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ON 2-ABSORBING FILTERS OF LATTICES

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Abstract

Let L be a lattice with 1. In this paper we study the concept of 2-absorbing filter which is a generalization of prime filter. A proper filter F of L is called a 2-absorbing filter (resp. a weakly 2-absorbing) if whenever $x_1 \vee x_2 \vee x_3 \in F$ (resp. $1 \neq x_1 \vee x_2 \vee x_3 \in F$), for $x_1, x_2, x_3 \in L$, then there are 2 of the x_i's whose join is in F. A basic number of results concerning 2-absorbing filters and weakly of 2-absorbing filters are given in the case when L is distributive.

Keywords: lattice, filter, 2-absorbing filter, weakly 2-absorbing filter.

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1. Introduction

Recently, the study of the 2-absorbing property in the rings, modules, and semigroups has become quite popular. In many ways this program began with the paper in 2007, by Ayman Badawi, [2]. He introduced, for a commutative ring R, the notion of 2-absorbing ideals of R. A proper ideal I of R is called a 2-absorbing ideal if whenever $x_1x_2x_3 \in I$ for $x_1, x_2, x_3 \in R$, then there are 2 of the x_i's whose product is in I. There have been several generalizations and extensions of this concept in the literature (see e.g. [1, 3, 5], and [10]).

In this paper, we are interested in investigating 2-absorbing filters to use other notions of 2-absorbing and associate which exist in the literature as laid forth in [2]. Now we summarize the content of the paper. Among many results in this paper, in Section 2, it is shown (Theorem 2.2) that the only weakly 2-absorbing filters of L that are not 2-absorbing can only be $\{1\}$ (so if L is an L-domain, then a filter is 2-absorbing if and only if it is weakly 2-absorbing), and F is a 2-absorbing filter of L if and only if whenever $F_1 \vee F_2 \vee F_3 \subseteq F$ for some filters F_1, F_2, F_3 of L, then $F_1 \vee F_2 \subseteq F$ or $F_1 \vee F_3 \subseteq F$ or $F_2 \vee F_3 \subseteq F$ (Theorem 2.5). It is shown (Theorem 2.8) that If F is a 2-absorbing filter of L, then either F is a prime filter or $F = \mathbf{p} \cap \mathbf{q} = \mathbf{p} \vee \mathbf{q}$, where $\mathbf{p}$, $\mathbf{q}$ are the only distinct filters of L that are minimal over F. Let G be a 2-absorbing subfilter of a filter F of L. It is shown (Theorem 2.14 and Theorem 2.15) that either $\mathrm{Ass}_L(G :_L F)$ is a totally ordered set or $\mathrm{Ass}_L(G :_L F)$ is the union of two totally ordered set. Payrovi and Babaei [10], using the technique of efficient covering of submodules (see [8]) proved the avoidance theorem for 2-absorbing submodules. They proved that if a submodule N of a module is contained in the union of a finite number of 2-absorbing submodules with some conditions, then N must be contained in one of them. Section 3 is devoted to prove that the 2-absorbing avoidance theorem. More precisely, let $F, F_1, F_2, \ldots, F_n$ $(n \geq 2)$ be filters of L such that at most two of $F_1, F_2, \ldots, F_n$ are not 2-absorbing. If $F \subseteq \cup_{i=1}^{n} F_i$ and $F_i \nsubseteq (F_j :_L x)$ for all $x \in L \setminus F_j$ whenever $i \neq j$, then $F \subseteq F_i$ for some i with $1 \leq i \leq n$ (Theorem 3.4).

Let us briefly review some definitions and tools that will be used later. A lattice is a poset $(L, \leq)$ in which every couple elements x, y has a g.l.b. (called the meet of x and y, and written $x \wedge y$) and a l.u.b. (called the join of x and y, and written $x \vee y$). A lattice L is complete when each of its subsets X has a l.u.b. and a g.l.b. in L. Setting $X = L$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that L is a lattice with 0 and 1). A lattice L is called a distributive lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in L (equivalently, L is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in L). A non-empty subset F of a lattice L is called a filter, if for $a \in F$, $b \in L$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if L is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of L). A lattice L with 1 is called L-domain if $a \vee b = 1$ $(a, b \in L)$, then $a = 1$ or $b = 1$. A proper filter F of L is called prime if $x \vee y \in F$, then $x \in F$ or $y \in F$. Let L be a lattice with 0 and 1. If $a \in L$, then a complement of a in L is an element $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. The lattice L is complemented if every element of L has a complement in L [4]. First we need the following well-known lemma.

Lemma 1.1. *Let L be a lattice.*

(i) *A non-empty subset F of L is a filter of L if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in L$ (so $0 \in F$ if and only if $F = L$). Moreover, since $x = x \vee (x \wedge y)$ and $y = y \vee (x \wedge y)$, F is a filter and $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in L$.*

(ii) *If* $F_1, \dots, F_n$ *are filters of* L *and* $a \in L$, *then* $\vee_{i=1}^{n} F_i = \{\vee_{i=1}^{n} a_i : a_i \in F_i\}$ *and* $a \vee F_i = \{a \vee a_i : a_i \in F_i\}$ *are filters of* L.

(iii) *If* D *is an arbitrary non-empty subset of* L, *then the set* $T(D)$ *consisting of all elements of* L *of the form* $(a_1 \wedge a_2 \wedge \cdots \wedge a_n) \vee x$ (*with* $a_i \in D$ *for all* $1 \leq i \leq n$ *and* $x \in L$) *is a filter of* L *containing* D (*so if* $D = \{a\}$, *then* $T(\{a\}) = T(a) = \{a \vee t : t \in L\}$).

(iv) *If* L *is distributive,* F, G *are filters of* L, *and* $x \in L$, *then* $(G :_L F) = \{x \in L : x \vee F \subseteq G\}$ *and* $(F :_L \{x\}) = (F :_L x) = \{a \in L : a \vee x \in F\}$ *are filters of* L.

