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CUBIC GENERALIZED BI-IDEALS IN SEMIGROUPS

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Abstract

In this paper, the concept of a cubic generalized bi-ideal in a semigroup is introduced, which is a generalization of the concept of a fuzzy generalized bi-ideal and interval-valued fuzzy generalized bi-ideal. Using this concept some characterization theorems are provided. In particular, we characterize regular semigroups by using cubic generalized bi-ideals. We show how the images or inverse images of a cubic generalized bi-ideal of a semigroup become a cubic generalized bi-ideal.

Keywords: semigroups, left (right) ideals, generalized bi-ideal, bi-ideal, cubic left (right) ideal, cubic bi-ideal, cubic generalized bi-ideal.

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1. Introduction

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation [1]. The formal study of semigroups started in the early 20th century. Semigroups are important in many branches of mathematics, for example coding and language theory, automata theory. The concept of fuzzy set was introduced by Zadeh [2]. The fuzzy algebraic structures play a prominent role in mathematics with extensive applications in many other branches such as theoretical physics, computer science, control engineering, information sciences, coding theory, topological spaces, logic, set theory, group theory, groupids, real analysis, measure theory etc. Kuroki [3–6] is the pioneer of fuzzy ideal theory of semigroups. The notion of fuzzy generalized bi-ideal in semigroups was introduced by Kuroki [6]. In [7], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set, i.e., a fuzzy set with an interval-valued membership function. Interval-valued fuzzy sets have been actively used in real-life applications. Using a fuzzy set and an interval-valued fuzzy set, Jun *et al.* [8] introduced a new notion, called a cubic set, and investigated several properties. In [9], Jun *et al.* introduced cubic subsemigroups and cubic left (resp. right) ideals. They studied several properties of cubic subsemigroups and cubic left (resp. right) ideals, and discussed the relation between them.

In this paper, we introduce cubic generalized bi-ideals in semigroups and study their several properties. We characterize regular semigroups in terms of cubic generalized bi-ideals. We prove that how the images or inverse images of a cubic generalized bi-ideal of a semigroup becomes a cubic generalized bi-ideal.

2. Preliminaries

A non-empty set S together with an associative binary operation "." is called a *semigroup*. A non-empty subset A of a semigroup S is called a *subsemigroup* if $xy \in A$ for all $x, y \in A$. Let S be a semigroup, by a *left* (resp. *right*) *ideal* of S, we mean a non-empty subset A of S such that $SA \subseteq A(AS \subseteq A)$. By a *two-sided ideal* or simply an ideal of S, we mean a non-empty subset A of S, which is both a left and a right ideal of S. Let S be a semigroup, by a *bi-ideal* of S, we mean a subsemigroup A of S such that $ASA \subseteq A$. A non-empty subset A of S is called a *generalized bi-ideal* of S if $ASA \subseteq A$.

A semigroup S is called *regular* if for each element $x \in S$, there exist y in S such that $x = xyx$.

An *interval valued fuzzy set* (*IVF* set for short) $\widetilde{\mu}$, defined on a non-empty set X is given by

$$\widetilde{\mu} = \{x, [\mu_A^+(x), \mu_A^-(x)]), x \in X\},$$

which is briefly denoted by $\widetilde{\mu} = [\mu_A^+, \mu_A^-]$, where μ_A^+ and μ_A^- are two fuzzy sets in X such that $\mu_A^+ \leq \mu_A^-$ for all $x \in X$.

A cubic set $\mathcal{A}$ in set X is a structure having the form

$$\mathcal{A} = \{\langle x, \widetilde{\mu}_A(x), f_A(x)\rangle : x \in X\},$$

which is briefly denoted by $\mathcal{A} = \langle \widetilde{\mu}, f\rangle$, where $\widetilde{\mu} = [\mu_A^+, \mu_A^-]$ is an IVF set in X and f is a fuzzy set in X.

Let $C(X)$ denotes the family of cubic sets in a set X. The order relation "$\sqsubseteq$" in the set of all cubic sets $\mathcal{A} = \langle \widetilde{\mu}_A, f_A\rangle$ and $\mathcal{B} = \langle \widetilde{\mu}_B, f_B\rangle$ is defined as follows: $\mathcal{A} \sqsubseteq \mathcal{B}$ if and only if $\widetilde{\mu}_A \preceq \widetilde{\mu}_B$, $f_A \geq f_B$ if and only if $\widetilde{\mu}_A(x) \preceq \widetilde{\mu}_B(x)$, $f_A(x) \geq f_B(x)$ for all $x \in X$.
Let $\mathcal{A}$ and $\mathcal{B}$ be two cubic sets in S. Then $\mathcal{A} = \mathcal{B}$ if and only if $\mathcal{A} \sqsubseteq \mathcal{B}$ and $\mathcal{B} \sqsubseteq \mathcal{A}$.

$$\mathcal{A} \sqcap \mathcal{B} = \langle \widetilde{\mu}_A \,\widetilde{\cap}\, \widetilde{\mu}_B, f_A \vee f_B\rangle,$$

where $(\widetilde{\mu}_A \widetilde{\cap} \widetilde{\mu}_B)(x) = \text{rmin}\{\widetilde{\mu}_A(x), \widetilde{\mu}_B(x)\}$ and $(f_A \vee f_B) = \max\{f_A(x), f_B(x)\}$

$$\mathcal{A} \sqcup \mathcal{B} = \langle \widetilde{\mu}_A \widetilde{\cup}\, \widetilde{\mu}_B, f_A \wedge f_B \rangle,$$

where $(\widetilde{\mu}_A \widetilde{\cup} \widetilde{\mu}_B)(x) = \text{rmax}\{\widetilde{\mu}_A(x), \widetilde{\mu}_B(x)\}$ and $(f_A \wedge f_B) = \min\{f_A(x), f_B(x)\}$ $\mathcal{A} \copyright \mathcal{B} = \{\langle x\colon (\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_B)(x), (f_A \circ f_B)(x) \rangle \colon x \in S\}$, which is briefly denoted by $\mathcal{A} \copyright \mathcal{B} = \langle (\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_B), (f_A \circ f_B) \rangle$, where

$$(\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_B)(x) = \begin{cases} \underset{x=yz}{\text{r}\sup}\,[\,\text{rmin}\{\widetilde{\mu}_A(y), \widetilde{\mu}_B(z)\}] & \text{if } x = yz \text{ for some } y, z \in S, \\ [0,0] & \text{otherwise} \end{cases}$$

and

$$(f_A \circ f_B)(x) = \begin{cases} \bigwedge\limits_{x=yz} [\,\max\{f_A(y), f_B(z)\}] & \text{if } x = yz \text{ for some } y, z \in S, \\ 1 & \text{otherwise.} \end{cases}$$

A cubic subset $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ of S is called a *cubic subsemigroup* [9] of S if

(i) $\widetilde{\mu}_A(xy) \succeq \text{rmin}\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\}$ for all $x, y \in S$.
(ii) $f_A(xy) \leq \max\{f_A(x), f_A(y)\}$ for all $x, y \in S$.

