Discussiones Mathematicae
General Algebra and Applications 36 (2016) 209–221
doi:10.7151/dmgaa.1258

RIGHT DERIVATION OF ORDERED Γ-SEMIRINGS

M. Murali Krishna Rao

Department of Mathematics
GIT, GITAM University
Visakhapatnam – 530 045, A.P., India

e-mail: mmarapureddy@gmail.com

and

B. Venkateswarlu

Department of Mathematics
GST, GITAM University
Bengaluru North – 562 163, Karnataka, India

e-mail: bvlmaths@gmail.com

Abstract

In this paper, we introduce the concept of a right derivation of ordered Γ-semirings, we study some of the properties of right derivations of ordered Γ-semirings and we prove that if d is a non-zero right derivation of additively commutative cancellative prime ordered Γ-semring M then M is a commutative ordered Γ-semiring.

Keywords: ordered Γ-semiring, right derivation, derivation, negatively ordered Γ-semigroup, positively ordered Γ-semigroup.

2010 Mathematics Subject Classification: 16Y60, 06F35, 08A30, 03G25.

1. Introduction

A semiring is an algebraic structure with two binary operations called addition and multiplication, where one of them distributive over the other. Bounded distributive lattices are commutative semirings which are both, additively idempotent and multiplicatively idempotent. A semiring is a common generalization of rings and distributive lattices and was first introduced by the American mathematician Vandiver [15] in 1934 but non trivial examples of semirings have appeared in the earlier studies on the theory of commutative ideals of rings by the

German mathematician Richard Dedekind in 19th century. A natural example of a semiring is the set of all natural numbers under usual addition and multiplication of numbers. In particular, if I is the unit interval on the real line then the algebraic structure $(I, \max, \min)$ is a semiring in which 0 is the additive identity and 1 is the multiplicative identity. The theory of rings and the theory of semigroups have considerable impact on the development of the theory of semirings. In structure, semirings lie between semigroups and rings. Additive and multiplicative structures of a semiring play an important role in determining the structure of a semiring. Semirings, as the basic algebraic structure, were used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata, coding theory and formal languages. Semiring theory has many applications in other branches.

The notion of a Γ-ring was introduced by Nobusawa [12] as a generalization of a ring in 1964. Sen [14] introduced the notion of a Γ-semigroup in 1981. The notion of a ternary algebraic system was introduced by Lehmer [4] in 1932. Lister [5] introduced ternary rings. Dutta and Kar [3] introduced the notion of a ternary semiring which is a generalization of a ternary ring and a semiring. In 1995, Murali Krishna Rao [8, 9] introduced the notion of a Γ-semiring which is a generalization of Γ-ring, ring, ternary semiring and semiring. After the paper [8] was published, many mathematicians obtained interesting results on Γ-semirings. Murali Krishna Rao and Venkateswarlu [10] introduced the unity element in a Γ-semiring and studied properties of Γ-incline and field Γ-semiring.

Over the last few decades several authors have investigated the relationship between the commutativity of ring R and the existence of certain specified derivations of R. The first result in this direction is due to Posner [13] in 1957. In the year 1990, Bresar and Vukman [2] established that a prime ring must be commutative if it admits a non-zero left derivation. Kim [6, 7] studied right derivation and generalized right derivations of incline algebras. The notion of derivation of a ring is useful for characterization of rings. In this paper, we introduce the concept of a right derivation of ordered Γ-semirings and we study some of the properties of right derivations of ordered Γ-semirings.

2. Preliminaries

In this section we recall some important definitions on semirings, Γ-semirings, ordered Γ-semirings that are necessary for this paper.

Definition 1. A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called a semiring provided

(i) addition is a commutative operation,

(ii) multiplication distributes over addition both from the left and from the right,

(iii) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.

Definition 2. Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. Then we call M a Γ-semiring, if there exists a mapping $M \times \Gamma \times M \to M$ where (x, α, y) is written as $x\alpha y$ such that it satisfies the follo w ing axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$

(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$

(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Every semiring R is a Γ-semiring with $\Gamma = R$ and the ternary operation $x\gamma y$ as the usual semiring multiplication.

We illustrate the definition of a Γ-semiring by the following example.