(v) *If* $\{F_i\}_{i \in \Delta}$ *is a chain of filters of* L, *then* $\cup_{i \in \Delta} F_i$ *is a filter of* L.

2. Some basic properties of 2-absorbing filters

In this section, we collect some properties concerning 2-absorbing filters of a lattice L. Throughout this paper, we shall assume unless otherwise stated, that L is a distributive lattice with 1 and 0.

Definition 2.1. A proper Filter F of L is called a 2-absorbing (resp. a weakly 2-absorbing) filter if whenever $a, b, c \in L$ and $a \vee b \vee c \in F$ (resp. $1 \neq a \vee b \vee c \in F$), then $a \vee b \in F$ or $a \vee c \in F$ or $b \vee c \in F$.

Clearly, every 2-absorbing filter of L is a weakly 2-absorbing. However, since $\{1\}$ is always weakly 2-absorbing "by definition", a weakly 2-absorbing filter need not be 2-absorbing.

Theorem 2.2. *If* F *is a weakly* 2*-absorbing of* L *that is not* 2*-absorbing, then* $F = \{1\}$. *In particular, the only weakly* 2*-absorbing filters of* L *that are not* 2*-absorbing can only be* $\{1\}$.

Proof. We suppose that $F \neq \{1\}$, and look for a contradiction. Let $x \vee y \vee z \in F$. If $x \vee y \vee z \neq 1$, then F weakly 2-absorbing gives $x \vee y \in F$ or $y \vee z \in F$ or $x \vee z \in F$; so F is 2-absorbing which is a contradiction. So assume that $x \vee y \vee z = 1$. Since $F \neq \{1\}$, there exists $b \in F$ with $b \neq 1$. Then $1 \neq b = b \wedge 1 = b \wedge (x \vee y \vee z) = ((b \wedge (x \vee y)) \vee ((b \wedge (x \vee z)) \vee ((b \wedge (y \vee z)) \in F$, so $b \wedge (x \vee y) \in F$ or $b \wedge (x \vee z) \in F$ or $b \wedge (y \vee z) \in F$. Thus $x \vee y \in F$ or $x \vee z \in F$ or $y \vee z \in F$ by Lemma 1.1 (i), and so F is 2-absorbing, a contradiction. Thus $F = \{1\}$. The "in particular" statement is clear. ∎

Remark 2.3. (i) If F, F_1, F_2 are filters of L with $F \subseteq F_1 \cup F_2$, then we show that either $F \subseteq F_1$ or $F \subseteq F_2$. Suppose that $F \subseteq F_1 \cup F_2$ such that $F \nsubseteq F_1$; we show that $F \subseteq F_2$. Let $a \in F$ be such that $a \notin F_1$. Let $x \in F \cap F_1$. Then F is a filter gives $a \wedge x \in F \subseteq F_1 \cup F_2$; so $a, x \in F_2$. Therefore $F \cap F_1 \subseteq F_2$. Thus $F = F \cap (F_1 \cup F_2) = (F \cap F_1) \cup (F \cap F_2) \subseteq F_2$.

(ii) Assume that $\mathbf{m}$ is a maximal filter of a lattice L with 0 and let $a \vee b \in \mathbf{m}$ with $a, b \notin \mathbf{m}$ for some $a, b \in L$. Then $T(\mathbf{m} \cup \{a\}) = T(\mathbf{m} \cup \{b\}) = L$ since $\mathbf{m}$ is maximal. An inspection will show that $0 \in L$ implies that $L = F$ which is a contradiction. Thus every maximal filter of L is prime [6].

(iii) If F is a filter of a L-domain L, then F is 2-absorbing if and only if it is weakly 2-absorbing.

Proposition 2.4. *Let F_1, F_2, F be filters of L such that F is 2-absorbing.*

(i) *If $a, b \in L$ and $(a \vee b) \vee F_1 \subseteq F$, then $a \vee b \in F$ or $a \vee F_1 \subseteq F$ or $b \vee F_1 \subseteq F$.*

(ii) *If $a \in L$ and $a \vee (F_1 \vee F_2) \subseteq F$, then $a \vee F_1 \subseteq F$ or $a \vee F_2 \subseteq F$ or $F_1 \vee F_2 \subseteq F$.*

Proof. (i) Let $a \vee b \notin F$ and $a \vee F_1 \nsubseteq F$. Then there is an element $c \in F_1$ such that $a \vee c \notin F$. Now $a \vee b \vee c \in F$ gives $b \vee c \in F$ since F is 2-absorbing. We have to show that $b \vee F_1 \subseteq F$. Let d be an arbitrary element of F_1. Then $(d \wedge c) \vee (a \vee b) = (a \vee b \vee c) \wedge (a \vee b \vee d) \in F$ since F is a filter; so either $(d \wedge c) \vee a = (a \vee c) \wedge (a \vee d) \in F$ or $(d \wedge c) \vee b = (b \vee c) \wedge (b \vee d) \in F$. If $(d \wedge c) \vee a \in F$, then $a \vee c \in F$ by Lemma 1.1 (i) that is a contradiction. If $(d \wedge c) \vee b \in F$, then $b \vee d \in F$. Thus $b \vee F_1 \subseteq F$.