Let A be a non-empty subset of S, the cubic characteristic function of A in S is defined by

$$\chi_A = \{\langle x, \widetilde{\mu}_{\chi_A}, f_{\chi_A} \rangle \colon x \in S\},$$

which is briefly denoted by $\chi_A(x) = \langle \widetilde{\mu}_{\chi_A}(x), f_{\chi_A}(x) \rangle$

$$\widetilde{\mu}_{\chi_A}(x) = \begin{cases} [1,1] & \text{if } x \in A, \\ [0,0], & \text{otherwise} \end{cases}$$

and

$$f_{\chi_A}(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1, & \text{otherwise.} \end{cases}$$

Definition 2.2 ([9]). A cubic subset $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ of S is called a *cubic left ideal* of S if

(i) $\widetilde{\mu}_A\ (xy) \succeq \widetilde{\mu}_A(y)$ for all $x, y \in S$,
(ii) $f_A(xy) \leq f_A(y)$ for all $x, y \in S$.

A cubic subset $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ of S is called a *cubic right ideal* of S if

(i) $\widetilde{\mu}_A(xy) \succeq \widetilde{\mu}_A(x)$ for all $x, y \in S$,

(ii) $f_A(xy) \leq f_A(x)$ for all $x, y \in S$.

A cubic subset $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ of S is called a *cubic ideal* of S if it is both a cubic left ideal and a cubic right ideal of S.

Theorem 2.3 ([9]). *For a cubic set $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ in a semigroup S the following are equivalent:*

(1) *$\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ is a cubic left (resp. right) ideal of S.*

(2) *$\chi_S \copyright \mathcal{A} \sqsubseteq \mathcal{A}$ (resp. $\mathcal{A} \copyright \chi_S \sqsubseteq \mathcal{A}$), where χ_S is the cubic characteristic function of S maps every element of S to $\langle [1,1], 0 \rangle$.*

Theorem 2.4 ([9]). *For every cubic right ideal $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ and every cubic left ideal $\mathcal{B} = \langle \widetilde{\mu}_B, f_B \rangle$ of S, if $\mathcal{A} \copyright \mathcal{B} = \mathcal{A} \sqcap \mathcal{B}$, then S is regular.*

Lemma 2.5 ([9]). *For a non-empty subset G of a semigroup S, we have G is a left (resp. right) ideal of S if and only if the cubic characteristic function $\chi_G = \langle \widetilde{\mu}_{\chi_G}, f_{\chi_G} \rangle$ of G in S is a cubic left (resp. right) ideal of S.*

Definition 2.6 ([12]). A cubic subsemigroup $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ of S is called a *cubic bi-ideal* of S if

(i) $\widetilde{\mu}_A(xyz) \succeq \text{rmin}\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(z)\}$ for all $x, y, z \in S$,

(ii) $f_A(xyz) \leq \max\{f_A(x), f_A(z)\}$ for all $x, y, z \in S$.

3. Cubic generalized bi-ideal

Definition 3.1. A cubic subset $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ of a semigroup S is called a *cubic generalized bi-ideal* of S if

(i) $\widetilde{\mu}_A(xyz) \succeq \text{rmin}\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(z)\}$ for all $x, y, z \in S$,

(ii) $f_A(xyz) \leq \max\{f_A(x), f_A(z)\}$ for all $x, y, z \in S$.

Theorem 3.2. *Let A be a non-empty subset of S and χ_A be the cubic characteristic function of A. Then $\chi_A = \langle \widetilde{\mu}_{\chi_A}, f_{\chi_A} \rangle$ is a cubic generalized bi-ideal of S if and only if A is a generalized bi-ideal of S.*

Proof. Let A be a generalized bi-ideal of S and $\chi_A = \langle \widetilde{\mu}_{\chi_A}, f_{\chi_A} \rangle$ be the cubic characteristic function of A. Let $x, y, z \in S$ be such that $x, z \in A$, then $xyz \in A$ it follows that $\widetilde{\mu}_{\chi_A}(xyz) = [1,1] = \text{rmin}\{\widetilde{\mu}_{\chi_A}(x), \widetilde{\mu}_{\chi_A}(z)\}$ and $f_{\chi_A}(xyz) = 0 = \max\{f_{\chi_A}(x), f_{\chi_A}(z)\}$.

If $x \notin A$ and $z \notin A$, then

Case (i) If $xyz \notin A$, then $\widetilde{\mu}_{\chi_A}(xyz) \succeq [0,0] = \text{rmin}\{\widetilde{\mu}_{\chi_A}(x), \widetilde{\mu}_{\chi_A}(z)\}$ and $f_{\chi_A}(xyz) \leq 1 = \max\{f_{\chi_A}(x), f_{\chi_A}(z)\}$.

Case (ii) If $xyz \in A$, then $\widetilde{\mu}_{\chi_A}(xyz) = [1,1] \succeq [0,0] = \text{rmin}\{\widetilde{\mu}_{\chi_A}(x), \widetilde{\mu}_{\chi_A}(z)\}$ and $f_{\chi_A}(xyz) = 0 \leq 1 = \max\{f_{\chi_A}(x), f_{\chi_A}(z)\}$.

Thus $\chi_A = \langle \widetilde{\mu}_{\chi_A}, f_{\chi_A} \rangle$ is a cubic generalized bi-ideal of S.

Conversely, let χ_A be a cubic generalized bi-ideal of S. We have to prove that A is a generalized bi-ideal of S. Let $x, z \in A$, then $\widetilde{\mu}_{\chi_A}(x) = [1,1] = \widetilde{\mu}_{\chi_A}(z)$ and $f_{\chi_A}(x) = 0 = f_{\chi_A}(z)$. Thus $\widetilde{\mu}_{\chi_A}(xyz) \succeq \text{rmin}\{\widetilde{\mu}_{\chi_A}(x), \widetilde{\mu}_{\chi_A}(z)\} = [1,1]$ and $f_{\chi_A}(xyz) \leq \max\{f_{\chi_A}(x), f_{\chi_A}(z)\} = 0$. So $\widetilde{\mu}_{\chi_A}(xyz) = [1,1]$ and $f_{\chi_A}(xyz) = 0 \Longrightarrow xyz \in A$ for all $x, y, z \in S$. Hence A is a generalized bi-ideal of S.

Let $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ be a cubic set in S. For any $r \in [0,1]$ and $[s,t] \in D[0,1]$, we define cubic level set of $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ as: $U(\mathcal{A}; [s,t], r) = \{x \in S \colon \widetilde{\mu}_A(x) \succeq [s,t], f_A(x) \leq r\}$, and is called a cubic level set of $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ (see [11]).

