - 7}{2} < \binom{n - 6}{2} + 10 = h_n.
\]

If \( m = 7 \), then \( n \geq 14 \). For \( n = 14 \), since \( H \not\subseteq 2K_6 \cup 2K_1 \in \mathcal{H}_{14} \), at least one of the components of \( H \) is not a sub-3-graph of \( K_6 \cup K_1 \) and therefore has at most \( \text{ex}^{(2)}(7; P) = 15 \) edges. Consequently,

\[
|H| \leq \text{ex}^{(1)}(7; P) + \text{ex}^{(2)}(7; P) = 20 + 15 = 35 < 39 = h_{14}.
\]

To bound the number of edges of \( H \) for \( n \geq 15 \) we use (2) to get

\[
|H| \leq \text{ex}^{(1)}(7; P) + \text{ex}^{(1)}(n - 7; P) = 20 + \binom{n - 8}{2} < \binom{n - 6}{2} + 10 \leq h_n.
\]

Finally, for \( m \geq 8 \) we have \( n \geq 16 \) and, by (2),

\[
|H| \leq \text{ex}^{(1)}(m; P) + \text{ex}^{(1)}(n - m; P) = \binom{m - 1}{2} + \binom{n - m - 1}{2} \\
\leq \binom{7}{2} + \binom{n - 9}{2} < \binom{n - 6}{2} + 10 \leq h_n.
\]

5. THE PROOF OF LEMMA 17

Recall that \( H \) is a connected, \( n \)-vertex, \( n \geq 12 \), \( \{P, C\} \)-free 3-graph such that \( M \subset H \), \( H \not\subseteq \text{Co}(n) \) and \( H \not\subseteq \text{Ro}(n) \). We need to show that

\[
|H| < \binom{n - 6}{2} + 10.
\]

Since for \( n \geq 11 \), by Lemma 15

\[
\text{ex}(\{n; P, C, P_2 \cup K_3\}|M) = 2n - 4 < \binom{n - 6}{2} + 10,
\]

we may assume that \( P_2 \cup K_3 \subset H \). Let us denote a copy of \( P_2 \) from \( P_2 \cup K_3 \) in \( H \) by \( Q \) and the vertex of degree two in \( Q \) by \( x \). We let \( U = V(Q), V = V(H) \) and \( W = V \setminus U \). Moreover, let \( W_0 \) be the set of vertices of degree zero in \( H[W] \) and \( W_1 = W \setminus W_0 \) (see Figure 3). Note that, by definition, \( H[W] = H[W_1] \) and \( |W_1| \geq 3 \).
We also split the set of edges of $H$. First, notice that since $H$ is $P$-free, there is no edge with one vertex in each $U$, $W_0$, and $W_1$. We define $H_i = \{h \in H : h \cap U \neq \emptyset, h \cap W_i \neq \emptyset\}$, where $i = 0, 1$. Then, clearly,

\begin{equation}
H = H[U] \cup H[W] \cup H_0 \cup H_1,
\end{equation}

with all four parts edge-disjoint. Since by definition $H[U] \cup H_0 = H[U \cup W_0]$, sometimes we will use the following equality

\begin{equation}
H = H[U \cup W_0] \cup H_1 \cup H[W].
\end{equation}

Recall that $H$ is $C$-free, and therefore one can use Theorem 6 to get the bounds, for $|W_0| \geq 1$

\begin{equation}
|H[U \cup W_0]| \leq \left(\frac{|U \cup W_0| - 1}{2}\right) = \left(\frac{|W_0| + 4}{2}\right)
\end{equation}

and for $|W_1| \geq 6$,

\begin{equation}
|H[W]| \leq \left(\frac{|W_1| - 1}{2}\right).
\end{equation}

Notice that for each edge $h \in H_0 \cup H_1$ with $|h \cap U| = 1$ we have $h \cap U = \{x\}$, because otherwise $h$ together with $Q$ would form a copy of $P$ in $H$. We let

\[ F^0 = \{h \in H_0 \cup H_1 : h \cap U = \{x\}\}. \]