(ii) Let $a \vee F_1 \nsubseteq F$ and $a \vee F_2 \nsubseteq F_2$. We have to show that $F_1 \vee F_2 \subseteq F$. Suppose that $x \in F_1$ and $y \in F_2$. By hypothesis, there exist $z \in F_1 \setminus F$ and $w \in F_2 \setminus F$ such that $a \vee z \notin F$ and $a \vee w \notin F$. As $a \vee z \vee w \in a \vee (F_1 \vee F_2) \subseteq F$, we get $z \vee w \in F$. Now $z \wedge x \in F_1$ and $y \wedge w \in F_2$ gives $a \vee (z \wedge x) \vee (y \wedge w) \in F$; so $(z \wedge x) \vee (y \wedge w) \in F$ since F is 2-absorbing (see Lemma 1.1 (i)). It follows that $(z \wedge x) \vee y \in F$; hence $x \vee y \in F$ by Lemma 1.1 (i). Therefore, $F_1 \vee F_2 \subseteq F$. ■

Theorem 2.5. *Let F be a proper filter of L. The following statements are equivalent:*

(i) *F is a 2-absorbing filter of L.*

(ii) *If $F_1 \vee F_2 \vee F_3 \subseteq F$ for some filters F_1, F_2, F_3 of L, then $F_1 \vee F_2 \subseteq F$ or $F_1 \vee F_3 \subseteq F$ or $F_2 \vee F_3 \subseteq F$.*

Proof. (i)⇒(ii) Suppose that $F_1 \vee F_2 \vee F_3 \subseteq F$ for some filters F_1, F_2, F_3 of L and $F_1 \vee F_2 \nsubseteq F$. Then by Proposition 2.4 for all $a \in F_3$ either $a \vee F_1 \subseteq F$ or $a \vee F_2 \subseteq F$. If $a \vee F_1 \subseteq F$, for all $a \in F_3$ we are done. Similarly, if $a \vee F_2 \subseteq F$, for all $a \in F_3$ we are done. Assume that $a, b \in L$ are such that $a \vee F_1 \nsubseteq F$ and $b \vee F_2 \nsubseteq F$. It follows that $b \vee F_1 \subseteq F$ and $a \vee F_2 \subseteq F$. Since $(a \wedge b) \vee (F_1 \vee F_2) \subseteq F$, we get either $(a \wedge b) \vee F_1 \subseteq F$ or $(a \wedge b) \vee F_2 \subseteq F$ by Proposition 2.4. If $(a \wedge b) \vee F_1 \subseteq F$, then $z \vee (a \wedge b) = (z \vee a) \wedge (z \vee b) \in F$ for all $z \in F_1$ which implies that $a \vee z \in F$ by Lemma 1.1 (i); so $a \vee F_1 \subseteq F$ which is a contradiction. Similarly, if $(a \wedge b) \vee F_2 \subseteq F$, we get a contradiction. Thus either $F_1 \vee F_3 \subseteq F$ or $F_2 \vee F_3 \subseteq F$.

(ii)⇒(i) Let $a, b, c \in L$ with $a \vee b \vee c \in F$. Then by (ii), $T(a) \vee T(b) \vee T(c) \subseteq F$ gives $a \vee b \in T(a) \vee T(b) \subseteq F$ or $a \vee c \in T(a) \vee T(c) \subseteq F$ or $b \vee c \in T(b) \vee T(c) \subseteq F$. Thus F is 2-absorbing. ■

We say that a subset $D \subseteq L$ is Join closed if $0 \in D$ and $a \vee b \in D$ for all $a, b \in D$. Clearly, if $\mathbf{p}$ is a prime filter of L, then $L \setminus \mathbf{p}$ is a join closed subset of L. The set of all prime filters of L is denoted by Spec(L). If F is a filter of L, then we set var(F) = {$\mathbf{p}$ ∈Spec (L): $F \subseteq \mathbf{p}$}, and the set of all prime filters of L that are minimal over F is denoted by min(F).

Lemma 2.6. (i) *Assume that F is a filter of L and let S be a join closed set of L such that $S \cap F = \emptyset$. Then the set $\sum = \{K : F \subseteq K, K \cap S = \emptyset\}$ of filters under the relation of inclusion has at least one maximal element, and any such maximal element of $\sum$ is a prime filter.*

(ii) *If F is a filter of L, then $F = \cap_{\mathbf{p} \in var(F)} \mathbf{p}$.*

(iii) *Let F, $\mathbf{p}$ be filters of L with $\mathbf{p}$ prime and $F \subseteq \mathbf{p}$. Then there exists a minimal prime filter $\mathbf{q}$ of F with $\mathbf{q} \subseteq \mathbf{p}$.*

(iv) *If F is a filter of L, then $F = \cap_{\mathbf{p} \in \mathrm{min(F)}} \mathbf{p}$.*

Proof. (i) Since $F \in \sum$, $\sum \neq \emptyset$. Of course, the relation of inclusion, $\subseteq$, is a partial order on $\sum$. Now $\sum$ is easily seen to be inductive under inclusion, so by Zorn's Lemma $\sum$ has a mximal element $\mathbf{q}$ with $\mathbf{q} \cap S = \emptyset$ and $F \subseteq \mathbf{q}$. It suffices to show that $\mathbf{q}$ is prime. Now let $x, x' \in L \setminus \mathbf{q}$; we must show that $x \vee x' \notin \mathbf{q}$. Since $x \notin \mathbf{q}$, we have $F \subseteq \mathbf{q} \subsetneqq T(\mathbf{q} \cup \{x\})$. By the maximality of $\mathbf{q}$, we have $T(\mathbf{q} \cup \{x\}) \cap S \neq \emptyset$, and so there exist $s \in S$, $c \in L$ and $q \in \mathbf{q}$ such that $s = (q \wedge x) \vee c$. Similarly, $s' = (q' \wedge x') \vee c'$ for some $s' \in S$, $q' \in \mathbf{q}$ and $c' \in L$. Set $z = c \vee c'$. Then $s \vee s' = (q \wedge x) \vee (q' \wedge x') \vee z = [(q \wedge x) \vee x'] \wedge [(q \wedge x) \vee q'] \vee z = [(x \vee x') \wedge (q \vee x')] \wedge [(q \wedge x) \vee q'] \vee z$. As $(q \wedge x) \vee q'$, $q \vee x' \in \mathbf{q}$, $S \cap \mathbf{q} = \emptyset$ and $\mathbf{q}$ is a filter, we have $x \vee x' \notin \mathbf{q}$. Thus $\mathbf{q}$ is a prime filter.