Theorem 3.3. *Let $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ be a cubic subset of S. Then the following are equivalent:*

(i) *$\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ is a cubic generalized bi-ideal of S.*

(ii) *Each non-empty cubic level set $U(\mathcal{A}; [s,t], r)$ is a generalized bi-ideal of S.*

Proof. Assume that $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ is a cubic generalized bi-ideal of S. Let $x, z \in U(\mathcal{A}; [s,t], r)$ for all $r \in [0,1]$ and $[s,t] \in D[0,1]$ and $y \in S$. Then $\widetilde{\mu}_A(x) \succeq [s,t], \widetilde{\mu}_A(z) \succeq [s,t], f_A(x) \leq r, f_A(z) \leq r$. It follows from Definition 3.1, that $\widetilde{\mu}_A(xyz) \succeq \text{rmin}\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(z)\} \succeq [s,t]$ and $f_A(xyz) \leq \max\{f_A(x), f_A(z)\} \leq r$. Hence $xyz \in U(\mathcal{A}; [s,t], r)$ and thus $U(\mathcal{A}; [s,t], r)$ is a generalized bi-ideal of S.

Conversely, let $r \in [0,1]$ and $[s,t] \in D[0,1]$ be such that $U(\mathcal{A}; [s,t], r) \neq \phi$, and $U(\mathcal{A}; [s,t], r)$ is a generalized bi-ideal of S. Suppose that (ii) does not hold. Then there exist $x, y, z \in S$ such that $\widetilde{\mu}_A(xyz) \not\succeq \text{rmin}\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(z)\}$ or $f_A(xyz) \not\leq \max\{f_A(x), f_A(z)\}$.

If $\widetilde{\mu}_A(xyz) \not\succeq \text{rmin}\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(z)\}$, then $\widetilde{\mu}_A(xyz) \prec [s_1, t_1] \preceq \text{rmin}\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(z)\}$ for some $[s_1, t_1] \in D[0,1]$. Hence $x, z \in U(\mathcal{A}; [s_1, t_1], \max\{f_A(x), f_A(z)\})$, but $xyz \notin U(\mathcal{A}; [s_1, t_1], \max\{f_A(x), f_A(z)\})$. This gives a contradiction.

If $f_A(xyz) \not\leq \max\{f_A(x), f_A(z)\}$, then there exists $r_1 \in [0,1]$ such that $f_A(xyz) > r_1 > \max\{f_A(x), f_A(z)\}$. Thus $x, z \in U(\mathcal{A}, \text{rmin}\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(z)\}, r_1)$ and $xyz \notin U(\mathcal{A}, \text{rmin}\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(z)\}, r_1)$. This is a contradiction. Thus (ii) holds and therefore $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ is a cubic generalized bi-ideal of S.

Example 3.4. Consider a semigroup $S = \{0, a, b, c\}$ with the following Cayley table.

.	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	b

Generalized bi-ideals of S are $\{0\}, \{0,a\}, \{0,b\}, \{0,a,b\}, \{0,a,c\}, S$. Define a cubic set $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ in S as follows:

S	$\widetilde{\mu}_A(x)$	$f_A(x)$
0	[0.8,0.9]	0.2
a	[0.7,0.7]	0.6
b	[0.5,0.6]	0.7
c	[0.2,0.2]	0.8

Then

$$U(\mathcal{A};[s,t]) = \begin{cases} S & \text{if } [0,0] \prec [s,t] \preceq [0.2,0.2] \\ \{0,a,b\} & \text{if } [0.2,0.2] \prec [s,t] \preceq [0.5,0.6] \\ \{0,a\} & \text{if } [0.5,0.6] \prec [s,t] \preceq [0.7,0.7] \\ \{0\} & \text{if } [0.7,0.7] \prec [s,t] \preceq [0.8,0.9] \\ \emptyset & \text{if } [0.8,0.9] \prec [s,t] \preceq [0.9,1] \end{cases}$$

$$U(\mathcal{A};r) = \begin{cases} S & \text{if } 0.8 \leq [s,t] < 0.9 \\ \{0,a,b\} & \text{if } 0.7 \leq [s,t] < 0.8 \\ \{0,a\} & \text{if } 0.6 \leq [s,t] < 0.7 \\ \{0\} & \text{if } 0.3 \leq [s,t] < 0.6 \\ \emptyset & \text{if } 0 \leq [s,t] < 0.3. \end{cases}$$

Then $U(\mathcal{A};[s,t],r)$ is a generalized bi-ideal of S and by Theorem 3.3, $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ is cubic generalized bi-ideal of S.

Remark 1. It is clear that each cubic bi-ideal of S is cubic generalized bi-ideal of S. But the converse does not hold, which is shown in the following example.

Example 3.5. Consider a semigroup $S = \{a,b,c,d\}$ with the following Cayley table

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define a cubic set $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ in S as follows

S	$\widetilde{\mu}_A(x)$	$f_A(x)$
0	[0.8,0.9]	0.1
a	[0,0]	0.7
b	[0.7,0.7]	0.3
c	[0.4,0.5]	0.4

It is easy to verify that $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ is a cubic generalized bi-ideal of S, but it is not a cubic bi-ideal of S. Since $\widetilde{\mu}_A(dc) = \widetilde{\mu}_A(b) = [0,0] \not\succeq [0.4, 0.5] = \text{rmin}$ $\{\widetilde{\mu}_A(d), \widetilde{\mu}_A(c)\}$ and $f_A(dc) = f_A(b) = 0.7 \nleq 0.4 = \max\{f_A(d), f_A(c)\}$. Hence $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ is not a cubic subsemigroup of S.

Remark 2. In the following Theorem, we give a condition in which every cubic generalized bi-ideal of S becomes a cubic bi-ideal of S.

Theorem 3.6. *Let S be a regular semigroup. Then every cubic generalized bi-ideal of S is a cubic bi-ideal of S.*

Proof. Let $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ be a cubic generalized bi-ideal of S. Let $a, b \in S$. Since S is regular, there exists $x \in S$ such that $b = bxb$

$$\begin{aligned} \widetilde{\mu}_A(ab) &= \widetilde{\mu}_A(abxb) \\ &= \widetilde{\mu}_A(a(bx)b) \\ &\succeq \text{rmin}\{\widetilde{\mu}_A(a), \widetilde{\mu}_A(b)\} \end{aligned}$$

and

$$\begin{aligned} f_A(ab) &= f_A(abxb) \\ &= f_A(a(bx)b) \\ &\leq \max\{f_A(a), f_A(b)\}. \end{aligned}$$

So $\mathcal{A}$ is a cubic subsemigroup and consequently $\mathcal{A}$ is a cubic bi-ideal of S.

