Discussiones Mathematicae
Probability and Statistics 36 (2016) 93–113
doi:10.7151/dmps.1182

BEST UNBIASED ESTIMATES FOR PARAMETERS OF THREE-LEVEL MULTIVARIATE DATA WITH DOUBLY EXCHANGEABLE COVARIANCE STRUCTURE AND STRUCTURED MEAN VECTOR

Arkadiusz Kozioł

Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra
Szafrana 4a, 65–516 Zielona Góra, Poland

e-mail: a.koziol@wmie.uz.zgora.pl

Abstract

In this article author obtain the best unbiased estimators of doubly exchangeable covariance structure. For this purpose the coordinate free-coordinate approach is used. Considered covariance structure consist of three unstructured covariance matrices for three-level $m-$variate observations with equal mean vector over v points in time and u sites under the assumption of multivariate normality. To prove, that the estimators are best unbiased, complete statistics are used. Additionally, strong consistency is proven. Under the proposed model the variances of the estimators of covariance components are compared with the ones in the model in [11].

Keywords: best unbiased estimator, doubly exchangeable covariance structure, three-level multivariate data, coordinate free approach, structured mean vector.

2010 Mathematics Subject Classification: 62H12, 62J10, 62F10.

1. Introduction

In this article doubly exchangeable covariance structure for three-level multivariate observations is considered. In this type of multivariate observations m dimensional observation vector is repeatedly measured over v time points and u locations. Additionally, author assumes that the mean vector remains constant over time points and over sites (locations) which means that vectors $\boldsymbol{\mu} = \mathbf{1}_{vu} \otimes \boldsymbol{\mu}$ with $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}^m$.

In this paper, optimal unbiased estimators for fixed and covariance parameters will be constructed from sufficient and complete statistics. These statistics will be derived using the free coordinate approach (see [2, 6, 14, 16, 4] and [17]).

2. Doubly exchangeable covariance structure

The $(vum \times vum)-$dimensional doubly exchangeable covariance structure is defined as

$$\begin{aligned} \boldsymbol{\Gamma} &= \begin{bmatrix} \boldsymbol{\Sigma}_0 & \boldsymbol{\Sigma}_1 & \dots & \boldsymbol{\Sigma}_1 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \boldsymbol{\Sigma}_1 & \boldsymbol{\Sigma}_1 & \dots & \boldsymbol{\Sigma}_0 \end{bmatrix} \\ &= \boldsymbol{I}_v \otimes (\boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}_1) + \boldsymbol{J}_v \otimes \boldsymbol{\Sigma}_1. \end{aligned} \tag{2.1}$$

The above doubly exchangeable covariance structure $\boldsymbol{\Gamma}$ can equivalently be written as follows

$$\boldsymbol{\Gamma} = \boldsymbol{I}_v \otimes \boldsymbol{\Sigma}_0 + (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{\Sigma}_1, \tag{2.2}$$

where $\boldsymbol{I}_v$ is the $v \times v$ identity matrix, $\mathbf{1}_v$ is a $v \times 1$ vector of ones, $\boldsymbol{J}_v = \mathbf{1}_v \mathbf{1}_v'$ is matrix of ones, $\otimes$ represents the Kronecker product and

$$\begin{aligned} \boldsymbol{\Sigma}_0 &= \begin{bmatrix} \boldsymbol{\Gamma}_0 & \boldsymbol{\Gamma}_1 & \dots & \boldsymbol{\Gamma}_1 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \boldsymbol{\Gamma}_1 & \boldsymbol{\Gamma}_1 & \dots & \boldsymbol{\Gamma}_0 \end{bmatrix} \\ &= \boldsymbol{I}_u \otimes (\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1) + \boldsymbol{J}_u \otimes \boldsymbol{\Gamma}_1. \end{aligned} \tag{2.3}$$

$$\begin{aligned} \boldsymbol{\Sigma}_1 &= \begin{bmatrix} \boldsymbol{\Gamma}_2 & \dots & \dots & \boldsymbol{\Gamma}_2 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \boldsymbol{\Gamma}_2 & \dots & \dots & \boldsymbol{\Gamma}_2 \end{bmatrix} \\ &= \boldsymbol{J}_u \otimes \boldsymbol{\Gamma}_2. \end{aligned} \tag{2.4}$$