(ii) It is enough to show that $\cap_{\mathbf{p} \in var(F)} \mathbf{p} \subseteq F$. Let $a \in \cap_{\mathbf{p} \in var(F)} \mathbf{p}$. We suppose that $a \notin F$, and look for a contradiction. Set $S = \{0, a\}$. Then S is a join closed set of L with $S \cap F = \emptyset$. Hence, by (i), there exists a prime filter $\mathbf{q}$ of L such that $F \subseteq \mathbf{q}$ and $\mathbf{q} \cap S = \emptyset$. It follows that $\mathbf{q} \in$ var(F), so that $a \in S \cap \mathbf{q}$, a contradiction.

(iii) Set $\Delta = \{\mathbf{q} \in \mathrm{Spec}(L) : F \subseteq \mathbf{q} \subseteq \mathbf{p}\}$. Then $\mathbf{p} \in \Delta$, and so $\Delta \neq \emptyset$. By an argument like that in (i) (take $S = L \setminus \mathbf{p}$), the set Δ of prime filters of L has a minimal member with respect to inclusion (by partially ordering Δ by reverse inclusion and using Zorn's Lemma) which is prime. (iv) follows from (iii) (since every prime filter in var(F) contains a minimal prime filter of F). ■

Compare the next Proposition with Theorem 2.1, p. 2 in [7].

Proposition 2.7. *Let $F \subseteq \mathbf{p}$ be filters of L, where $\mathbf{p}$ is a prime filter. Then the following conditions are equivalent:*

(i) $\mathbf{p}$ *is a minimal prime filter of F.*

(ii) $L \setminus \mathbf{p}$ *is a join closed set that is maximal with $(L \setminus \mathbf{p}) \cap F = \emptyset$.*

(iii) *For each $x \in \mathbf{p}$, there is a $y \notin \mathbf{p}$ such that $y \vee x \in F$.*

Proof. (i)⇒(ii) Since $(L \setminus \mathbf{p}) \cap F = \emptyset$, the set Δ of all join closed sets, say H, with $H \cap F = \emptyset$ is not empty. Of course, the relation of inclusion, $\subseteq$, is a partial order on Δ. Now Δ is easily seen to be inductive under inclusion, so by Zorn's Lemma Δ has a mximal element S. Again by Zorn's Lemma, there is a filter $\mathbf{q}$ of L containing F that is maximal with respect to being disjoint from S which is prime by Lemma 2.6 (i). Note that $\mathbf{q}$ is disjoint from $L \setminus \mathbf{p}$ which implies that $\mathbf{p} = \mathbf{q}$. Thus $S = L \setminus \mathbf{p}$.

(ii)⇒(iii) Assume that $1 \neq x \in \mathbf{p}$ and let $S = \{y \vee (\wedge_{j=1}^{i} x) : y \in L \setminus \mathbf{p}, i = 0, 1, \dots\}$ (Note that $\wedge_{j=1}^{0} x$ is interpreted as 0, and clearly, $\wedge_{j=1}^{i} x = x$). Then S is a join closed set that properly contains $L \setminus \mathbf{p}$; so $F \cap S \neq \emptyset$ by maximality of $L \setminus \mathbf{p}$. Thus there exists $y \in L \setminus \mathbf{p}$ such that $x \vee y \in F$.

(iii)⇒(i) Let $\mathbf{q}$ be a prime filter such that $F \subsetneqq \mathbf{q} \subseteq \mathbf{p}$. If $\mathbf{p} \neq \mathbf{q}$, then there is an element $x \in \mathbf{p}$ with $x \notin \mathbf{q}$; so $x \vee y \in F \subsetneqq \mathbf{q}$ for some $y \notin \mathbf{p}$ which is a contradiction. Therefore $\mathbf{p} = \mathbf{q}$. ■

The following theorem is a lattice counterpart of Theorem 2.4 in [2] describing the structure of 2-absorbing ideals.

Theorem 2.8. (i) *If F is a 2-absorbing filter of L, then there exist at most two prime filters of L that are minimal over F.*

(ii) *If F is a 2-absorbing filter of L, then either F is a prime filter of L or $F = \mathbf{p} \cap \mathbf{q} = \mathbf{p} \vee \mathbf{q}$, where $\mathbf{p}$, $\mathbf{q}$ are the only distinct filters of L that are minimal over F.*

(iii) *If either F is a prime filter of L or F is an intersection of two prime filter of L, then F is 2-absorbing.*

Proof. (i) Assume that that Δ is the set of prime filters of L which are minimal over F and let Δ has at least three elements. Let $\mathbf{p}, \mathbf{q} \in \Delta$ with $\mathbf{p} \neq \mathbf{q}$. Then there exist $x_1, x_2 \in L$ such that $x_1 \in \mathbf{p} \setminus \mathbf{q}$ and $x_2 \in \mathbf{q} \setminus \mathbf{p}$. First we show that $x_1 \vee x_2 \in F$. By Proposition 2.7, there exist $a \notin \mathbf{p}$ and $b \notin \mathbf{q}$ such that $a \vee x_1, b \vee x_2 \in F$. Since $x_1, x_2 \notin \mathbf{p} \cap \mathbf{q}$ and $a \vee x_1, b \vee x_2 \in F \subseteq \mathbf{p} \cap \mathbf{q}$, we conclude that $a \in \mathbf{q} \setminus \mathbf{p}$ and $b \in \mathbf{p} \setminus \mathbf{q}$; so $a, b \notin \mathbf{p} \cap \mathbf{q}$. Since $a \vee x_1, b \vee x_2 \in F$, we have $(a \wedge b) \vee (x_1 \vee x_2) = [(a \vee x_1) \vee x_2] \wedge [(b \vee x_2) \vee x_1] \in F$ since F is a filter. By Lemma 1.1 (i), $a \wedge b \notin \mathbf{p}$ and $a \wedge b \notin \mathbf{q}$. Since $(a \wedge b) \vee x_1 \notin \mathbf{q}$ and $(a \wedge b) \vee x_2 \notin \mathbf{p}$, F is a 2-absorbing filter gives $x_1 \vee x_2 \in F$. Now suppose there is a $\mathbf{r} \in \Delta$ such that $\mathbf{r}$ is neither $\mathbf{p}$ nor $\mathbf{q}$. Then we can choose $z_1 \in \mathbf{p} \setminus (\mathbf{q} \cup \mathbf{r})$, $z_2 \in \mathbf{q} \setminus (\mathbf{p} \cup \mathbf{r})$, and $z_3 \in \mathbf{r} \setminus (\mathbf{p} \cup \mathbf{q})$. By an argument like that as above, we have $z_1 \vee z_2 \in F$. Since $F \subseteq \mathbf{p} \cap \mathbf{q} \cap \mathbf{r}$ and $z_1 \vee z_2 \in F$, we get either $z_1 \in \mathbf{r}$ or $z_2 \in \mathbf{r}$ that is a contradiction, as required.