Also, to avoid a copy of $C$ in $H$, if for $h \in H_0 \cup H_1$ we have $|h \cap U| = 2$, then the pair $h \cap U$ is contained in an edge of $Q$. For $k = 1, 2$, we define

\[ F^k = \{h \in H_0 \cup H_1 : |h \cap U \setminus \{x\}| = k\}. \]
Clearly, $H_0 \cup H_1 = F^0 \cup F^1 \cup F^2$ (see Figure 4). Further, for $i = 0,1$ and $k = 0, 1, 2$, we set

$$F^k_i = F^k \cap H_i.$$  

It is easy to see that, as $H$ is $P$-free, $F^1 = \emptyset$ and therefore,

$$H_1 = F^0 \cup F^2.$$  

Moreover, for all $v \in W$ we have

$$F^0(v) = \emptyset \quad \text{or} \quad F^2(v) = \emptyset,$$

and, by the definition of $F^1$ and $F^2$,

$$|F^1(v)| \leq 4 \quad \text{and} \quad |F^2(v)| \leq 2,$$

where for a given subset of edges $G \subseteq H$ and for a vertex $v \in V(H)$ we set $G(v) = \{ h \in G : v \in h \}$.

In the whole proof we will be using the fact that for all edges $e \in F^0$, the pair $e \cap W_1$ is nonseparable in $H[W]$, that is, every edge of $H[W]$ must contain both these vertices or none. Consequently, for each $v \in W_0$, $|F^0(v)| \leq |W_0| - 1$ and thus, by (8) and (9),

$$|H(v)| = |F^0(v)| + |F^1(v)| + |F^2(v)| \leq 4 + \max\{2, |W_0| - 1\}.$$  

Moreover, if $F^0 \neq \emptyset$, then there exists at least one nonseparable pair in $W_1$, and therefore one can show the following fact.

**Fact 1.** If $F^0 \neq \emptyset$, then $|H[W]| \leq \binom{|W_1| - 2}{3} + |W_1| - 2$. Moreover, if in addition $H[W_1] \subseteq S_{|W_1|}$, then $|H[W]| \leq \binom{|W_1| - 3}{2} + 1$. 

![Figure 4. Three types of edges in $H_0 \cup H_1$.](image)
To prove another fact let us define an auxiliary graph $G$ for nonseparable pairs on the set of vertices $W_1$, $G = \{e \setminus \{x\} : e \in F_1^0\}$. Then each component of $G$ has size at most 3. This gives the proof of the following inequality. For any $W'_1 \subseteq W_1$ we let $F_1^0[W'_1] \subseteq F_1^0$ to be the set of edges $h \in F_1^0$ such that $h \cap W_1 \subseteq W'_1$. Then,

$$|	ext{F}_1^0[W'_1]| \leq |W'_1|.$$  \hspace{1cm} (11)

Observe also that, because $H$ is connected, $H_1 \neq \emptyset$. Consequently, since the presence of any edge of $H_1$ forbids at least 4 edges of $H[U]$, \hspace{1cm} (12)

$$|H[U]| \leq 6.$$

Moreover, in [11] the authors have proved the following bounds on the number of edges in $H_1$.

$$\text{For } |W_1| \geq 4, \quad |F_1^2| \leq 2|W_1| - 4.$$ \hspace{1cm} (13)

$$\text{For } |W_1| \geq 3, \quad |H_1| \leq 2|W_1| - 3.$$ \hspace{1cm} (14)

As a consequence of these inequalities one can prove the following.

$$\text{For } |W_1| \geq 7, \quad |H[U]| + |H_1| \leq 2|W_1| - 1.$$ \hspace{1cm} (15)

Indeed, if $|H_1| \leq |W_1|$, then (15) results from (12) and the inequality $|W_1| - 1 \geq 7 - 1 = 6$. Otherwise, by (11), (7) and (8), there exists a vertex $v \in W_1$ such that $|F_1^2(v)| = 2$. As expected, assume $|H_1| > |W_1|$, and there does not exist the desired vertex $v$, i.e., for any vertex $v \in W_1$, $|F_1^2(v)| \leq 1$. Further, let $W''_1 \subseteq W_1$ be the set of vertices $v$ such that $F_1^2(v) = \emptyset$, and let $W''_1 = W_1 \setminus W'_1$. Then by (7), (8), (11) and the definition of $W''_1$, we have

$$|H_1| = |F_1^0| + |F_1^2| = |F_1^0[W'_1]| + \sum_{v \in W''_1} |F_1^2(v)| \leq |W'_1| + |W''_1| = |W_1|,$$

a contradiction. As $H$ is $\{P, C\}$-free, by the definition of $F_1^2(v)$, this implies that $|H[U]| = 2$ and (15) follows from (14).