Example 1. Let S be a semiring and $M_{p,q}(S)$ denote the additive abelian semigroup of all $p \times q$ matrices with identity element whose entries are from S. Then $M_{p,q}(S)$ is a Γ-semiring with $\Gamma = M_{p,q}(S)$. A ternary operation is defin ed by $x\alpha z = x(\alpha^t)z$ as the usual matrix multiplication, where α^t denotes the transpose of the matrix α, for all x, y and $\alpha \in M_{p,q}(S)$.

A Γ-semiring M is said to have a zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0$, for all $x \in M, \alpha \in \Gamma$. A Γ-semiring M is said to be commutative if $x\alpha y = y\alpha x$, for all $x, y \in M$ and $\alpha \in \Gamma$. An element $a \in M$ is said to be idempotent of M if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$ and $a + a = a$. If every element of M is an idempotent of M, then M is said to be an idempotent Γ-semiring M. An element $a \in M$ is said to be regular element of M if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. If every element of M is a regular element of M, then M is said to be a regular Γ-semiring M. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 3. Let M be a Γ-semiring. A function $d : M \to M$ is called a right derivation of M if

(i) $d(x + y) = d(x) + d(y)$

(ii) $d(x\alpha y) = d(x)\alpha y + d(y)\alpha x, \quad$ for all $x, y \in M, \alpha \in \Gamma$

Definition 4. A Γ-semiring M is called an ordered Γ-semiring if it admits a compatible relation $\leq$ i.e., $\leq$ is a partial ordering on M which satisfies the following conditions.

If $a \leq b$ and $c \leq d$, then

(i) $a + c \leq b + d$ (ii) $a\alpha c \leq b\alpha d$ (iii) $c\alpha a \leq d\alpha b$, for all $a, b, c, d \in M, \alpha \in \Gamma$.

Example 2. Let $M = [0, 1], \Gamma = N$, $+$ a ternary operation be defined as $x + y = \max\{x, y\}, x\gamma y = \min\{x, \gamma, y\}$ for all $x, y \in M, \gamma \in \Gamma$. Then M is an ordered Γ-semiring with respect to the usual ordering.

Definition 5. An ordered Γ-semiring M is said to be totally ordered if any two elements of M are comparable.

Definition 6. In an ordered Γ-semiring M

(i) $(M, +)$ is positively ordered if $a + b \geq a, b$, for all $a, b \in M$.

(ii) $(M, +)$ is negatively ordered if $a + b \leq a, b$, for all $a, b \in M$.

(iii) Γ-semigroup M is positively ordered if $a\alpha b \geq a, b$, for all $\alpha \in \Gamma, a, b \in M$.

(iv) Γ-semigroup M is negatively ordered if $a\alpha b \leq a, b$, for all $\alpha \in \Gamma, a, b \in M$.

A non-empty subset A of the ordered Γ-semiring M is called a Γ-subsemiring M if $(A, +)$ is a subsemigroup of $(M, +)$ and $a\alpha b \in A$ for all $a, b \in A$ and $\alpha \in \Gamma$. A non-empty subset A of an ordered Γ-semiring M is called a left (right) ideal of an ordered Γ-semiring M if A is closed under addition, $M\Gamma A \subseteq A$ ($A\Gamma M \subseteq A$) and if for any $a \in M$, $b \in I$, $a \leq b \Rightarrow a \in I$. A is called an ideal of M if it is both, a left ideal and a right ideal of M. A non-empty subset A of an ordered Γ-semiring M is called a k-ideal if A is an ideal and $x \in M$, $x + y \in A$, $y \in A$ then $x \in A$.

Definition 7. Let M be an ordered Γ-semiring. A Γ-subsemiring P of M is called a prime ideal of M if

(i) $a \leq b, a \in M, b \in P \Rightarrow a \in P$

(ii) $a\gamma b \in P, a, b \in M, \gamma \in \Gamma \Rightarrow a \in P$ or $b \in P$.

3. Right derivation of ordered Γ-semirings

In this section, we introduce the concept of a right derivation of an ordered Γ-semiring, we study some of the properties of right derivations of ordered Γ-semirings and we prove that if d is a non-zero right derivation of an additively commutative cancellative prime ordered Γ-semring M then M is a commutative ordered Γ-semiring.