(ii) By (i) and Lemma 2.6 (iv), we conclude that either F is a prime filter or $F = \mathbf{p} \cap \mathbf{q}$, where $\mathbf{p}$, $\mathbf{q}$ are the only distinct filters of L that are minimal over F. An inspection will show that $\mathbf{p} \cap \mathbf{q} = \mathbf{p} \vee \mathbf{q}$.

(iii) The first assertion is clear. Let $\mathbf{p}$ and $\mathbf{p}$ be two prime filters of L; we have to show that $F = \mathbf{p} \cap \mathbf{q}$ is a 2-absorbing filter of L. Let $a, b, c \in L$ such that $a \vee b \vee c \in \mathbf{p} \cap \mathbf{q}$. Therefore $a \vee b \vee c \in \mathbf{p}$ and $a \vee b \vee c \in \mathbf{q}$. If $a \in \mathbf{p} \cap \mathbf{q}$, then $a \vee b \in \mathbf{p} \cap \mathbf{q}$. If $a \in \mathbf{p}$ and $b \in \mathbf{p}$, then $a \vee b \in \mathbf{p} \cap \mathbf{q}$ since $\mathbf{p}$ and $\mathbf{q}$ are filters of L. The other cases we do the same. ■

The collection of ideals of Z, the ring of integers, form a lattice under set inclusion which we shall denote by $L(Z)$ with respect to the following definitions: $mZ \vee nZ = (m, n)Z$ and $mZ \wedge nZ = [m, n]Z$ for all ideals mZ and nZ of Z, where (m, n) and $[m, n]$ are greatest common divisor and least common multiple of m, n, respectively. Note that $L(Z)$ is a distributive complete lattice with least element the zero ideal and the greatest element Z.

Theorem 2.9. *The following hold:*

(i) *If p is a prime number and k is a positive integer, then the set $F_{p^k} = \{mZ \in L(Z) : p^k \nmid m\}$ is a prime filter of $L(Z)$.*
(ii) *$L(Z) \setminus \{0\}$ is the only maximal filter of $L(Z)$.*
(iii) *Every prime filter of $L(Z)$ is of the form either F_{p^k} for some prime number p and positive integer k or $L(Z) \setminus \{0\}$.*
(iv) *Every 2-absorbing filter of $L(Z)$ is of the form $L(Z) \setminus \{0\}$ or F_{p^m} or $F_{p^m} \cap F_{q^n}$ for some positive integers m, n and prime numbers p, q with $p \neq q$.*

Proof. (i) Let $mZ, nZ \in F_{p^k}$ and $sZ \in L(Z)$. Now $p^k \nmid m$ and $p^k \nmid n$ gives $p^k \nmid [m, n]$; so $[m, n]Z \in F_{p^k}$. As $p^k \nmid m$, we get $p^k \nmid (m, s)$ which implies that $(m, s)Z \in F_{p^k}$. Thus F_{p^k} is a filter of $L(Z)$. Let $mZ \vee nZ = (m, n)Z \in F_{p^k}$ with $mZ \notin F_{p^k}$. Then $p^k \nmid (m, n)$ and $p^k \mid m$ gives $p^k \nmid n$; so $nZ \in F_{p^k}$. Thus F_{p^k} is prime.

(ii) is clear.

(iii) Let F be a prime filter of $L(Z)$. First we show that there exist at most one prime number p and positive integer k such that for every $mZ \in F$ implies that $p^k \nmid m$. Otherwise, there are distinct prime numbers p, q and positive integers k, s such that for every $mZ \in F$ implies that $p^k \nmid m$ and $q^s \nmid m$. Then $p^k Z \vee q^s Z = Z \in F$ gives either $p^k Z \in F$ or $q^s Z \in F$ which is a contradiction. If there exists p^k such that for every $mZ \in F$ implies that $p^k \nmid m$. Let t be least positive integer such that for every $mZ \in F$ implies that $p^t \nmid m$; we show that $F = F_{p^t}$. It suffices to show that for every mZ with $p^t \nmid m$, $mZ \in F$. There are distinct prime numbers $q_1, \dots, q_n$ such that $m = p^l q_1^{s_1} \cdots q_n^{s_n}$, where $0 \leq l < t$, $p \neq q_j$ with $1 \leq j \leq n$, and s_j is a positive integer for $1 \leq j \leq n$. As $l < t$, there exist $m'Z \in F$ such that $p^l \mid m'$, so $m'Z \subseteq p^l Z$. Thus $p^l Z \in F$ since F is a filter. Moreover, $p^l Z \vee q_i^{s_i} Z = Z \in F$ gives $q_i^{s_i} Z \in F$ with $1 \leq i \leq n$. Thus $mZ = p^l Z \wedge (\wedge_{i=1}^n q_i^{s_i} Z) \in F$. Suppose that there is not such p^k; we show that $F = L(Z) \setminus \{0\}$. Let m be a non-zero integer. It is enough to show that $mZ \in F$. We can write $m = p_1^{s_1} \cdots p_n^{s_n}$, where $p_i \neq p_j$ with $i \neq j$ and for each i, s_i is a positive integer. Then for each i, there exists $m_i Z \in F$ such that $p_i^{s_i} \mid m_i$, so $m_i Z \subseteq p_i^{s_i} Z \in F$ since F is a filter. Thus $mZ = \wedge_{i=1}^n p_i^{s_i} Z \in F$.