Thus writing the doubly exchangeable covariance structure in terms of gammas

$$\boldsymbol{\Gamma} = \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes (\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1) + \boldsymbol{I}_v \otimes \boldsymbol{J}_u \otimes (\boldsymbol{\Gamma}_1 - \boldsymbol{\Gamma}_2) + \boldsymbol{J}_v \otimes \boldsymbol{J}_u \otimes \boldsymbol{\Gamma}_2, \tag{2.5}$$

which can equivalently be written as

$$\boldsymbol{\Gamma} = \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes \boldsymbol{\Gamma}_0 + \boldsymbol{I}_v \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{\Gamma}_1 + (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{\Gamma}_2. \tag{2.6}$$

The last form will be used to build orthogonal with respect to trace of inner product base for components of matrix $\boldsymbol{\Gamma}$.

$\boldsymbol{\Gamma}_0$ is assumed to be a positive definite symmetric $m \times m$ matrix, $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ are assumed to be a symmetric $m \times m$ matrices, and $\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1$, $\boldsymbol{\Gamma}_0 + (u-1)\boldsymbol{\Gamma}_1 - u\boldsymbol{\Gamma}_2$, $\boldsymbol{\Gamma}_0 + (u-1)\boldsymbol{\Gamma}_1 + (v-1)u\boldsymbol{\Gamma}_2$ are positive definite matrices, so that the $vum \times vum$ matrix $\boldsymbol{\Gamma}$ is positive definite (for a proof, see [8] and [9]). The $m \times m$ block diagonals $\boldsymbol{\Gamma}_0$ in $\boldsymbol{\Gamma}$ represent the variance-covariance matrix of the m response variables at any given time point and at any given site, whereas the $m \times m$ block off diagonals $\boldsymbol{\Gamma}_1$ in $\boldsymbol{\Gamma}$ represent the covariance matrix of the m response variables at any given time point and between any two sites. The $m \times m$ block off diagonals $\boldsymbol{\Gamma}_2$ in $\boldsymbol{\Gamma}$ represent the covariance matrix of the m response variables between any two time points. In view of form of matix $\boldsymbol{\Gamma}$ presented in (2.6) is clear that $\boldsymbol{\Gamma}_0$ is constant for all time points and sites, $\boldsymbol{\Gamma}_1$ is same between any two sites and for all time points and $\boldsymbol{\Gamma}_2$ is assumed to be the same for any pair of time points, irrespective of the same site or between any two sites.

3. BEST UNBIASED ESTIMATORS OF $\boldsymbol{\mu}$ AND $\boldsymbol{\Gamma}$

Let $\boldsymbol{y}_{r,ts}$ be a m-variate vector of measurements on the rth individual at the tth time point and at the sth site; $r = 1, \ldots, n$, $t = 1, \ldots, v$, $s = 1, \ldots, u$. The n individuals are all independent. Let $\boldsymbol{y}_r = (\boldsymbol{y}'_{r,11}, \ldots, \boldsymbol{y}'_{r,vu})'$ be the vum-variate vector of all measurements corresponding to the rth individual. Finally, let $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n$ be a random sample of size n drawn from the population $N_{vum}(\boldsymbol{1}_{vu} \otimes \boldsymbol{\mu}, \boldsymbol{\Gamma})$, where $E[\boldsymbol{y}_r] = \boldsymbol{1}_{vu} \otimes \boldsymbol{\mu} \in \mathbb{R}^{vum}$ and $\boldsymbol{\Gamma}$ is assumed to be a $vum \times vum$ positive definite matrix.