Definition 8. Let M be an ordered Γ-semiring. If the mapping $d : M \to M$ satisfies the following conditions

(i) $d(x+y) = d(x) + d(y)$

(ii) $d(x\alpha y) = d(x)\alpha y + d(y)\alpha x$

(iii) If $x \leq y$ then $d(x) \leq d(y)$, for all $x, y \in M$ and $\alpha \in \Gamma$,

then d is called a right derivation of M.

Example 3. Let $M = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \Big| a, b \in Q \right\}$ where Q is the set of all rational numbers and $\Gamma = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \Big| a, b \in N \right\}$ where N is the set of all natural numbers. Then M and Γ are additive abelian semigroups with respect to the usual matrix addition of 2×2 matrices and a ternary operation is defined as $M \times \Gamma \times M \to M$ by $(x, \alpha, y) \to x\alpha y$ using the usual matrix multiplication for all $x, y \in M$ and $\alpha \in \Gamma$. Let $A = (a_{ij}), B = (b_{ij}) \in M$, we define $A \leq B \Leftrightarrow a_{ij} \leq b_{ij}$, for all i, j. Then M is an ordered Γ-semiring.

Define a map $d : M \to M$ by $d\begin{pmatrix} a & 0 \\ b & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2b & 0 \end{pmatrix}$. Then d is a right derivation but not a derivation of M.

Theorem 4. *Let M be an ordered Γ-semiring. If $(M, +)$ is a positively ordered semigroup then 0 is the minimal element of M.*

Proof. Let $a \in M$.
Now $a + 0 = a$
$\Rightarrow a + 0 \geq 0$
$\Rightarrow a \geq 0$.
Therefore 0 is the minimal element.
Hence the theorem is proved. ■

Theorem 5. *Let M be a commutative ordered Γ-semiring and $(M, +)$ be idempotent. Then for a fixed element $a \in M$, the mapping $d_a : M \to M$ is given by $d_a(x) = x\alpha a$, for all $x \in M, \alpha \in \Gamma$ is a right derivation of M.*

Proof. Let $a \in M$. Suppose $x, y \in M$ and $\alpha \in \Gamma$.

$$\begin{aligned} d_a(x+y) &= (x+y)\alpha a \\ &= x\alpha a + y\alpha a \\ &= d_a(x) + d_a(y) \end{aligned}$$

$$\begin{aligned}\text{and } d_a(x\alpha y) &= (x\alpha y)\alpha a \\ &= (x\alpha y)\alpha a + (x\alpha y)\alpha a \\ &= (x\alpha a)\alpha y + (y\alpha a)\alpha x \\ &= d_a(x)\alpha y + d_a(y)\alpha x.\end{aligned}$$

Suppose $x \leq y \Rightarrow x\alpha a \leq y\alpha a$
$\Rightarrow d_a(x) \leq d_a(y).$

Hence d_a is a right derivation of M. ■

Theorem 6. *Let d be a right derivation of an ordered Γ-semiring M. Then $d(0) = 0$.*

Proof. We have

$$\begin{aligned}d(0) &= d(0\alpha 0) \\ &= d(0)\alpha 0 + d(0)\alpha 0 \\ &= 0 + 0 \\ &= 0.\end{aligned}$$

Therefore $d(0) = 0$. ■

Theorem 7. *Let d be a right derivation of the idempotent ordered Γ-semiring M in which the Γ-semigroup M is negatively ordered. Then $d(x) \leq x$, for all $x \in M$.*

Proof. Let $x \in M$. Then there exists $\alpha \in \Gamma$ such that $x = x\alpha x$

$$\begin{aligned}\Rightarrow\ d(x) &= d(x\alpha x) \\ &= d(x)\alpha x + d(x)\alpha x \\ &= d(x)\alpha x \\ \Rightarrow d(x) &= d(x)\alpha x \\ \Rightarrow d(x) &\leq x.\end{aligned}$$

Hence the theorem is proved. ■

Theorem 8. *Let M be an ordered Γ-semiring in which Γ-semigroup M is negatively ordered and d be a right derivation of M. Then $d(x\alpha y) \leq d(x + y)$.*

Proof. Suppose $x, y \in M, \alpha \in \Gamma$.