Proposition 3.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be two cubic generalized bi-ideals of a semigroup S. Then $\mathcal{A} \sqcap \mathcal{B}$ is a cubic generalized bi-ideal of S, provided $\mathcal{A} \sqcap \mathcal{B}$ is non-empty.*

Proof. Let $\mathcal{A}(x) = \langle \widetilde{\mu}_A(x), f_A(x)\rangle$ and $\mathcal{B}(x) = \langle \widetilde{\mu}_B(x), f_B(x)\rangle$ be two cubic generalized bi-ideals of S. Let $x, y, z \in S$. $\mathcal{A} \sqcap B = \langle \widetilde{\mu}_A \widetilde{\cap}\, \widetilde{\mu}_B, f_A \vee f_B\rangle$

$$\begin{aligned}(\widetilde{\mu}_A \widetilde{\cap} \widetilde{\mu}_B)(xyz) &= \text{rmin}\,\{\widetilde{\mu}_A(xyz), \widetilde{\mu}_B(xyz)\}\\ &\succeq \text{rmin}\,\{\text{rmin}\,\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(z),\ \text{rmin}\,\{\widetilde{\mu}_B(x), \widetilde{\mu}_B(z)\}\}\\ &= \text{rmin}\,\{\text{rmin}\,\{\widetilde{\mu}_A(x), \widetilde{\mu}_B(x),\ \text{rmin}\,\{\widetilde{\mu}_A(z), \widetilde{\mu}_B(z)\}\}\\ &= \text{rmin}\,\{\widetilde{\mu}_A \widetilde{\cap}\, \widetilde{\mu}_B)(x), \widetilde{\mu}_A \widetilde{\cap}\, \widetilde{\mu}_B)(z)\}\end{aligned}$$

$$\begin{aligned}(f_A \vee f_B)(xyz) &= \max\,\{f_A(xyz), f_B(xyz)\}\\ &\le \max\,\{\max\,\{f_A(x), f_A(z),\ \max\,\{f_B(x), f_B(z)\}\}\\ &= \max\,\{\max\,\{f_A(x), f_B(x),\ \max\,\{f_A(z), f_B(z)\}\}\\ &= \max\,\{(f_A \vee f_B)(x), (f_A \vee f_B)(z)\}.\end{aligned}$$

Hence $\mathcal{A} \sqcap \mathcal{B}$ is a cubic generalized bi-ideal of S.

Lemma 3.8. *A cubic subset $\mathcal{A} = \langle \widetilde{\mu}_A, f_A\rangle$ of a semigroup S is a cubic generalized bi-ideal of S if and only if $\mathcal{A} Ⓒ \chi_S Ⓒ \mathcal{A} \sqsubseteq \mathcal{A}$, where χ_S is the cubic characteristic function of S.*

Proof. Let $\mathcal{A} = \langle \widetilde{\mu}_A, f_A\rangle$ be a cubic generalized bi-ideal of S. Suppose there exist $x, y, p, q \in S$ such that $x = yz$ and $y = pq$. Since $\mathcal{A}$ is a cubic generalized bi-ideal of S, we obtain

$$\begin{aligned}(\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_{\chi_S} \widetilde{\circ} \widetilde{\mu}_A)(x) &= \underset{x=yz}{\text{r}\sup}\,[\text{rmin}\{(\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_{\chi_S})(y), \widetilde{\mu}_A(z)\}]\\ &= \underset{x=yz}{\text{r}\sup}\,[\text{rmin}\{\underset{y=pq}{\text{r}\sup}\{\text{rmin}\{\widetilde{\mu}_A(p), \widetilde{\mu}_{\chi_S})(q)\}\}, \widetilde{\mu}_A(z)\}]\\ &= \underset{y=pqz}{\text{r}\sup}[\text{rmin}\{\widetilde{\mu}_A(p), \widetilde{\mu}_{\chi_S})(q), \widetilde{\mu}_A(z)\}]\\ &= \underset{y=pqz}{\text{r}\sup}\,[\text{rmin}\{\widetilde{\mu}_A(p), [1,1], \widetilde{\mu}_A(z)\}]\\ &= \underset{y=pqz}{\text{r}\sup}\,[\text{rmin}\{\widetilde{\mu}_A(p), \widetilde{\mu}_A(z)\}] \preceq \underset{y=pqz}{\text{r}\sup}[\widetilde{\mu}_A(pqz)] = \widetilde{\mu}_A(x),\end{aligned}$$

so we have $(\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_{\chi_S} \widetilde{\circ} \widetilde{\mu}_A)(x) \preceq \widetilde{\mu}_A(x)$ and

$$\begin{aligned}(f_A \circ f_{\chi_S} \circ f_A)(x) &= \bigwedge_{x=yz} [\max\{(f_A \circ f_{\chi_S})(y), f_A)(z)\}]\\ &= \bigwedge_{x=yz} \Big[\max\Big\{ \bigwedge_{y=pq} \{\max\{(f_A(p), f_{\chi_S}(q))\}\}, f_A(z)\Big\}\Big]\\ &= \bigwedge_{x=yz} \Big[\max\Big\{ \bigwedge_{y=pq} \{\max\{f_A(p), 0\}\}, f_A(z)\Big\}\Big]\\ &= \bigwedge_{x=pqz} [\max\,\{f_A(p), f_A(z)\}] \ge \bigwedge_{x=pqz} [f_A(pqz)] = f_A(x),\end{aligned}$$

so $(f_A \circ f_S \circ f_A)(x) \geq f_A(x)$. Hence $\mathcal{A}ⓒ\chi_Sⓒ\mathcal{A} \sqsubseteq \mathcal{A}$.

Conversely, assume that $\mathcal{A}ⓒ\chi_Sⓒ\mathcal{A} \sqsubseteq \mathcal{A}$

$$\begin{aligned}
\widetilde{\mu}_A(a) &\succeq (\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_{\chi_S} \widetilde{\circ} \widetilde{\mu}_A)(a) \\
&= \operatorname{r}\sup_{a=yz} [\operatorname{rmin}\{(\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_{\chi_S})(y), \widetilde{\mu}_A(z)\}] \\
&= \operatorname{r}\sup_{a=yz} [\operatorname{rmin}\{\operatorname{r}\sup_{y=pq}\{\operatorname{rmin}\{\widetilde{\mu}_A(p), \widetilde{\mu}_{\chi_S})(q)\}\}, \widetilde{\mu}_A(z)\}] \\
&= \operatorname{r}\sup_{a=yz} \operatorname{r}\sup_{y=pq} [\operatorname{rmin}\{\widetilde{\mu}_A(p), [1,1], \widetilde{\mu}_A(z)\}] \\
&= \operatorname{r}\sup_{a=pqz} [\operatorname{rmin}\{\widetilde{\mu}_A(p), \widetilde{\mu}_A(z)\}]
\end{aligned}$$

and

$$\begin{aligned}
f_A(a) &\leq (f_A \circ f_{\chi_S} \circ f_A)(a) \\
&= \bigwedge_{a=yz} [\max\{(f_A \circ f_{\chi_S})(y), \circ f_A)(z)\}] \\
&= \bigwedge_{a=yz} \Big[\max\Big\{ \bigwedge_{y=pq} \{\max\{(f_A(p), f_{\chi_S}(q))\}\}, f_A(z)\Big\}\Big] \\
&= \bigwedge_{a=yz} \bigwedge_{y=pq} [\max\{(f_A(p), 0, f_A(z)\}] \\
&= \bigwedge_{a=pqz} [\max\{(f_A(p), f_A(z)\}].
\end{aligned}$$

Thus $\mathcal{A} = \langle \widetilde{\mu}_A, f_A \rangle$ is a cubic generalized bi-ideal of S.