In this section, optimal properties of unbiased estimators for parameters of the probability distribution of the following column vector

$$\underset{nvum \times 1}{\boldsymbol{y}} = vec(\underset{vum \times n}{\boldsymbol{Y}'}) \sim N\big((\boldsymbol{1}_{nvu} \otimes \boldsymbol{I}_m)\boldsymbol{\mu}, \boldsymbol{I}_n \otimes \boldsymbol{\Gamma}_{vum}\big)$$

are presented. This means that n independent random column vectors are identically distributed $(vum \times vum)-$dimensional variance covariance matrix

$$\boldsymbol{\Gamma} = \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes \boldsymbol{\Gamma}_0 + \boldsymbol{I}_v \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{\Gamma}_1 + (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{\Gamma}_2.$$

Define the projection matrix $\boldsymbol{P}$ as follows

$$\boldsymbol{P} = \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m. \tag{3.7}$$

It is clear that $\boldsymbol{P}$ is an orthogonal projector on the subspace of the mean vector of $\boldsymbol{y}$. If $\boldsymbol{I}_n \otimes \boldsymbol{I}_{vum} \in \boldsymbol{\vartheta}$, from [3] it follows that $\boldsymbol{Py}$ is the best linear unbiased estimator (BLUE) if and only if $\boldsymbol{P}$ commutes with all covariance matrices $\boldsymbol{V}$. Therefore, we have the following results.

Result 1. The projection matrix $\boldsymbol{P}$ commutes with the covariance matrix $\boldsymbol{V}$, i.e., $\boldsymbol{PV} = \boldsymbol{VP}$, where $\boldsymbol{V} = \boldsymbol{I}_n \otimes \boldsymbol{\Gamma}$, the covariance matrix of $\boldsymbol{y}$.

Proof. Now,

$$\begin{aligned}\boldsymbol{P} &= (\mathbf{1}_n \otimes \mathbf{1}_v \otimes \mathbf{1}_u \otimes \boldsymbol{I}_m)(\mathbf{1}_n \otimes \mathbf{1}_v \otimes \mathbf{1}_u \otimes \boldsymbol{I}_m)^+ \\ &= (\mathbf{1}_n \otimes \mathbf{1}_v \otimes \mathbf{1}_u \otimes \boldsymbol{I}_m)\left(\frac{1}{n}\mathbf{1}'_n \otimes \frac{1}{v}\mathbf{1}'_v \otimes \frac{1}{u}\mathbf{1}'_u \otimes \boldsymbol{I}_m\right) \\ &= \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m.\end{aligned}$$

Note that

$$\begin{aligned}\boldsymbol{PV} &= \left(\frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\right)(\boldsymbol{I}_n \otimes \boldsymbol{\Gamma}) \\ &= \frac{1}{n}\boldsymbol{J}_n \otimes \Bigg[\left(\frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\right) \\ &\quad (\boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes \boldsymbol{\Gamma}_0 + \boldsymbol{I}_v \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{\Gamma}_1 + (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{\Gamma}_2)\Bigg] \\ &= \frac{1}{n}\boldsymbol{J}_n \otimes \Bigg[\frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{\Gamma}_0 + \frac{1}{v}\boldsymbol{J}_v \otimes \frac{u-1}{u}\boldsymbol{J}_u \otimes \boldsymbol{\Gamma}_1 \\ &\quad + \frac{v-1}{v}\boldsymbol{J}_v \otimes \boldsymbol{J}_u \otimes \boldsymbol{\Gamma}_2\Bigg]\end{aligned}$$

is symmetric. It implies that matrix $\boldsymbol{P}$ commutes with the covariance matrix of $\boldsymbol{y}$. ■

Lemma 1. *Let $\boldsymbol{\vartheta}$ denote the subspace spanned by $\boldsymbol{V}$, i.e., $\boldsymbol{\vartheta} = sp\{\boldsymbol{V}\}$. Then, $\boldsymbol{\vartheta}$ is a quadratic subspace, meaning that ϑ is a linear space and if $\boldsymbol{V} \in \vartheta$ then $\boldsymbol{V}^2 \in \vartheta$ (see [12] for the definition).*