Then $d(x)\alpha y \leq d(x)$ and $d(y)\alpha x \leq d(y)$. Therefore

$$\begin{aligned} d(x\alpha y) &= d(x)\alpha y + d(y)\alpha x \\ &\leq d(x) + d(y) \\ &= d(x+y). \end{aligned}$$

Hence the theorem is proved. ■

Theorem 9. *Let M be an idempotent ordered Γ-semiring and d be a right derivation of M. If $d^2(x) = d(d(x)) = d(x)$ then $d(x\alpha d(x)) = d(x)$, $\alpha \in \Gamma$ and for all $x \in M$.*

Proof. Let $x \in M$. Then there exists $\alpha \in \Gamma$ such that $x = x\alpha x$,

$$\begin{aligned} d(x\alpha d(x)) &= d(x)\alpha d(x) + d(d(x))\alpha x \\ &= d(x) + d(x)\alpha x \\ &= d(x) + d(x) \\ &= d(x). \end{aligned}$$

Therefore $d(x\alpha d(x)) = d(x)$. ■

Theorem 10. *Let M be a commutative ordered Γ-semiring and d_1, d_2 be right derivations of M. Define $d_1d_2(x) = d_1(d_2(x))$, for all $x \in M$. If $d_1d_2 = 0$ then d_2d_1 is a right derivation of M.*

Proof. Suppose $d_1d_2 = 0, x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} & d_1d_2(x\alpha y) = 0 \\ \Rightarrow\ & d_2(x)\alpha d_1(y) + d_2(y)\alpha d_1(x) = 0 \qquad \cdots \quad (1) \\ \text{and } & d_2d_1(x\alpha y) = d_2(d_1(x\alpha y)) \\ &= d_2(d_1(x)\alpha y + d_1(y)\alpha x) \\ &= d_2(d_1(x)\alpha y) + d_2(d_1(y)\alpha x) \\ &= d_2d_1(x)\alpha y + d_2(y)\alpha d_1(x) + d_2d_1(y)\alpha x + d_2(x)\alpha d_1(y) \\ &= d_2d_1(x)\alpha y + d_2d_1(y)\alpha x (\text{from } (1)). \\ \text{Obviously } & d_2d_1(x+y) = d_2d_1(x) + d_2d_1(y). \end{aligned}$$

Hence d_2d_1 is a right derivation of commutative ordered Γ-semiring M. ■

Let d be a right derivation of ordered Γ-semiring M. Define
$kerd = \{x \in M \mid d(x) = 0\}$.

Theorem 11. *Let d be a right derivation of ordered Γ-semiring M in which $(M, +)$ is positively ordered. Then $kerd$ is a k-ideal of ordered Γ-semiring M.*

Proof. Suppose $x, y \in kerd, \alpha \in \Gamma$. Then $d(x) = 0$ and $d(y) = 0 \Rightarrow d(x\alpha y) = 0$ and $d(x+y) = 0$. Therefore $x + y \in kerd$ and $x\alpha y \in kerd$. Hence $kerd$ is a Γ-subsemiring of M. Let $x \in M$ and $y \in kerd$ such that $x \leq y$. Then $d(y) = 0$.

$$\begin{aligned} & x \leq y \\ \Rightarrow & d(x) \leq d(y) \\ \Rightarrow & d(x) = 0. \end{aligned}$$

Therefore $x \in kerd$.
Let $x \in M$. $x + y \in kerd$ and $y \in kerd$. Then

$$\begin{aligned} & d(x+y) = d(y) = 0 \\ \Rightarrow & d(x) + d(y) = 0 \\ \Rightarrow & d(x) = 0. \end{aligned}$$

Therefore $x \in kerd$. Hence $kerd$ is a k-ideal of M. ■

Theorem 12. *Let M be an idempotent ordered Γ-semiring in which Γ-semigroup M is negatively ordered and d be a right derivation of M. Then $d(x\alpha y) \leq x$ and $d(x\alpha y) \leq y$, for all $x, y \in M, \alpha \in \Gamma$.*

Proof. We have

$$\begin{aligned} x\alpha y &\leq x, \text{ for all } x, y \in M \text{ and } \alpha \in \Gamma. \\ \Rightarrow d(x\alpha y) &\leq d(x) \\ &\leq x, \text{ by Theorem7.} \end{aligned}$$

Similarly we can prove $d(x\alpha y) \leq y$. ■

The proof of following theorem is the routine verification.