We also need the following fact proven in [15].

**Fact 2** [15]. If $F_1^2 \neq \emptyset$, then

$$|H[U \cup W_0]| \leq \begin{cases} 8 & \text{for } |W_0| = 1, \\ 3|W_0| + 7 & \text{for } 2 \leq |W_0| \leq 4, \\ (|W_0| + 2) + 1 & \text{for } |W_0| \geq 5. \end{cases}$$ \hspace{1cm} (16)
We split the whole proof of Lemma 17 into a few short parts, Facts 3–7.

**Fact 3.** For \( n \geq 13 \), if \( W_0 = \emptyset \) and \( H_1 \neq \emptyset \), then \( |H| < 10 + \binom{n-6}{2} \).

**Proof.** Let us consider two cases, whether or not \( H[W] \subseteq S_{n-5} \). First assume that \( H[W] \subseteq S_{n-5} \), the set of the star \( S_{n-5} \) with the center \( y \in W_1 \). Further assume also that \( F^2(y) \neq \emptyset \), say, then exists \( v \in W_1, v \neq y \), such that \( F^2(v) \neq \emptyset \). Let \( h \in H[W] \) be the edge containing \( y \) and \( v \) (because \( H[W] \subseteq S_{n-5} \)). Since \( H \) is \( P \)-free, for any \( h' \in H[W], h' \neq h \), it holds that \(|h' \cap h| \geq 2 \) and \( y \in h' \cap h \). Thus \(|H[W]| \leq 1 + 2(|W_1| - 3) \). By (15), we have \(|H| \leq 2|W_1| - 1 + (1 + 2(|W_1| - 3)) = 4n - 26 < \binom{n-6}{2} + 10 \). Otherwise, \(|F^2| = |F^2(y)| \leq 2 \). Additionally, if \( F_1^0 \neq \emptyset \), then by (3), (12), (7) and (6),

\[
|H| = |H[U]| + |H_1| + |H[W]| \leq 6 + 2 + \binom{n-6}{2} = \binom{n-6}{2} + 8 < \binom{n-6}{2} + 10.
\]

Otherwise, \( F_1^0 \neq \emptyset \) and therefore by Fact 1, \(|H[W]| \leq \binom{n-8}{2} + 1 \). By (11), \(|F_1^0| \leq |W_1| = n - 5 \) and hence by (7), \(|H_1| \leq n - 5 + 2 = n - 3 \). Consequently, by (3) and (12),

\[
|H| = |H[U]| + |H_1| + |H[W]| \leq 6 + n - 3 + \binom{n-8}{2} + 1
\]

\[
= \binom{n-7}{2} + 12 < \binom{n-6}{2} + 10.
\]

Now we move to the case \( H[W] \not\subseteq S_{n-5} \). Since for \( n \geq 13 \), by Theorem 7, \( Ex(1)(n-5; P) = \{S_{n-5}\} \), we may bound the number of edges in \( H[W] \) by \( ex(2)(n-5; P) \). Moreover, by (15), \(|H[U]| + |H_1| \leq 2(n-5) - 1 = 2n - 11 \). Consequently, by (3) and Theorem 8,

\[
|H| = |H[U]| + |H_1| + |H[W]| \leq 2 \cdot n - 11 + \text{ex}(2)(n-5; P) < \binom{n-6}{2} + 10,
\]

where the last inequality is valid for \( n \geq 15 \). For \( 13 \leq n \leq 14 \) we have to strengthen the bound of \( H[W] \). Since \( W \) does not contain isolated vertices, we have \( H[W] \not\subseteq K_6 \cup K_2 \). Therefore by Theorems 7 and 8 we get \(|H[W]| < \text{ex}(2)(n-5; P) \) and consequently, for \( n = 14 \), \(|H| < \binom{n-6}{2} + 10 \). In addition, for \( n = 13 \), we use the fact that for \( i = 1, 2, C \subseteq G_i(8) \) and \( H \) is \( C \)-free, hence \( H[W] \not\subseteq G_i(8) \). Then by Theorems 7, 8 and 9 we have \(|H[W]| < \text{ex}(3)(8; P) = 16 \) and therefore

\[
|H| < 2 \cdot 13 - 11 + 16 = 31 = \binom{13-6}{2} + 10.
\]