Theorem 3.9. *Let S be a semigroup. The following are equivalent*

(1) *S is regular.*

(2) *For every cubic generalized bi-ideal $\mathcal{A}$ of S, $\mathcal{A}ⓒ\chi_Sⓒ\mathcal{A} = \mathcal{A}$.*

Proof. (1)$\Longrightarrow$(2) Let $\mathcal{A}$ be a cubic generalized bi-ideal of S and $a \in S$. Then there exist $x \in S$ such that $a = axa$

$$\begin{aligned}
(\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_{\chi_S} \widetilde{\circ} \widetilde{\mu}_A)(a) &= \operatorname{r}\sup_{a=axa} [\operatorname{rmin}\{(\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_{\chi_S})(ax), \widetilde{\mu}_A(a)\}] \\
&\succeq \operatorname{rmin}\{ \operatorname{r}\sup_{ax=axax} \{\operatorname{rmin}\{\widetilde{\mu}_A(a), \widetilde{\mu}_{\chi_S})(xax)\}\}, \widetilde{\mu}_A(a)\}] \\
&\succeq \operatorname{rmin}\{\operatorname{rmin}\{\widetilde{\mu}_A(a), [1,1]\}, \widetilde{\mu}_A(a)\} \\
&= \operatorname{rmin}\{\widetilde{\mu}_A(a), \widetilde{\mu}_A(a)\} \\
&= \widetilde{\mu}_A(a)\}
\end{aligned}$$

and

$$\begin{aligned}(f_A \circ f_{\chi_S} \circ f_A)(a) &= \bigwedge_{a=axa} [\max\{(f_A \circ f_{\chi_S})(ax), f_A(a)\}] \\ &\leq \max\Big\{ \bigwedge_{ax=axax} \max\{f_A(a), f_{\chi_S}(xax)\}, f_A(a)\Big\} \\ &\leq \max\{\max\{f_A(a), 0\}, f_A(a)\} \\ &= \max\{f_A(a), f_A(a)\} = f_A(a).\end{aligned}$$

So $\mathcal{A} \sqsubseteq ©\chi_S©\mathcal{A}$ and by Lemma 3.8, $\mathcal{A}©\chi_S©\mathcal{A} \sqsubseteq \mathcal{A}$. Hence $\mathcal{A}©\chi_S©\mathcal{A} = \mathcal{A}$.

(2)$\Longrightarrow$(1) Let A be a generalized bi-ideal of S. Then by Theorem 3.2, χ_A is a cubic generalized bi-ideal of S. By hypothesis, $\chi_A©\chi_s©\chi_A = \chi_A$. Let $a \in A$, then $\widetilde{\mu}_{\chi_A}(a) = [1,1]$ and $f_{\chi_A}(a) = 0$. Thus

$$\begin{aligned}(\widetilde{\mu}_{\chi_A} \widetilde{\circ} \widetilde{\mu}_{\chi_S} \widetilde{\circ} \widetilde{\mu}_{\chi_A})(a) &= [1,1] \\ \operatorname{r}\sup_{a=bc} [\operatorname{rmin}\{(\widetilde{\mu}_{\chi_A} \widetilde{\circ} \widetilde{\mu}_{\chi_S})(b), \widetilde{\mu}_{\chi_A}(c)\}] &= [1,1] \\ \operatorname{r}\sup_{a=bc} [\operatorname{rmin}\{\operatorname{r}\sup_{b=pq}\{\operatorname{rmin}\{\widetilde{\mu}_{\chi_A}(p), \widetilde{\mu}_{\chi_S})(q)\}\}, \widetilde{\mu}_{\chi_A}(c)\}] &= [1,1] \\ \operatorname{r}\sup_{a=bc} [\operatorname{rmin}\{\operatorname{r}\sup_{b=pq}\{\operatorname{rmin}\{\widetilde{\mu}_{\chi_A}(p), [1,1]\}\}, \widetilde{\mu}_{\chi_A}(c)\}] &= [1,1] \\ \operatorname{r}\sup_{a=pqc} [\operatorname{rmin}\{\widetilde{\mu}_{\chi_A}(p), \{\widetilde{\mu}_{\chi_A}(c)\}] &= [1,1]\end{aligned}$$

and

$$\begin{aligned}(f_{\chi_A} \circ f_{\chi_S} \circ f_{\chi_A})(a) &= 0 \\ \bigwedge_{a=bc} [\max\{(f_{\chi_A} \circ f_{\chi_S})(b), \circ f_{\chi_A}(c)\}] &= 0 \\ \bigwedge_{a=bc} \Big[\max\Big\{ \bigwedge_{b=pq} \{\max\{(f_{\chi_A}(p), f_{\chi_S}(q))\}\}, f_{\chi_A}(c)\Big\}\Big] &= 0 \\ \bigwedge_{a=bc} \Big[\max\Big\{ \bigwedge_{b=pq} \{\max\{f_{\chi_A}(p), 0\}\}, f_{\chi_A}(c)\Big\}\Big] &= 0 \\ \bigwedge_{a=pqc} [\max\{f_{\chi_A}(p), f_{\chi_A}(c)\}] &= 0.\end{aligned}$$

Thus we get $p, c \in S$ such that $a = pqc$ with $\widetilde{\mu}_{\chi_A}(p) = [1,1] = \widetilde{\mu}_{\chi_A}(c)$ and $f_{\chi_A}(p) = 0 = f_{\chi_A}(c)$ whence $p, c \in A$. So $a = pqc \in ASA$, consequently $A \subseteq ASA$. Since A is generalized bi-ideal of S, we have $ASA \subseteq A$.

Theorem 3.10. *A semigroup S is regular if and only if for each cubic generalized bi-ideal $\mathcal{A}$ of S and each cubic ideal $\mathcal{B}$ of S, $\mathcal{A} \sqcap \mathcal{B} = \mathcal{A}©\mathcal{B}©\mathcal{A}$.*