Theorem 13. *Let M be an ordered Γ-semiring. If d_1, d_2 are right derivations of M and $(d_1 + d_2)(x) = d_1(x) + d_2(x)$, for all $x \in M$ then $d_1 + d_2$ is also a right derivation of M.*

Theorem 14. *Let d be a right derivation of an additively cancellative commutative idempotent ordered Γ-semiring M in which $(M, +)$ is positively ordered. Define a set $Fix_d(M) = \{x \in M \mid d(x) = x\}$. Then $Fix_d(M)$ is a k-ideal of M.*

Proof. Suppose $x, y \in Fix_d(M)$ and $\alpha \in \Gamma$.
Then $d(x) = x, d(y) = y \Rightarrow d(x+y) = d(x) + d(y) = x + y$.
Therefore $x + y \in Fix_d(M)$.
$d(x\alpha y) = d(x)\alpha y + d(y)\alpha x = x\alpha y + y\alpha x = x\alpha y + x\alpha y = x\alpha y$.
Therefore $Fix_d(M)$ is a Γ-subsemiring of M.

Suppose $x \leq y$ and $y \in Fix_d(M)$.

$$\begin{aligned} & x \leq y \\ \Rightarrow & x + y \leq y + y \\ \Rightarrow & x + y \leq y \leq x + y. \end{aligned}$$

Therefore $x + y = y$.

$$\begin{aligned} \Rightarrow & d(x + y) = x + y \\ \Rightarrow & d(x) + d(y) = x + y \\ \Rightarrow & d(x) + y = x + y. \end{aligned}$$

Hence $d(x) = x$.

Suppose $x + y \in Fix_d(M)$ and $y \in Fix_d(M)$. Then

$$\begin{aligned} \Rightarrow & d(x + y) = x + y \text{ and } d(y) = y \\ \Rightarrow & d(x) + d(y) = x + y. \\ \Rightarrow & d(x) + y = x + y. \end{aligned}$$

Therefore $d(x) = x$. Hence $Fix_d(M)$ is a k-ideal of M. ■

Theorem 15. *Let d be a right derivation of ordered Γ-semiring M and I, J be any two ideals of M and $I \subseteq J$. Then $d(I) \subseteq d(J)$.*

Proof. Suppose I, J be any two ideals of ordered Γ-semiring M and $I \subseteq J$ and $x \in d(I)$. Then there exists $y \in I$ such that $x = d(y)$. Since $y \in I$ and $I \subseteq J \Rightarrow y \in J$. Therefore $x \in d(J)$. Hence $d(I) \subseteq d(J)$. ■

Definition 9. Let M be an ordered Γ-semiring and d be a right derivation of M. An ideal I of ordered Γ-semiring M is called a d-ideal if $d(I) = I$.

Example 16. The zero ideal $\{0\}$ is a d-ideal of ordered Γ-semiring. Since $d(0) = 0$. If d is onto derivation then $d(M) = M$. Hence M is a d-ideal.

Theorem 17. *Let M be an ordered Γ-semiring and I, J be d-ideals of M. Then $I + J$ is also a d-ideal of M.*

Proof. I and J are subsets of $I + J$. Therefore

$$\begin{aligned} I = d(I) &\subseteq d(I + J) \\ J = d(J) &\subseteq d(I + J) \\ I + J \subseteq d(I + J) &\subseteq d(I) + d(J) = I + J. \end{aligned}$$

Therefore $d(I + J) = I + J$. Hence $I + J$ is also a d-ideal of M. ■

Theorem 18. *Let M be an ordered Γ-semiring in which $(M,+)$ is positively ordered with unity 1 and d be a right derivation of M. Then*

(i) $d(1)\alpha x \leq d(x)$.

(ii) *If* $d(1) = 1$ *then* $x \leq d(x)$.