**Fact 4.** For \( n \geq 13 \), if \( H_1 \neq \emptyset, H \not\subseteq Co(n) \) and \(|W_1| = 3 \), then \(|H| < 10 + \binom{n-6}{2} \).
Proof. We have \(|H[W]| = 1, |U \cup W_0| = n - 3\) and by (14), \(|H_1| \leq 3\). Therefore, by (4),

\[
|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq |H[U \cup W_0]| + 3 + 1 = |H[U \cup W_0]| + 4.
\]

Consequently, all we need to do is to bound the number of edges in \(H[U \cup W_0]\). Since \(H \not\subseteq \text{Co}(n)\), either \(F_1^2 \neq \emptyset\) or \(H[U \cup W_0] \not\subseteq S_{n-3}\). In the former case we use Fact 2 to get 

\[
|H[U \cup W_0]| \leq \binom{n-6}{2} + 1 \quad \text{and therefore}
\]

\[
|H| \leq \binom{n-6}{2} + 1 + 4 = \binom{n-6}{2} + 5 < \binom{n-6}{2} + 10.
\]

Otherwise, \(H[U \cup W_0] \not\subseteq S_{n-3}\), so by Theorem 7, \(|H[U \cup W_0]| \leq \text{ex}(2)(n - 3; P)\). Consequently, by Theorem 8, for \(13 \leq n \leq 15\), \(|H[U \cup W_0]| \leq 20 + \binom{n-3-6}{3}\) and therefore,

\[
|H| \leq 20 + \binom{n-9}{3} + 4 = \binom{n-9}{3} + 24 < \binom{n-6}{2} + 10.
\]

Whereas for \(n \geq 16\) we get \(|H[U \cup W_0]| \leq 4 + \binom{n-3-4}{2}\), and hence

\[
|H| \leq \binom{n-7}{2} + 4 + 4 = \binom{n-7}{2} + 8 < \binom{n-6}{2} + 10.
\]

\[\blacksquare\]

Fact 5. For \(n \geq 13\), if \(H_1 \neq \emptyset\), \(H \not\subseteq \text{Ro}(n)\) and \(|W_1| = 4\), then \(|H| < 10 + \binom{n-6}{2}\).

Proof. The proof goes along the lines of the previous one. We have \(|H[W]| \leq \binom{13}{3} = 4, |U \cup W_0| = n - 4\) and by (14), \(|H_1| \leq 5\). Therefore, by (4),

\[
|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq |H[U \cup W_0]| + 5 + 4 = |H[U \cup W_0]| + 9.
\]

Consequently, to finish the proof we need to bound \(|H[U \cup W_0]|\). Assume that \(F_1^2 = \emptyset\) and \(H[U \cup W_0] \subseteq S_{n-4}\). Then, since \(F_1^0 \neq \emptyset\), \(|W_1| = 4\) and for \(e \in F_1^0\) the pair \(e \cap W_1\) is nonseparable in \(H[W]\), we get \(H \not\subseteq \text{Ro}(n)\), a contradiction. Therefore \(F_1^2 \neq \emptyset\) or \(H[U \cup W_0] \not\subseteq S_{n-4}\). In the former case we use Fact 2 to get for \(n = 13\), \(|H[U \cup W_0]| \leq 19\) and consequently,

\[
|H| \leq 19 + 9 = 28 < 31 = 10 + \binom{13-6}{2}.
\]

Whereas for \(n \geq 14\), \(|H[U \cup W_0]| \leq \binom{n-7}{2} + 1\) and hence,

\[
|H| \leq \binom{n-7}{2} + 1 + 9 = \binom{n-7}{2} + 10 < \binom{n-6}{2} + 10.
\]
Otherwise, $H[U \cup W_0] \not\subseteq S_{n-4}$ so we use Theorem 7 to get $|H[U \cup W_0]| \leq \text{ex}(2)(n-4; P)$. Consequently, by Theorem 8, for $13 \leq n \leq 16$, $H[U \cup W_0] \leq 20 + (\frac{n-4}{3})$ and hence

$$|H| \leq 20 + \binom{n-10}{3} + 9 = \binom{n-10}{3} + 29 < \binom{n-6}{2} + 10.$$  