(iv) This follows from (i), (ii), (iii), and Theorem 2.8. ■

Remark 2.10 shows that prime filters which are maximals are abundant.

Remark 2.10. (i) Assume that F is a prime filter of a complemented lattice L with 0 and 1 and let F' be a filter of L such that $F \subsetneqq F' \subseteq L$. Then there exist $x \in F' \setminus F$ and $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1 \in F$. Then F is prime gives $y \in F \subseteq F'$, and so $x \wedge y = 0 \in F'$; hence $F' = L$. Thus F is maximal.

(ii) Let $D = \{1, \ldots, n\}$. Then the set $L = \{X : X \subseteq D\}$ forms a complemented distributive lattice under set inclusion with greatest element D and least element $\emptyset$ (note that if $x, y \in L$, then $x \vee y = x \cup y$ and $x \wedge y = x \cap y$). Then every prime filter of L is a maximal filter by (i).

Corollary 2.11. *The following statements are equivalent:*

(i) *Every prime filter of L is maximal;*

(ii) *If F is a 2-absorbing filter of L, then either F is maximal or $F = \mathbf{p_1} \cap \mathbf{p_2} = \mathbf{p_1} \vee \mathbf{p_2}$, where $\mathbf{p_1}, \mathbf{p_2}$ are some maximal filters of L.*

Proof. (i)⇒(ii) follows from Theorem 2.8. To see that (ii)⇒(i), assume that F is a prime filter of L. By (ii), if F is maximal, then we are done. So we assume that $F = \mathbf{p_1} \vee \mathbf{p_2}$, where $\mathbf{p_1}, \mathbf{p_2}$ are some maximal filters of L. Then either $\mathbf{p_1} \subseteq F$ or $\mathbf{p_2} \subseteq F$; hence either $F = \mathbf{p_1}$ or $F = \mathbf{p_1}$ (otherwise, there exist $a \in \mathbf{p_1} \setminus F$ and $b \in \mathbf{p_2} \setminus F$ with $a \vee b \notin F$ since F is a prime filter, and this contradicts the statements of (ii)). ■

Proposition 2.12. *If G is a 2-absorbing subfilter of a filter F of L, then $(G :_L F)$ is a 2-absorbing filter of L.*

Proof. Let $a, b, c \in L$, $a \vee b \vee c \in (G :_L F)$, $a \vee c \notin (G :_L F)$, and $b \vee c \notin (G :_L F)$. We must to show that $a \vee b \in (G :_L F)$. There exist $x_1, x_2 \in L$ such that $a \vee c \vee x_1, b \vee c \vee x_2 \notin G$ but $(a \vee b) \vee [(c \vee x_1) \wedge (c \vee x_2)] = (a \vee b \vee c) \vee (x_1 \wedge x_2) \in G$ since G is a filter. Now G is a 2-absorbing filter gives $a \vee [(c \vee x_1) \wedge (c \vee x_2)] = (a \vee c \vee x_1) \wedge (a \vee c \vee x_2) \in G$ or $b \vee [(c \vee x_1) \wedge (c \vee x_2) = (b \vee c \vee x_1) \wedge (b \vee c \vee x_2) \in G$ or $a \vee b \in G$. If $a \vee b \in G$, we are done. If $a \vee [(c \vee x_1) \wedge (c \vee x_2)] \in G$, then by Lemma 1.1 (i), $a \vee c \vee x_1 \in G$ which is a contradiction. Similarly, $b \vee [(c \vee x_1) \wedge (c \vee x_2)] \notin G$. This completes the proof. ■

Proposition 2.13. *If G is a 2-absorbing subfilter of a filter F of L, then $(G :_L F)$ is a prime filter if and only if $(G :_L x)$ is a prime filter for all $x \in F \setminus G$.*

Proof. Let $a, b \in L$, $x \in F \setminus G$, and $a \vee b \in (G :_L x)$. Then $a \vee b \vee x \in G$ gives $a \vee x \in G$ or $b \vee x \in G$ or $a \vee b \in G$. If $a \vee x \in G$ or $b \vee x \in G$ we are done. If $a \vee b \in G$, then $(a \vee b) \vee F \subseteq G$ since G is a filter; so $a \vee b \in (G :_L F)$. By assumption, $a \in (G :_L F)$ or $b \in (G :_L F)$; hence $a \in (G :_L x)$ or $b \in (G :_L x)$. Thus $(G :_L x)$ is a prime filter of L. Conversely, suppose that $a \vee b \in (G :_L F)$ for some $a, b \in L$ with $a, b \notin (G :_L F)$. It follows that $a \vee x \notin G$ and $b \vee y \notin G$ for some $x, y \in F \setminus G$ (so $x \wedge y \notin G$ by Lemma 1.1 (i)). As $a \vee b \vee (x \wedge y) = (a \vee b \vee x) \wedge (a \vee b \vee y) \in G$, we have $a \vee b \in (G :_L (x \wedge y))$; hence $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y) \in G$ or $b \vee (x \wedge y) = (b \vee x) \wedge (b \vee y) \in G$ since $(G :_L (x \wedge y))$ is a prime filter which is a contradiction. Thus $a \in (G :_L F)$ or $b \in (G :_L F)$ which implies that $(G :_L F)$ is a prime filter of L. ■

Let G be a proper subfilter of a filter F of L. We say that $\mathbf{p} \in \mathrm{Spec}(L)$ is an associated prime filter of F with respect to G if there is an element $x \in F \setminus G$ such that $(G :_L x) = \mathbf{p}$. The set of associated prime filters of F with respect to G is denoted $\mathrm{Ass}_L(G :_L F)$.

Compare the next Theorem with Theorem 2.6 in [10].