Proof. See Lemma 4.1 in [13]. ■

Because orthogonal projector on the space generated by the mean vector commutes with all covariances matrices, there exists BLUE for each estimable function of mean. Moreover BLUE are least squares estimators (LSE), in view of Result 1. Thus, $\widetilde{\boldsymbol{\mu}}$ is the unique solution of the following normal equation:

$$(\mathbf{1}_n \otimes \mathbf{1}_v \otimes \mathbf{1}_u \otimes \boldsymbol{I}_m)'(\mathbf{1}_n \otimes \mathbf{1}_v \otimes \mathbf{1}_u \otimes \boldsymbol{I}_m)\boldsymbol{\mu} = (\mathbf{1}_n \otimes \mathbf{1}_v \otimes \mathbf{1}_u \otimes \boldsymbol{I}_m)'\boldsymbol{y} \quad \text{or}$$

$$nvu\boldsymbol{I}_m\boldsymbol{\mu} = [\boldsymbol{I}_m, \boldsymbol{I}_m, \ldots, \boldsymbol{I}_m]\boldsymbol{y},$$

which means that

$$\widetilde{\boldsymbol{\mu}} = \frac{1}{nvu}\sum_{r=1}^{n}\sum_{t=1}^{v}\sum_{s=1}^{u}\boldsymbol{y}_{r,ts}.$$

Let $\boldsymbol{M} = \boldsymbol{I}_n \otimes \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes \boldsymbol{I}_m - \boldsymbol{P}$. So, $\boldsymbol{M}$ is idempotent. Now, since $\boldsymbol{PV} = \boldsymbol{VP}$, and $\boldsymbol{\vartheta}$ is a quadratic space, $\boldsymbol{M\vartheta M} = \boldsymbol{M\vartheta}$ is also a quadratic space. We now construct a base for the quadratic subspace $\boldsymbol{\vartheta}$. Define

$$\boldsymbol{A}_{ii} = \boldsymbol{E}_{ii} \quad \text{and} \quad \boldsymbol{A}_{ij} = \boldsymbol{E}_{ij} + \boldsymbol{E}_{ji}, \quad \text{for } i < j; \text{ and } j = 1, \ldots, m,$$

as a base for symmetric matrices $\boldsymbol{\Gamma}$. The $(m \times m)-$dimensional matrices $\boldsymbol{E}_{ij}$ has 1 only at the ijth element, and 0 at all other elements. Then it is clear that the base for diagonal matrices of the form $\boldsymbol{I}_n \otimes \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes \boldsymbol{\Gamma}_0$ is constituted by matrices

$$\boldsymbol{K}_{ij}^{(0)} = \boldsymbol{I}_n \otimes \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes \boldsymbol{A}_{ij}, \quad \text{for } i \leq j, \quad j = 1, \ldots, m, \tag{3.8}$$

the base for matrices of the form $\boldsymbol{I}_n \otimes \boldsymbol{I}_v \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{\Gamma}_1$ is constituted by matrices

$$\boldsymbol{K}_{ij}^{(1)} = \boldsymbol{I}_n \otimes \boldsymbol{I}_v \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{A}_{ij}, \quad \text{for } i \leq j, \quad j = 1, \ldots, m \tag{3.9}$$

and the base for matrices of the form $\boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{\Gamma}_2$ is constituted by matrices

$$\boldsymbol{K}_{ij}^{(2)} = \boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{A}_{ij}, \quad \text{for } i \leq j, \quad j = 1, \ldots, m. \tag{3.10}$$

It is clear from (2.2) that above base is orthogonal with respect to trace of inner product. See also [11].