Whereas for $n \geq 17$ we have $|H[U \cup W_0]| \leq 4 + (\frac{n-4}{2})$ and therefore,

$$|H| \leq 4 + \binom{n-8}{2} + 9 = \binom{n-8}{2} + 13 < \binom{n-6}{2} + 10.$$  

\[\text{Fact 6.}\] If $n = 12$, $H_1 \neq \emptyset$ and $H \not\subseteq \text{Co}(12)$, then $|H| < 10 + \binom{12-6}{2} = 25$.

\[\text{Proof.}\] Let us split the proof into five parts according to the size of the set $W_1$. We start with $|W_1| = 3$. Then $|W_0| = 4$, $|U \cup W_0| = 9$, $|H[W]| = 1$ and by (14), $|H_1| \leq 3$. Consequently, by (4),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq |H[U \cup W_0]| + 3 + 1 = |H[U \cup W_0]| + 4.$$  

Further, as $H \not\subseteq \text{Co}(12)$, either $F_1^0 \neq \emptyset$ or $H[U \cup W_0] \not\subseteq S_{n-3}$. In the former case we use Fact 2 to get $|H[U \cup W_0]| \leq 19$. Otherwise, $H[U \cup W_0] \not\subseteq S_{n-3}$, and since $H[U \cup W_0] \neq K_6 \cup K_3$, by Theorems 7 and 8, $|H[U \cup W_0]| < 21$. In both cases $|H[U \cup W_0]| \leq 20$ and therefore

$$|H| \leq |H[U \cup W_0]| + 4 \leq 20 + 4 = 24 < 25.$$  

For $|W_1| = 4$ we have $|W_0| = 3$, $|U \cup W_0| = 8$ and $|H[W]| \leq \binom{4}{3} = 4$. If $F_1^2 = \emptyset$, then $H_1 = F_1^0 \neq \emptyset$ and as for each $h \in F_1^0$ the pair $h \cap W_1$ is nonseparable, $|H_1| = 1$ and by Fact 1, $|H[W]| \leq 2$. Consequently, by (4) and (5),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq \binom{4}{2} + 2 + 2 = 24 < 25.$$  

Otherwise, $F_1^2 \neq \emptyset$ and we can use Fact 2 to get $|H[U \cup W_0]| \leq 16$. For $F_1^0 \neq \emptyset$, $|H[W]| = 2$ and consequently, by (4) and (14),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 16 + 5 + 2 = 23 < 25.$$  

Whereas for $F_1^0 = \emptyset$ we use (4), (7) and (13) to get

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 16 + 4 + 4 = 24 < 25.$$  

Now let $|W_1| = 5$, $|W_0| = 2$, $|U \cup W_0| = 7$ and $|H[W]| \leq \binom{5}{3} = 10$. For $F_1^2 \neq \emptyset$, by Fact 2 we get $|H[U \cup W_0]| \leq 13$ and moreover $|H[W]| \leq 6$, because otherwise
we would not be able to avoid a path $P$ in $H$. If additionally $P_2 \subseteq H[W]$, then again by $P \not\subseteq H$, $|H_1| = |F_0^1| + |F_2^1| \leq 2 + 2 = 4$. Hence, by (4)

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 13 + 4 + 6 = 23 < 25.$$ 

Otherwise, $P_2 \not\subseteq H[W]$ and consequently one can show that $|H[W]| \leq |W_1| - 2 = 3$. Therefore, by (4) and (14),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 13 + 7 + 3 = 23 < 25.$$ 

For $F_2^1 = \emptyset$ we have $F_0^1 \neq \emptyset$. Hence, since for each $h \in F_0^1$ the pair $h \cap W_1$ is nonseparable, $|H_1| = |F_0^1| \leq 2$ and by Fact 1, $|H[W]| \leq 4$. Consequently, by (4) and (5),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq \left(\frac{7 - 1}{2}\right) + 2 + 4 = 21 < 25.$$ 