Proof. Let $x \in M$. Then there exists $\alpha \in \Gamma$ such that $x\alpha 1 = x$ and $1\alpha x = x$.

$$\begin{aligned} d(x) &= d(x\alpha 1) \\ &= d(x)\alpha 1 + d(1)\alpha x \\ \Rightarrow d(1)\alpha x &\leq d(x)\alpha 1 + d(1)\alpha x = d(x). \end{aligned}$$

Suppose $d(1) = 1$. Then

$$\begin{aligned} & d(1)\alpha x \leq d(x) \\ \Rightarrow\ & 1\alpha x \leq d(x) \\ \Rightarrow\ & x \leq d(x). \end{aligned}$$

Hence the theorem follows. ■

Theorem 19. *Let M be an idempotent ordered Γ-semiring in which $(M,+)$ is positively ordered, Γ-semigroup M is negatively ordered with unity and d be a right derivation. Then $d(1) = 1$ if and only if $d(x) = x$.*

Proof. Suppose $x \in M$ and $\alpha \in \Gamma$.

$$\begin{aligned} d(x) &= d(x\alpha x) \\ &= d(x)\alpha x + d(x)\alpha x \\ &= d(x)\alpha x \leq x \\ d(x) &\leq x. \end{aligned}$$

Suppose $d(1) = 1$. By Theorem 18, we have $x \leq d(x)$. Therefore $d(x) = x$. Converse is obvious. ■

Theorem 20. *Let M be an ordered Γ-semiring and d be a right derivation. Then for an element $a \in M, d\big((a\alpha)^n a\big) = nd(a)\alpha(a\alpha)^{n-1}a$, for all $\alpha \in \Gamma$.*

Proof. We prove this result by mathematical induction. Let d be a right derivation of $M, a \in M$ and $\alpha \in \Gamma$. Suppose $n = 1$.

$$d(a\alpha a) = d(a)\alpha a + d(a)\alpha a = 2d(a)\alpha a.$$

Suppose that the statement is true for n, i.e., $d\big((a\alpha)^n a\big) = nd(a)\alpha(a\alpha)^{n-1}a$.

$$\begin{aligned}\text{Then } d\big((a\alpha)^{n+1}a\big) &= d\big((a)\alpha(a\alpha)^n a\big)\\ &= d(a)\alpha(a\alpha)^n a + d\big((a\alpha)^n a\big)\alpha a\\ &= d(a)\alpha(a\alpha)^n a + nd(a)\alpha(a\alpha)^n a\\ &= (n+1)d(a)\alpha(a\alpha)^n a.\end{aligned}$$

Hence by mathematical induction, the theorem is true for all $n \in \mathbb{N}$. ■

Definition 10. An ordered Γ-semiring M is called a prime ordered Γ-semiring if $a\Gamma M\Gamma b = 0$ then $a = 0$ or $b = 0$, for all $a, b \in M$.

We write $[x\alpha y]$ for $x\alpha y - y\alpha x,\ x, y \in M$ and $\alpha \in \Gamma$.

Theorem 21. *Let M be an additively commutative cancellative prime ordered Γ-semiring and d be a non-zero right derivation of M. Then M is a commutative ordered Γ-semiring.*

Proof. Suppose $a, b \in M$ and $\alpha \in \Gamma$.

$$\begin{aligned}d(a\alpha b\alpha a) &= d(a)\alpha b\alpha a + d(b\alpha a)\alpha a\\ &= d(a)\alpha b\alpha a + (d(b)\alpha a + d(a)\alpha b)\alpha a.\\ &= d(a)\alpha b\alpha a + d(b)\alpha a\alpha a + d(a)\alpha b\alpha a \cdots \qquad (1)\\ d(a\alpha b\alpha a) &= d(a\alpha b)\alpha a + d(a)\alpha(a\alpha b)\\ &= \big(d(a)\alpha b + d(b)\alpha a\big)\alpha a + d(a)\alpha a\alpha b\\ &= d(a)\alpha b\alpha a + d(b)\alpha a\alpha a + d(a)\alpha a\alpha b \cdots \qquad (2).\end{aligned}$$

From (1) and (2) ,

$$d(a)\alpha b\alpha a = d(a)\alpha a\alpha b \cdots \qquad (3).$$

$$\Rightarrow\ d(a)\alpha[a\alpha b] = 0, \text{ for all } a, b \in M, \alpha \in \Gamma.$$