Proof. Let S be regular. Let $\mathcal{A}$ be a cubic generalized bi-ideal of S and $\mathcal{B}$ be a cubic ideal of S. Then by Lemma 3.8, $\mathcal{A}©\mathcal{B}©\mathcal{A} \sqsubseteq \mathcal{A}©\chi_S©\mathcal{A} \sqsubseteq \mathcal{A}$. Again by Theorem 2.3, $\mathcal{A}©\mathcal{B}©\mathcal{A} \sqsubseteq \chi_S©\mathcal{B}©\chi_S \sqsubseteq \chi_S©\mathcal{B} \sqsubseteq \mathcal{B}$. So $\mathcal{A}©\mathcal{B}©\mathcal{A} \sqsubseteq \mathcal{A} \sqcap \mathcal{B}$. Now let $a \in S$. Since S is regular, there exist $x \in S$ such that $a = axa = axaxa$. Then

$$\begin{aligned}
(\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_B \widetilde{\circ} \widetilde{\mu}_A)(a) &= \underset{a=axa}{\mathrm{r}\sup}\,[\mathrm{rmin}\{\widetilde{\mu}_A(a), (\widetilde{\mu}_B \widetilde{\circ} \widetilde{\mu}_A)(xa)\}] \\
&\succeq \mathrm{rmin}\,\{\widetilde{\mu}_A(a), (\widetilde{\mu}_B \widetilde{\circ} \widetilde{\mu}_A)(xa)\} \\
&= \mathrm{rmin}\,[\widetilde{\mu}_A(a), \underset{xa=xaxa}{\mathrm{r}\sup}\,\{\mathrm{rmin}\{\widetilde{\mu}_B(xax), \widetilde{\mu}_A(a)\}\}] \\
&\succeq \mathrm{rmin}\,\{\widetilde{\mu}_A(a), \widetilde{\mu}_B(xax), \widetilde{\mu}_A(a)\} \\
&= \mathrm{rmin}\,\{\widetilde{\mu}_A(a), \widetilde{\mu}_B(xax)\} \\
&\succeq \mathrm{rmin}\,\{\widetilde{\mu}_A(a), \widetilde{\mu}_B(a)\} \\
&= \{\widetilde{\mu}_A(a) \,\widetilde{\cap}\, \widetilde{\mu}_B(a)\}
\end{aligned}$$

and

$$\begin{aligned}
(f_A \circ f_B \circ f_A)(a) &= \bigwedge_{a=axa} [\max\{f_A(a), (f_B \circ f_A)(xa)\}] \\
&\leq \max\{f_A(a), (f_B \circ f_A)(xa)\} \\
&= \max\{f_A(a), \bigwedge_{xa=xaxa} \{\max\{f_B(xax), f_A(a)\}\}\} \\
&\leq \max\{f_A(a), f_B(xax), f_A(a)\} \\
&= \max\{f_A(a), f_B(xax)\} \\
&\leq \max\{f_A(a), f_B(a)\} \\
&= \{f_A(a) \vee f_B(a)\}.
\end{aligned}$$

So $\mathcal{A} \sqcap \mathcal{B} \sqsubseteq \mathcal{A}©\mathcal{B}©\mathcal{A}$. Hence $\mathcal{A} \sqcap \mathcal{B} = \mathcal{A}©\mathcal{B}©\mathcal{A}$.

Conversely, assume that $\mathcal{A}$ be a cubic generalized bi-ideal of S. Since χ_S is a cubic generalized bi-ideal, by hypothesis, $\mathcal{A} = \mathcal{A} \sqcap \chi_S = \mathcal{A}©\chi_S©\mathcal{A}$. Hence by Theorem 3.9, S is regular.

Theorem 3.11 [10]**.** *Let S be a semigroup. Then the following are equivalent:*

(1) *S is regular.*

(2) *$A \cap L \subseteq AL$ for each generalized bi-ideal A of S and each left ideal L of S.*

(3) *$R \cap A \cap L \subseteq RAL$ for each generalized bi-ideal A of S, each right ideal R of S and each left ideal L of S.*

Theorem 3.12. *Let S be a semigroup. Then the following are equivalent:*

(1) *S is regular,*

(2) *$\mathcal{A} \sqcap \mathcal{B} \sqsubseteq \mathcal{A}©\mathcal{B}$ for each cubic generalized bi-ideal $\mathcal{A}$ of S and for each cubic left ideal $\mathcal{B}$ of S,*

(3) *$\mathcal{A} \sqcap \mathcal{B} \sqcap \mathcal{C} \sqsubseteq \mathcal{A}©\mathcal{B}©\mathcal{C}$ for each cubic generalized bi-ideal $\mathcal{B}$ of S, for each cubic left ideal $\mathcal{C}$ of S and for each cubic right ideal $\mathcal{A}$ of S.*

Proof. (1)$\Longrightarrow$(2). Assume that S is regular, then for each $a \in S$ there exist x in S such that $a = axa = axaxa$. Then

$$\begin{aligned}(\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_B)(a) &= \underset{a=axaxa}{\mathrm{r}\sup}\,[\mathrm{rmin}\{\widetilde{\mu}_A(axa), \widetilde{\mu}_B(xa)\}] \\ &\succeq \mathrm{rmin}\{\widetilde{\mu}_A(axa), \widetilde{\mu}_B(xa)\} \text{ since } a = axa = axaxa \\ &\succeq \mathrm{rmin}\{\widetilde{\mu}_A(a), \widetilde{\mu}_B(a)\} \text{ since } \mathcal{A} \text{ is cubic generalized bi-ideal of } S \\ &\quad \text{and } \mathcal{B} \text{ is a cubic left ideal of } S \\ &= (\widetilde{\mu}_A \widetilde{\cap} \widetilde{\mu}_B)(a)\end{aligned}$$

and

$$\begin{aligned}(f_A \circ f_B)(a) &= \bigwedge_{a=axaxa} [\max\{f_A(axa), f_B(xa)\}] \\ &\leq \max\{f_A(axa), f_B(xa)\} \\ &\leq [\max\{f_A(a), f_B(a)\}] \\ &= (f_A(a) \vee f_B)(a).\end{aligned}$$

Hence $\mathcal{A} \sqcap \mathcal{B} \sqsubseteq \mathcal{A}©\mathcal{B}$.

(2)$\Longrightarrow$(1) Let A be a generalized bi-ideal of S, B be a left ideal of S and $a \in A \cap B$. Then $a \in A$ and $a \in B$. Since A is a generalized bi-ideal of S, so by Theorem 3.2, χ_A is cubic generalized bi-ideal of S and by Theorem 2.3, χ_B is cubic left ideal of S. Hence by hypothesis,

$$\begin{aligned}\chi_A \sqcap \chi_B &\sqsubseteq \chi_A © \chi_B \\ (\widetilde{\mu}_{\chi_A} \widetilde{\circ} \widetilde{\mu}_{\chi_B})(a) &\succeq (\widetilde{\mu}_{\chi_A} \widetilde{\cap} \widetilde{\mu}_{\chi_B})(a) \\ &= \mathrm{rmin}\{\widetilde{\mu}_{\chi_A}(a), \widetilde{\mu}_{\chi_B}(a)\} \\ &= [1,1].\end{aligned}$$

Thus $\underset{a=bc}{\mathrm{r}\sup}\,[\mathrm{rmin}\{\widetilde{\mu}_{\chi_A}(b), \widetilde{\mu}_{\chi_B}(c)\}] = [1,1]$ and

$$\begin{aligned}(f_{\chi_A} \circ f_{\chi_B})(a) &\leq (f_{\chi_A} \vee f_{\chi_B})(a) \\ &= \max\ \{f_{\chi_A}(a), f_{\chi_B}(a)\} = 0.\end{aligned}$$

Thus $\bigwedge_{a=bc} [\max\{f_{\chi_A}(b), f_{\chi_B}(c)\}]$. So there exist $b, c \in S$ with $a = bc$ such that $\widetilde{\mu}_{\chi_A}(b) = \widetilde{\mu}_{\chi_B}(c) = [1,1]$ and $f_{\chi_A}(b) = f_{\chi_B}(c) = 0$.