Theorem 2.14. *Let G be a 2-absorbing subfilter of a filter F of L. If $(G :_L F)$ is a prime filter of L, then $\mathrm{Ass}_L(G :_L F)$ is a totally ordered set.*

Proof. Let $\mathbf{p}, \mathbf{q} \in \mathrm{Ass}_L(G :_L F)$. Then there are elements $x, y \in F \setminus G$ such that $(G :_L x) = \mathbf{p}$ and $(G :_L y) = \mathbf{q}$. Suppose that $\mathbf{q} \nsubseteq \mathbf{p}$. We have to show that $(G :_L x) \subseteq (G :_L y)$. Let $z \in (G :_L x)$ (so $z \vee x \in G$). There exists $w \in (G :_L y)$ such that $w \notin (G :_L x)$; so $w \vee y \in G$ and $w \vee x \notin G$. Clearly, $x \wedge y \notin G$. If $z \vee (x \wedge y) = (z \vee x) \wedge (z \vee y) \in G$, then $z \vee y \in G$ by Lemma 1.1 (i) and so $z \in (G :_L y)$. Now assume that $z \vee (x \wedge y) \notin G$, so $(z \vee w) \vee (x \wedge y) = (z \vee w \vee x) \wedge (z \vee w \vee y) \in G$ since G is a filter; hence $z \vee w \in (G :_L (x \wedge y))$. By Proposition 2.13 and Lemma 1.1 (i), $(G :_L (x \wedge y))$ is a prime filter gives $z \vee (x \wedge y) = (z \vee x) \wedge (z \vee y) \in G$ and $w \vee (x \wedge y) = (w \vee x) \wedge (w \vee y) \notin G$. Thus $z \vee y \in G$ and so $z \in (G :_L y)$. ■

Compare the next Theorem with Theorem 2.7 in [10].

Theorem 2.15. *Let G be a 2-absorbing subfilter of a filter F of L such that $(G :_L F) = \mathbf{p} \cap \mathbf{q}$ for some prime filters $\mathbf{p}, \mathbf{q}$ of L.*

(i) *If $x \in F \setminus G$ and $\mathbf{p} \subseteq (G :_L x)$, then $(G :_L x)$ is a prime filter of L.*
(ii) *If $x, y \in F \setminus G$ and $\mathbf{p} \subseteq (G :_L x) \cap (G :_L y)$, then either $(G :_L x) \subseteq (G :_L y)$ or $(G :_L y) \subseteq (G :_L x)$. Therefore $\mathrm{Ass}_L(G :_L F)$ is the union of two totally ordered sets.*

Proof. (i) Let $a, b \in L$ and $a \vee b \in (G :_L x)$. Then $a \vee b \vee x \in G$ gives $a \vee x \in G$ or $b \vee x \in G$ or $a \vee b \in G$. If $a \vee x \in G$ or $b \vee x \in G$ we are done. If $a \vee b \in G$, then $(a \vee b) \vee F \subseteq G$ since G is a filter; so $a \vee b \in (G :_L F) \subseteq \mathbf{p}$. thus either $a \in \mathbf{p} \subseteq (G :_L x)$ or $b \in \mathbf{p} \subseteq (G :_L x)$.

(ii) Suppose that $(G :_L y) \nsubseteq (G :_L x)$. We have to show that $(G :_L x) \subseteq (G :_L y)$. Let $z \in (G :_L x)$ (so $z \vee x \in G$). There exists $w \in (G :_L y)$ such that $w \notin (G :_L x)$; so $w \vee y \in G$ and $w \vee x \notin G$. Clearly, $x \wedge y \notin G$. If $z \vee (x \wedge y) = (z \vee x) \wedge (z \vee y) \in G$, then $z \vee y \in G$ by Lemma 1.1 (i) and so $z \in (G :_L y)$. Now assume that $z \vee (x \wedge y) \notin G$, so $(z \vee w) \vee (x \wedge y) = (z \vee w \vee x) \wedge (z \vee w \vee y) \in G$ since G is a filter; hence $z \vee w \in G$ since $w \vee (x \wedge y) = (w \vee x) \wedge (w \vee y) \notin G$ and $z \vee (x \wedge y) \notin G$. Thus $z \vee w \in (G :_L F) \subseteq \mathbf{p}$. If $w \in \mathbf{p} \subseteq (G :_L x)$, then $w \vee x \in G$ that is a contradiction; hence $z \in \mathbf{p} \subseteq (G :_L y)$. ∎

Theorem 2.16. *If G is a 2-absorbing subfilter of a filter F of L, then $(G :_L F)$ is a prime filter if and only if $(G :_L H)$ is a prime filter of L for all subfilters H of F containing G.*

Proof. By Proposition 2.13 and Theorem 2.14, the set $\{(G :_L x) : x \in H \setminus G\}$ is a totally ordered set of prime filters of L; so $(G :_L H) = \cap_{x \in H}(G :_L x)$ is a prime filter of L. Conversely, suppose that $x \vee y \in (G :_L F)$ with $x, y \notin (G :_L F)$. Then there exist $a, b \in F \setminus G$ (so $a \wedge b \notin G$) such that $x \vee a, y \vee b \notin G$, so $x \vee y \in (G :_L (a \wedge b))$. Now $(G :_L (a \wedge b))$ is a prime filter gives $x \vee (a \wedge b) = (x \vee a) \wedge (x \vee b) \in G$ or $y \vee (a \wedge b) = (y \vee a) \wedge (y \vee b) \in G$ which is a contradiction. Thus $(G :_L F)$ is prime. ∎

3. 2-Absorbing Avoidance Theorem

Let $F, F_1, F_2, \ldots, F_n$ be filters of L. We call a covering $F \subseteq \cup_{i=1}^{n} F_i$ efficient if no F_i is superfluous. Analogously, we say that $F = \cup_{i=1}^{n} F_i$ is an efficient union if none of the F_i may be excluded. Any cover or union consisting of filters of L can be reduced to an efficient one, called an efficient reduction, by deleting any unnecessary terms.