Result 2. The complete and minimal sufficient statistics for the mean vector and the variance-covariance matrix are

$$(\mathbf{1}'_{nvu} \otimes \boldsymbol{I}_m)\boldsymbol{y} \tag{3.11}$$

and

$$\boldsymbol{y}'\boldsymbol{M}\boldsymbol{K}_{ij}^{(l)}\boldsymbol{M}\boldsymbol{y},\ l = 0, 1, 2, \tag{3.12}$$

where $\boldsymbol{M} = \boldsymbol{I}_{nvum} - \boldsymbol{P}$ and $\boldsymbol{P}$ is given in (3.7), see [1, 14] and [17].

Now it is necessary to prove that $\widetilde{\mathbf{\Gamma}}_{vum}$ is the best quadratic unbiased estimator (BQUE) for $\mathbf{\Gamma}$. Since $\boldsymbol{P}$ commutes with the covariance matrix of $\boldsymbol{y}$, for each parameter of covariance there exists BQUE if and only if

$$sp\{\boldsymbol{MVM}\}, \quad \text{where} \quad \boldsymbol{M} = \boldsymbol{I}_{nvum} - \boldsymbol{P},$$

is a quadratic subspace (see [15, 17] and [3]) or Jordan algebra (see [5]), where $\boldsymbol{V}$ stands for covariance matrix of $\boldsymbol{y}$. It is clear that if $sp\{\boldsymbol{V}\}$ is a quadratic subspace and if for each $\mathbf{\Sigma} \in sp\{\boldsymbol{V}\}$ commutativity $\boldsymbol{P}\mathbf{\Sigma} = \mathbf{\Sigma}\boldsymbol{P}$ holds, then $sp\{\boldsymbol{MVM}\} = sp\{\boldsymbol{MV}\}$ is also a quadratic subspace. According to the coordinate free approach, the expectation of $\boldsymbol{Myy'M}$ can be written as a linear combination of matrices $\boldsymbol{MK}_{ij}^{(0)}$, $\boldsymbol{MK}_{ij}^{(1)}$ and $\boldsymbol{MK}_{ij}^{(2)}$ with unknown coefficients $\sigma_{ij}^{(0)}$, $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$, respectively. Note also that identity covariance operator of $\boldsymbol{yy'}$ belongs to $sp\{\mathrm{cov}(\boldsymbol{yy'})\}$. It implies that the ordinary best quadratic estimators are least square estimators for corresponding parameters $\sigma_{ij}^{(0)}$, $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$. They cannot be calculated independently (as in [11]) because $\boldsymbol{MK}_{ij}^{(0)}$, $\boldsymbol{MK}_{ij}^{(1)}$ and $\boldsymbol{MK}_{ij}^{(2)}$ are not orthogonal. Defining $\frac{m(m+1)}{2}$ column vectors $\boldsymbol{\sigma}^{(l)} = [\sigma_{ij}^{(l)}]$ for $i \leq j = 1, \ldots, m$; $l = 0, 1, 2$, the normal equations have the following block diagonal structure

$$\left(\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \otimes \boldsymbol{I}_{\frac{m(m+1)}{2}}\right) \begin{bmatrix} \boldsymbol{\sigma}^{(0)} \\ \boldsymbol{\sigma}^{(1)} \\ \boldsymbol{\sigma}^{(2)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}^{(0)} \\ \boldsymbol{r}^{(1)} \\ \boldsymbol{r}^{(2)} \end{bmatrix}, \tag{3.13}$$