We move to $|W_1| = 6$. Then $|W_0| = 1$, $|U \cup W_0| = 6$ and by (6), $|H[W]| \leq \binom{6 - 1}{\frac{2}{2}} = 10$. Let us again start with the case $F_1^2 \neq \emptyset$. By (16) we get $|H[U \cup W_0]| \leq 8$. If $P_2 \subseteq H[W]$, then since $H$ is $P$-free, $|H_1| = |F_0^1| + |F_2^1| \leq 2 + 4 = 6$. Consequently, by (4),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 8 + 6 + 10 = 24 < 25.$$ 

Otherwise, $P_2 \not\subseteq H[W]$ and therefore one can show that $|H[W]| \leq |W_1| - 2 = 4$. By (14), $|H_1| \leq 9$ and consequently by (4),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 8 + 9 + 4 = 21 < 25.$$ 

For $F_2^1 = \emptyset$ we have $F_0^1 \neq \emptyset$, thus by Fact 1, $|H[W]| \leq 8$ and by (11), $|H_1| = |F_0^1| \leq 6$. Therefore, by (4) and (5),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 10 + 6 + 8 = 24 < 25.$$ 

Finally, $|W_1| = 7$, $W_0 = \emptyset$ and by (6), $|H[W]| \leq \binom{7 - 1}{\frac{2}{2}} = 15$. First assume that $H[W]$ is a subset of the star $S_7$ with the center $y \in W_1$. Further assume also that $F_2 \setminus F_2(y) \neq \emptyset$, say, there exists $v \in W_1$, $v \neq y$, such that $F_2(v) \neq \emptyset$. Let $h \in H[W_1]$ be the edge containing $y$ and $v$ (because $H[W] \subseteq S_7$). Since $H$ is $P$-free, for any $h' \in H[W_1]$, $h' \neq h$, it holds that $|h' \cap h| \geq 2$ and $y \in h' \cap h$. Thus $|H[W_1]| \leq 1 + 2(|W_1| - 3) = 9$. By (15), we have $|H| \leq 2|W_1| - 1 + 9 = 22 < 25$. Otherwise, $|F_2^1| = |F_2^1(y)| \leq 2$. If additionally $F_1^0 = \emptyset$, then by (3), (7) and (12),

$$|H| = |H[U]| + |H_1| + |H[W]| \leq 6 + 2 + 15 = 23 < 25.$$
Otherwise $F_1^0 \neq \emptyset$, thus $|H_1| = |F_1^0| + |F_1^2| \leq 3 + 2 = 5$ and by Fact 1, $|H[W]| \leq 7$. Therefore by (3) and (12),

$$|H| = |H[U]| + |H_1| + |H[W]| \leq 6 + 5 + 7 = 18 < 25.$$  

The last case we have to consider is $H[W] \not\subseteq S_7$. If $M \subseteq H[W]$, then by Lemma 14, $|H[W]| \leq \text{ex}(7; \{P, C\} | M) = 10$. Otherwise, by Lemma 16, $|H[W]| \leq \text{ex}(2)(7; \{M, C\}) = 10$. Hence by (3) and (15),


**Fact 7.** For $n \geq 12$, if $|W_1| \geq 5$ and $H_1 \neq \emptyset$, then

(17)  

$$|H| < \left(\frac{n - 6}{2}\right) + 10.$$  

**Proof.** The proof is by induction on $n$ with the initial step $n = 12$ done in Fact 6. Let $n \geq 13$. For $W_0 = \emptyset$ the inequality (17) results from Fact 3. Otherwise, there exist a vertex $v \in W_0$. Notice that since $|W_1| \geq 5$, we have $|W_0| \leq n - 10$ and consequently, by (10), $|H(v)| \leq 4 + \max\{2, |W_0| - 1\} \leq 4 + n - 11 = n - 7$. Finally, by the induction assumption we get $|H - v| < \left(\frac{n - 7}{2}\right) + 10$. Therefore,

$$|H| = |H(v)| + |H - v| < n - 7 + \left(\frac{n - 7}{2}\right) + 10 = \left(\frac{n - 6}{2}\right) + 10.$$  

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**References**


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