Consequently, $b \in A$ and $c \in B$. So $a = bc \in AB$. So $A \cap B \subseteq AB$. Hence by Theorem 3.11, S is regular.

(1)$\Longrightarrow$(3) Let S be regular. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be a cubic right ideal, cubic generalized bi-ideal and cubic left ideal of S, respectively. Let there exist $x \in S$ such that $a = axa = axaxa = axaxaxa$. Then

$$\begin{aligned}
(\widetilde{\mu}_A \widetilde{\circ} \widetilde{\mu}_B \widetilde{\circ} \widetilde{\mu}_C)(a) &= \underset{a=axa}{\mathrm{r}\sup}[\mathrm{rmin}\ \{\widetilde{\mu}_A(ax), (\widetilde{\mu}_B \widetilde{\circ} \widetilde{\mu}_C)(a)\}] \\
&\succeq \mathrm{rmin}\ \{\widetilde{\mu}_A(ax), (\widetilde{\mu}_B \widetilde{\circ} \widetilde{\mu}_C)(a)\} \\
&= \mathrm{rmin}\ \{\widetilde{\mu}_A(ax), \underset{a=axaxa}{\mathrm{r}\sup}\ \{\mathrm{rmin}\ \{\widetilde{\mu}_B(axa), \widetilde{\mu}_C(xa)\}\}\} \\
&\succeq \mathrm{rmin}\ [\widetilde{\mu}_A(ax), \mathrm{rmin}\ \{\widetilde{\mu}_B(axa), \widetilde{\mu}_C(xa)\}] \\
&= \mathrm{rmin}\ [\widetilde{\mu}_A(ax), \mathrm{rmin}\ \{\widetilde{\mu}_B(axa), \widetilde{\mu}_A(xa)\}] \\
&= \mathrm{rmin}\ \{\widetilde{\mu}_A(ax), \widetilde{\mu}_B(axa), \widetilde{\mu}_C(xa)\} \\
&\succeq \mathrm{rmin}\ \{\widetilde{\mu}_A(a), \widetilde{\mu}_B(a), \widetilde{\mu}_C(a)\} \\
&= (\widetilde{\mu}_A \widetilde{\cap} \widetilde{\mu}_B \,\widetilde{\cap}\, \widetilde{\mu}_C)(a)
\end{aligned}$$

and

$$\begin{aligned}
(f_A \circ f_B \circ f_C)(a) &= \bigwedge_{a=axa} [\max\{f_A(ax), (f_B \circ f_C)(a)\}] \\
&\leq \max\{f_A(ax), (f_B \circ f_C))(a)\} \\
&= \max[f_A(ax), \bigwedge_{a=axaxa} \{\max\{f_B(axa), f_C(xa)\}\}] \\
&\leq \max[f_A(ax), \max\{f_B(axa), f_C(xa)\}] \\
&= \max[f_A(ax), \{\max\{f_B(axa), f_C(xa)\}\}] \\
&= \max\{f_A(ax), f_B(axa), f_C(xa)\} \\
&\leq \max\{f_A(a), f_B(a), f_C(a)\} \\
&= (f_A \vee f_B \vee f_C)(a).
\end{aligned}$$

Hence $\mathcal{A} \sqcap \mathcal{B} \sqcap \mathcal{C} \sqsubseteq \mathcal{A} \copyright \mathcal{B} \copyright \mathcal{C}$.

(3)$\Longrightarrow$(1) Let B be a generalized bi-ideal of S, C be a left ideal of S and A be a right ideal of S. Let $a \in A \cap B \cap C$. Then $a \in A, a \in B$ and $a \in C$. Since B is a generalized bi-ideal of S, so by Theorem 3.2, χ_B is cubic generalized bi-ideal of S. By Theorem 2.3, χ_C is a cubic left ideal of S and χ_A is a cubic right ideal of S. Hence by hypothesis, $\chi_A \sqcap \chi_B \sqcap \chi_C \sqsubseteq \chi_A \copyright \chi_B \copyright \chi_C$. Then

$$\begin{aligned}
(\widetilde{\mu}_{\chi_A} \widetilde{\circ} \widetilde{\mu}_{\chi_B} \widetilde{\circ} \widetilde{\mu}_{\chi_C})(a) &\succeq (\widetilde{\mu}_{\chi_A} \widetilde{\cap} \widetilde{\mu}_{\chi_B} \widetilde{\cap} \widetilde{\mu}_{\chi_C})(a) \\
&= \mathrm{rmin}\{\widetilde{\mu}_{\chi_A}(a), \widetilde{\mu}_{\chi_B}(a), \widetilde{\mu}_{\chi_C}(a)\} \\
&= [1, 1].
\end{aligned}$$

Thus $\operatorname{r\,sup}_{a=yz}[\operatorname{rmin}\{(\widetilde{\mu}_{\chi_A}\widetilde{\circ}\widetilde{\mu}_{\chi_B})(y), \widetilde{\mu}_{\chi_C})(z)\}] = [1,1]$ and

$$\begin{aligned}(f_{\chi_A} \circ f_{\chi_B} \circ f_{\chi_C})(a) &\leq (f_{\chi_A} \vee f_{\chi_B} \vee f_{\chi_C})(a)\\ &= \max\{f_{\chi_A}(a) \circ f_{\chi_B}(b) \circ f_{\chi_C}(c)\}\\ &= 0.\end{aligned}$$

Thus $\bigwedge_{a=yz}[\max\{(f_{\chi_A} \circ f_{\chi_B})(y), f_{\chi_C}(z)\}] = 0$. So there exist $b, c \in S$ with $a = bc$ such that $(\widetilde{\mu}_{\chi_A}\widetilde{\circ}\widetilde{\mu}_{\chi_B})(b) = \widetilde{\mu}_{\chi_C}(c) = [1,1]$ and $(f_{\chi_A} \circ f_{\chi_B})(b) = f_{\chi_C}(c) = 0$. Then $c \in L$ and $\operatorname{r\,sup}_{b=pq}[\operatorname{rmin}\{\widetilde{\mu}_{\chi_A}(p), \widetilde{\mu}_{\chi_B}(q)\}] = [1,1]$ and $\bigwedge_{b=pq}[\max\{f_{\chi_A}(p), f_{\chi_B}(q)\}] = 0$. Then $b = pq$ for some $p, q \in S$ with $\widetilde{\mu}_{\chi_A}(p) = \widetilde{\mu}_{\chi_B}(q) = [1,1]$ and $f_{\chi_A}(p) = f_{\chi_B}(q) = 0$. Consequently, $p \in A$ and $q \in B$. So $a = bc = pqc \in ABC$. So $A \cap B \cap C \subseteq ABC$. Hence by Theorem 3.11, S is regular.