where for $i \leq j = 1, \ldots, m$; $a = \mathrm{tr}\big(\boldsymbol{M}(\boldsymbol{K}_{ij}^{(0)})^2\big)$, $b = \mathrm{tr}\big(\boldsymbol{MK}_{ij}^{(0)}\boldsymbol{K}_{ij}^{(1)}\big)$, $c = \mathrm{tr}\big(\boldsymbol{MK}_{ij}^{(0)}\boldsymbol{K}_{ij}^{(2)}\big)$, $d = \mathrm{tr}\big(\boldsymbol{M}(\boldsymbol{K}_{ij}^{(1)})^2\big)$, $e = \mathrm{tr}\big(\boldsymbol{MK}_{ij}^{(1)}\boldsymbol{K}_{ij}^{(2)}\big)$, $f = \mathrm{tr}\big(\boldsymbol{M}(\boldsymbol{K}_{ij}^{(2)})^2\big)$ while $\boldsymbol{r}^{(l)} = \frac{1}{2-\boldsymbol{\delta}_{ij}}\big[r'\boldsymbol{K}_{ij}^{(l)}r\big]$ for $l = 0, 1, 2$ is $\frac{m(m+1)}{2} \times 1$ vector, $\boldsymbol{\delta}_{ij}$ is the Kronecker delta and $\boldsymbol{r}$ stands for the residual vector, i.e., $\boldsymbol{r} = \boldsymbol{My} = (\boldsymbol{I}_{nvum} - \boldsymbol{P})\boldsymbol{y}$. Now to prove (3.13), consider the following six cases

Case 1. for $l = 0$, $i = j$,

$$\begin{aligned}
\mathrm{tr}\Big(\boldsymbol{M}(\boldsymbol{K}_{ii}^{(0)})^2\Big) &= \mathrm{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big)(\boldsymbol{I}_{nvu} \otimes \boldsymbol{A}_{ii})^2\Big] \\
&= \mathrm{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big)(\boldsymbol{I}_{nvu} \otimes \boldsymbol{A}_{ii}^2)\Big] \\
&= \mathrm{tr}(\boldsymbol{A}_{ii}^2)\mathrm{tr}\Big(\boldsymbol{I}_{nvu} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u\Big) \\
&= nvu - 1.
\end{aligned}$$

Case 2. for $l = 0$, $i < j$,

$$\begin{aligned}
\mathrm{tr}\Big(\boldsymbol{M}(\boldsymbol{K}_{ij}^{(0)})^2\Big) &= \mathrm{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big)(\boldsymbol{I}_{nvu} \otimes \boldsymbol{A}_{ij})^2\Big] \\
&= \mathrm{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big)(\boldsymbol{I}_{nvu} \otimes \boldsymbol{A}_{ij}^2)\Big] \\
&= \mathrm{tr}(\boldsymbol{A}_{ij}^2)\mathrm{tr}\Big(\boldsymbol{I}_{nvu} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u\Big) = 2(nvu - 1).
\end{aligned}$$

Case 3. for $l = 1$, $i = j$,

$$\begin{aligned}
\mathrm{tr}\Big(\boldsymbol{M}(\boldsymbol{K}_{ii}^{(1)})^2\Big) &= \mathrm{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_{nv} \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{A}_{ii})^2\Big] \\
&= \mathrm{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_{nv} \otimes ((u-2)\boldsymbol{J}_u + \boldsymbol{I}_u) \otimes \boldsymbol{A}_{ii}^2)\Big] \\
&= \mathrm{tr}(\boldsymbol{A}_{ii}^2)\mathrm{tr}\Big(\boldsymbol{I}_{nv} \otimes ((u-2)\boldsymbol{J}_u + \boldsymbol{I}_u) \\
&\quad - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{(u-1)^2}{u}\boldsymbol{J}_u\Big) \\
&= nvu(u-1) - (u-1)^2 \\
&= [(nv-1)u + 1](u-1).
\end{aligned}$$

Case 4. for $l = 1$, $i < j$,

$$\begin{aligned}
\mathrm{tr}\Big(\boldsymbol{M}(\boldsymbol{K}_{ij}^{(1)})^2\Big) &= \mathrm{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_{nv} \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{A}_{ij})^2\Big] \\
&= \mathrm{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_{nv} \otimes ((u-2)\boldsymbol{J}_u + \boldsymbol{I}_u) \otimes \boldsymbol{A}_{ij}^2)\Big] \\
&= \mathrm{tr}(\boldsymbol{A}_{ij}^2)\mathrm{tr}\Big(\boldsymbol{I}_{nv} \otimes ((u-2)\boldsymbol{J}_u + \boldsymbol{I}_u) \\
&\quad - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{(u-1)^2}{u}\boldsymbol{J}_u\Big) \\
&= 2(nvu(u-1) - (u-1)^2) \\
&= 2[(nv-1)u + 1](u-1).
\end{aligned}$$