Theorem 3.1. *If G is a 2-absorbing subfilter of a filter F of L and $x \in F \setminus G$, then either $(G :_L x)$ is a prime filter of L or there exists an element $a \in L$ such that $(G :_L a \vee x)$ is a prime filter of L.*

Proof. By Proposition 2.12 and Theorem 2.8 (iii), $(G :_L F)$ is a prime filter of L or $(G :_L F)$ is an intersection of two prime filter of L. We split the proof into two cases:

Case 1. $(G :_L F) = \mathbf{p}$, where $\mathbf{p}$ is a prime filter of L. We show that $(G :_L x)$ is a prime filter of L. Clearly, $\mathbf{p} \subseteq G :_L x)$. Suppose that $a, b \in L$ and

$a \vee b \in G :_L x)$. Thus $a \vee b \vee x \in G$; hence $a \vee x \in G$ or $b \vee x \in G$ or $a \vee b \in G$. If either $a \vee x \in G$ or $b \vee x \in G$, we are done. So we may assume that $a \vee b \in G$. As G is a filter, $(a \vee b) \vee F \subseteq G$; thus $a \vee b \in \mathbf{p}$ and so $a \in \mathbf{p}$ or $b \in \mathbf{p}$. Therefore, $a \in G :_L x)$ or $b \in G :_L x)$ and the assertion follows.

Case 2. $(G :_L F) = \mathbf{p} \cap \mathbf{q}$, where $\mathbf{p}$ and $\mathbf{p}$ are distinct prime filters of L. If $\mathbf{p} \subseteq (G :_L x)$, then the result follows by an argument like that in the Case 1. So we may assume that $\mathbf{p} \nsubseteq (G :_L x)$. There is an element $a \in \mathbf{p}$ such that $a \vee x \notin G$. By Theorem 2.8 (ii), $\mathbf{p} \vee \mathbf{q} \subseteq (G :_L x)$; so $\mathbf{q} \subseteq G :_L a \vee x)$ and the result follows by a similar proof to that of Case 1. ■

Compare the next lemma with Lemma 1 in [7].

Lemma 3.2. *Let F and F_i $(i = 1, 2, \dots, n)$ be filters such that $F \subseteq \cup_{i=1}^{n} F_i$ is an efficient covering of filters of L, where $n \geq 3$. Then The intersection of any $n-1$ of the filters $F \cap F_i$ coincides with $H = \cap_{i=1}^{n}(F \cap F_i)$.*

Proof. It suffices to show that the intersection of any $n-1$ of the filters $F \cap F_i$ is contained in H. Since $F \subseteq \cup_{i=1}^{n} F_i$ is an efficient covering, we have $F = \cup_{i=1}^{n}(F \cap F_i)$ is an efficient union consisting of subfilters of F, so F is not contained in the union of any $n-1$ of the filters $F \cap F_i$; hence there exists an element $c_n \in F_n$ which is not in $\cup_{i=1}^{n-1}(F \cap F_i)$. If $x \in \cap_{i=1}^{n-1}(F \cap F_i)$, then the element $x \wedge c_n$ in F can not be in F_i for $1 \leq i \leq n-1$; thus $x \wedge c_n \in F_n$. By Lemma 1.1 (i), $x \in F_n$ and so $x \in H$, as needed. ■

Proposition 3.3. *Let F and F_i $(i = 1, 2, \dots, n)$ be filters such that $F \subseteq \cup_{i=1}^{n} F_i$ is an efficient covering of filters of L, where $n \geq 3$. If $F_i \nsubseteq (F_j :_L x)$ for all $x \in L \setminus F_j$ whenever $i \neq j$, then no F_i for $1 \leq i \leq n$ is a 2-absorbing filter of L.*

Proof. Assume to the contrary, F_k is a 2-absorbing filter of L for some $k = 1, \dots, n$. By Lemma 3.2, $\cap_{i \neq j}(F_i \cap F) \subseteq F \cap F_k$. Clearly, $F \nsubseteq F_k$, so there is an element $b \in F$ with $b \notin F_k$. Now Theorem 3.1 gives either $(F_k :_L b)$ is a prime filter or there exists $a \in L$ such that $(F_k :_L (a \vee b))$ is a prime filter of L. Suppose first that $(F_k :_L b)$ is a prime filter. By assumption, there is $a_i \in F_i \setminus (F_k :_L b)$ for all $i \neq k$; so $(\vee_{i \neq j} a_i) \vee b \in \cap_{i \neq j}(F \cap F_i) \setminus (F \cap F_k)$ since $(\vee_{i \neq j} a_i) \vee b \in F \cap F_k$ implies that $(\vee_{i \neq j} a_i) \in (F_k :_L b)$ and so there is $a_i \in (F_k :_L b)$ for some $i \neq k$ that is a contradiction. If $(F_k :_L (a \vee b))$ is a prime filter of L for some $a \in L$, then there exists $c_i \in F_i \setminus (F_k :_L (a \vee b))$ for all $i \neq k$. Therefore $(\vee_{i \neq j} c_i) \vee (a \vee b \in \cap_{i \neq j}(F \cap F_i) \setminus (F \cap F_k)$ which is a contradiction. Thus F_k is not a 2-absorbing filter, as required. ■

The following theorem is a lattice counterpart of Theorem 3.2 in [10] describing the structure of 2-absorbing submodules.

Theorem 3.4 (2-Absorbing Avoidance Theorem)**.** *Let $F, F_1, F_2, \ldots, F_n$ $(n \geq 2)$ be filters of L such that at most two of $F_1, F_2, \ldots, F_n$ are not 2-absorbing. If $F \subseteq \cup_{i=1}^{n} F_i$ and $F_i \nsubseteq (F_j :_L x)$ for all $x \in L \setminus F_j$ whenever $i \neq j$, then $F \subseteq F_i$ for some i with $1 \leq i \leq n$.*

Proof. By Remark 2.3 (i), we may assume that $n \geq 3$. Let $F \nsubseteq F_i$ for all i with $1 \leq i \leq n$. Then $F \subseteq \cup_{i=1}^{n} F_i$ is an efficient covering of filters of L. Then by Proposition 3.3, no F_i is 2-absorbing that contradicts the assumption. Therefore $F \subseteq F_i$ for some i with $1 \leq i \leq n$. ■

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