Denote the set of all cubic sets by $C(X)$. Let X and Y be given classical sets. A mapping $f : X \longrightarrow Y$ induces two mappings $C_f : C(X) \longrightarrow C(Y)$, $\mathcal{A} \longrightarrow C_f(\mathcal{A})$, and $C_f^{-1} : C(Y) \longrightarrow C(X)$, $\mathcal{B} \longrightarrow C_f^{-1}(\mathcal{B})$, where $C_f(\mathcal{A})$ is given by

$$C_f(\widetilde{\mu}_A)(y) = \begin{cases} \operatorname{r\,sup}_{y=f(x)} \widetilde{\mu}_A(y) & \text{if } f^-1(y) \neq \emptyset \\ [0,0] & \text{otherwise}\end{cases}$$

$$C_f(f_A)(y) = \begin{cases} \inf_{y=f(x)} f_A(y) & \text{if } f^-1(y) \neq \emptyset \\ 1 & \text{otherwise}\end{cases}$$

for all $y \in Y$; and $\mathcal{C}_f^{-1}(\mathcal{B})$ is defined by $\mathcal{C}_f^{-1}(\widetilde{\mu}_B)(x) = (\widetilde{\mu}_B)(f(x))$ and $\mathcal{C}_f^{-1}(f_B)(x) = (f_B)(f(x))$ for all $x \in X$. Then the mapping C_f (resp. C_f^{-1}) is called a cubic transformation (resp. inverse cubic transformation) induced by f. A cubic set $\mathcal{A} = \langle \widetilde{\mu}_A, f_A\rangle$ in X has the cubic property if for any subset T of X there exists $x_0 \in T$ such that $\widetilde{\mu}_A(x_0) = \operatorname{r\,sup}_{x\in T} \widetilde{\mu}_A(x)$ and $f_A(x_0) = \inf_{x\in T} f_A(x)$.

Theorem 3.13. *For a homomorphism $h : X \longrightarrow Y$ of semigroups, let $C_h : C(X) \longrightarrow C(Y)$ and $C_h^{-1} : C(Y) \longrightarrow C(X)$ be the cubic transformation and inverse cubic transformation, respectively, induced by h.*

(1) *If $\mathcal{A} = \langle \widetilde{\mu}_A, f_A\rangle \in C(X)$ is a cubic generalized bi-ideal of X, which has the cubic property, then $C_h(\mathcal{A})$ is a cubic generalized bi-ideal of Y.*

(2) *If $\mathcal{B} = \langle \widetilde{\mu}_B, f_B\rangle \in C(Y)$ is a cubic generalized bi-ideal of Y, which has the cubic property, then $C_h^{-1}(\mathcal{B})$ is a cubic generalized bi-ideal of X.*

Proof. (1) Given $h(x), h(y)$ and $h(z) \in h(X)$, let $x_0 \in h^{-1}(h(x))$ and $z_0 \in h^{-1}(h(z))$ be such that $\widetilde{\mu}_A(x_0) = \operatorname{r\,sup}_{a\in h^-1(h(x))} \widetilde{\mu}_A(a)$, $f_A(x_0) = \inf_{a\in h^-1(h(x))} f_A(a)$, and $\widetilde{\mu}_A(z_0) = \underset{c\in h^{-1}(h(z))}{\mathrm{r\,sup}} \widetilde{\mu}_A(c)$, $f_A(y_0) = \inf_{c\in h^{-1}(h(z))} f_A(c)$, respectively. Then

$$\begin{aligned} C_h(\widetilde{\mu}_A)(h(x)h(y)h(z)) &= \underset{p\in h^{-1}(h(x)h(y)h(z))}{\mathrm{r\,sup}} \widetilde{\mu}_A(p) \\ &\succeq \widetilde{\mu}_A(x_\circ y_\circ z_\circ) \\ &\succeq \mathrm{rmin}\{\widetilde{\mu}_A(x_\circ), \widetilde{\mu}_A(z_\circ)\} \\ &= \mathrm{rmin}\{\underset{a\in h^{-1}(h(x))}{\mathrm{r\,sup}} \widetilde{\mu}_A(a), \underset{c\in h^{-1}(h(z))}{\mathrm{r\,sup}} \widetilde{\mu}_A(c) \\ &= \mathrm{rmin}\{C_h(\widetilde{\mu}_A)(h(x), C_h(\widetilde{\mu}_A)(h(z)\} \end{aligned}$$

and

$$\begin{aligned} C_h(f_A)(h(x)h(y)h(z)) &= \inf_{p\in h^{-1}(h(x)h(y)h(z))} f_A(p) \\ &\leq f_A(x_\circ y_\circ z_\circ) \\ &\leq \max\{f_A(x_\circ), f_A(z_\circ)\} \\ &= \max\{\inf_{a\in h^{-1}(h(x))} f_A(a), \inf_{c\in h^{-1}(h(z))} f_A(c)\} \\ &= \max\{C_h(f_A)(h(x)), C_h(f_A)((h(z))\}. \end{aligned}$$

Therefore, $C_h(\mathcal{A})$ is a cubic generalized bi-ideal of Y.

(2) For any $x, y, z \in X$, we have

$$\begin{aligned} C_h^{-1}(\widetilde{\mu}_B)(xyz) &= (\widetilde{\mu}_B)(h(xyz)) \\ &= (\widetilde{\mu}_B)(h(x)h(y)h(z)) \\ &\succeq \mathrm{rmin}\{(\widetilde{\mu}_B)(h(x)), (\widetilde{\mu}_B)(h(z))\} \\ &= \mathrm{rmin}\{C_h^{-1}(\widetilde{\mu}_B)(x), C_h^{-1}(\widetilde{\mu}_B)(z)\} \end{aligned}$$

and

$$\begin{aligned} C_h^{-1}(f_B)(xyz) &= (f_B)(h(xyz)) \\ &= (f_B)(h(x)h(y)h(z)) \\ &\leq \max\{(f_B)(h(x)), (f_B)(h(z))\} \\ &= \max\{C_h^{-1}(f_B)(x), C_h^{-1}(f_B)(z)\}. \end{aligned}$$

Hence $C_h^{-1}(\mathcal{B})$ is a cubic generalized bi-ideal of X.

Conclusions. We have considered the following items:

1. To define cubic generalized bi-ideals in semigroups.
2. To study the basic properties of generalized bi-ideals of semigroups by using cubic generalized bi-ideals.

3. To characterize different classes of semigroups using cubic generalized bi-ideals.
4. To study the concepts of images and inverse images of cubic generalized bi-ideals.

Work in this direction is on going. Some points for future work may be the following.

1. To define cubic interior ideals of semigroups and to study the properties of semigroups by using this notion.
2. To study the concept of a cubic quasi-ideal in semigroups and to investigate the related properties of semigroups in terms of cubic quasi-ideals.

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