Case 5. for $l = 2$, $i = j$,

$$\begin{aligned}
\operatorname{tr}\Big(\boldsymbol{M}(\boldsymbol{K}_{ii}^{(2)})^2\Big) &= \operatorname{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{A}_{ii})^2\Big] \\
&= \operatorname{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_n \otimes ((v-2)\boldsymbol{J}_v + \boldsymbol{I}_v) \otimes u\boldsymbol{J}_u \otimes \boldsymbol{A}_{ii}^2)\Big] \\
&= \operatorname{tr}(\boldsymbol{A}_{ii}^2)\operatorname{tr}\Big(\boldsymbol{I}_n \otimes ((v-2)\boldsymbol{J}_v + \boldsymbol{I}_v) \otimes u\boldsymbol{J}_u \\
&\quad - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{(v-1)^2}{v}\boldsymbol{J}_v \otimes u\boldsymbol{J}_u\Big) \\
&= nv(v-1)u^2 - (v-1)^2u^2 \\
&= [(n-1)v+1](v-1)u^2.
\end{aligned}$$

Case 6. for $l = 2$, $i < j$,

$$\begin{aligned}
\operatorname{tr}\Big(\boldsymbol{M}(\boldsymbol{K}_{ij}^{(2)})^2\Big) &= \operatorname{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{A}_{ij})^2\Big] \\
&= \operatorname{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_n \otimes ((v-2)\boldsymbol{J}_v + \boldsymbol{I}_v) \otimes u\boldsymbol{J}_u \otimes \boldsymbol{A}_{ij}^2)\Big] \\
&= \operatorname{tr}(\boldsymbol{A}_{ij}^2)\operatorname{tr}\Big(\boldsymbol{I}_n \otimes ((v-2)\boldsymbol{J}_v + \boldsymbol{I}_v) \otimes u\boldsymbol{J}_u \\
&\quad - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{(v-1)^2}{v}\boldsymbol{J}_v \otimes u\boldsymbol{J}_u\Big) \\
&= 2(nv(v-1)u^2 - (v-1)^2u^2) \\
&= 2[(n-1)v+1](v-1)u^2.
\end{aligned}$$

Case 7. for $l_1 = 0$, $l_2 = 1$, $i = j$,

$$\begin{aligned}
\operatorname{tr}\Big(\boldsymbol{M}\boldsymbol{K}_{ii}^{(0)}\boldsymbol{K}_{ii}^{(1)}\Big) &= \operatorname{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_{nvu} \otimes \boldsymbol{A}_{ii})(\boldsymbol{I}_{nv} \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{A}_{ii})\Big] \\
&= \operatorname{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_{nv} \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{A}_{ii}^2)\Big] \\
&= \operatorname{tr}(\boldsymbol{A}_{ii}^2)\operatorname{tr}\Big(\boldsymbol{I}_{nv} \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{u-1}{u}\boldsymbol{J}_u\Big) \\
&= -(u-1).
\end{aligned}$$

Case 8. for $l_1 = 0$, $l_2 = 1$, $i < j$,

$$\begin{aligned}
\mathrm{tr}\Big(\boldsymbol{M}\boldsymbol{K}_{ij}^{(0)}\boldsymbol{K}_{ij}^{(1)}\Big) &= \mathrm{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_{nvu} \otimes \boldsymbol{A}_{ij})(\boldsymbol{I}_{nv} \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{A}_{ij})\Big] \\
&= \mathrm{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_{nv} \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{A}_{ij}^2)\Big] \\
&= \mathrm{tr}(\boldsymbol{A}_{ij}^2)\mathrm{tr}\Big(\boldsymbol{I}_{nv} \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{u-1}{u}\boldsymbol{J}_u\Big) \\
&= -2(u-1).
\end{aligned}$$