Replacing b by $c\alpha b$ in (3), we get

$$\begin{aligned}d(a)\alpha c\alpha b\alpha a &= d(a)\alpha a\alpha c\alpha b\\ &= d(a)\alpha c\alpha a\alpha b, \text{ since } d(a)\alpha a\alpha c = d(a)\alpha c\alpha a\\ \Rightarrow\ d(a)\alpha c\alpha[a\alpha b] &= 0, \text{ for all } a, b \in M, \alpha \in \Gamma\\ \Rightarrow\ d(a)\alpha M\alpha[a\alpha b] &= 0, \text{ for all } a, b \in M, \alpha \in \Gamma.\end{aligned}$$

Since $d(a) \neq 0$, we get $[a\alpha b] = 0$, for all $a, b \in M, \alpha \in \Gamma$.
Hence M is a commutative ordered Γ-semiring. ■

4. Conclusion

In this paper, we introduced the notion of a right derivation of an ordered Γ-semirings and we studied some of the properties of right derivations of ordered Γ-semiring and we proved that, if d is a non-zero right derivation of an additively commutative cancellative prime ordered Γ-semring M then M is a commutative ordered Γ-semiring. In continuation of this paper, we propose to introduce the notion of derivation of fuzzy ideal of ordered Γ-semiring and study the properties of derivations of fuzzy ideals of ordered Γ-semirings.

Acknowledgments

The authors are highly grateful to the chief editor and referee for thier careful reading, valuable suggestions and comments, which helped to improve the presentation of this paper.

References

[1] P.J. Allen, *A fundamental theorem of homomorphism for semirings*, Proc. Amer. Math. Soc. **21** (1969) 412–416. doi:10.1090/S0002-9939-1969-0237575-4

[2] M. Bresar and J. Vukman, *On the left derivation and related mappings*, Proc. Amer. Math. Soc. **10** (1990) 7–16. doi:10.2307/2048234

[3] T.K. Dutta and S. Kar, *On regular ternary semirings*, in: Advances in Algebra, Proceedings of the ICM Satellite Conference in Algebra and Related Topics (Ed(s)), (World Scientific, 2003) 343–355.

[4] H. Lehmer, *A ternary analogue of abelian groups*, Amer. J. Math. **59** (1932) 329–338. doi:10.2307/2370997

[5] W.G. Lister, *Ternary rings*, Tran. Amer. Math. Soc. **154** (1971) 37–55. doi:10.2307/1995425

[6] K.H. Kim, *On right derivation of incline algebras*, J. Chungcheong Math. Soc. **26** (2013) 683–690. doi:10.14403/jcms.2013.26.4.683

[7] K.H. Kim, *On generalized right derivation of incline algebras*, Gulf J. Math. **3** (2015) 36–46.
gjom.org/wp-content/uploads/2015/02/v7-4-H20149.pdf

[8] M. Murali Krishna Rao, Γ-*semirings-I*, Southeast Asian Bull. Math. **19** (1995) 49–54.

[9] M. Murali Krishna Rao, Γ-*semirings-II*, Southeast Asian Bull. of Math. **21** (1997) 281–287.

[10] M. Murali Krishna Rao and B. Venkateswarlu, *Regular* Γ-*semirings and field* Γ-*semirings*, Novi Sad J. Math. **45** (2015) 155–171.
emis.ams.org/journals/NSJOM/Papers/45-2 /NSJOM-45-2-155-171.pdf

[11] V. Neumann, *On regular rings*, Proc. Nat. Acad. Sci. **22** (1936) 707–713. doi:org/10.1073/pnas.22.12.707

[12] N. Nobusawa, *On a genralization of the ring theory*, Osaka. J. Math. **1** (1964) 81–89. ir.library.osaka-u.ac.jp/dspace/bitstream/11094/12354/1/ojm 01-01-08.pdf

[13] E.C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957) 1093–1100. doi:10.2307/2032686

[14] M.K. Sen, *On* Γ*-semigroup*, in: Proc. of International Conference of algebra and its application, Decker Publication (Ed(s)), (New York, 1981) 301–308..

[15] H.S. Vandiver, *Note on a simple type of algebra in which the cancellation law of addition does not hold*, Bull. Amer. Math. **40** (1934) 914–921. doi:10.1090/s0002-9904-1934-06003-8

Received 15 August 2016
Revised 24 October 2016