Case 9. for $l_1 = 0$, $l_2 = 2$, $i = j$,

$$\begin{aligned}
\mathrm{tr}\Big(\boldsymbol{M}\boldsymbol{K}_{ii}^{(0)}\boldsymbol{K}_{ii}^{(2)}\Big) &= \mathrm{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_{nvu} \otimes \boldsymbol{A}_{ii})(\boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{A}_{ii})\Big] \\
&= \mathrm{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad \boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{A}_{ii}^2)\Big] \\
&= \mathrm{tr}(\boldsymbol{A}_{ii}^2)\mathrm{tr}\Big(\boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{v-1}{v}\boldsymbol{J}_v \otimes \boldsymbol{J}_u\Big) \\
&= -(v-1)u.
\end{aligned}$$

Case 10. for $l_1 = 0$, $l_2 = 2$, $i < j$,

$$\begin{aligned}
\mathrm{tr}\Big(\boldsymbol{M}\boldsymbol{K}_{ij}^{(0)}\boldsymbol{K}_{ij}^{(2)}\Big) &= \mathrm{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_{nvu} \otimes \boldsymbol{A}_{ij})(\boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{A}_{ij})\Big] \\
&= \mathrm{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{A}_{ij}^2)\Big] \\
&= \mathrm{tr}(\boldsymbol{A}_{ij}^2)\mathrm{tr}\Big(\boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{v-1}{v}\boldsymbol{J}_v \otimes \boldsymbol{J}_u\Big) \\
&= -2(v-1)u.
\end{aligned}$$

Case 11. for $l_1 = 1$, $l_2 = 2$, $i = j$,

$$\begin{aligned}
\operatorname{tr}\Big(\boldsymbol{M}\boldsymbol{K}_{ii}^{(1)}\boldsymbol{K}_{ii}^{(2)}\Big) &= \operatorname{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_{nv} \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{A}_{ii})(\boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{A}_{ii})\Big] \\
&= \operatorname{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes (u-1)\boldsymbol{J}_u \otimes \boldsymbol{A}_{ii}^2)\Big] \\
&= \operatorname{tr}(\boldsymbol{A}_{ii}^2)\operatorname{tr}\Big(\boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes (u-1)\boldsymbol{J}_u \\
&\quad -\frac{1}{n}\boldsymbol{J}_n \otimes \frac{v-1}{v}\boldsymbol{J}_v \otimes (u-1)\boldsymbol{J}_u\Big) \\
&= -(v-1)u(u-1).
\end{aligned}$$

Case 12. for $l_1 = 1$, $l_2 = 2$, $i < j$,

$$\begin{aligned}
\operatorname{tr}\Big(\boldsymbol{M}\boldsymbol{K}_{ij}^{(1)}\boldsymbol{K}_{ij}^{(2)}\Big) &= \operatorname{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_{nv} \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{A}_{ij})(\boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{A}_{ij})\Big] \\
&= \operatorname{tr}\Big[\Big(\boldsymbol{I}_{nvum} - \frac{1}{n}\boldsymbol{J}_n \otimes \frac{1}{v}\boldsymbol{J}_v \otimes \frac{1}{u}\boldsymbol{J}_u \otimes \boldsymbol{I}_m\Big) \\
&\quad (\boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes (u-1)\boldsymbol{J}_u \otimes \boldsymbol{A}_{ij}^2)\Big] \\
&= \operatorname{tr}(\boldsymbol{A}_{ij}^2)\operatorname{tr}\Big(\boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes (u-1)\boldsymbol{J}_u \\
&\quad -\frac{1}{n}\boldsymbol{J}_n \otimes \frac{v-1}{v}\boldsymbol{J}_v \otimes (u-1)\boldsymbol{J}_u\Big) \\
&= -2(v-1)u(u-1).
\end{aligned}$$

Thus, to find the best quadratic unbiased estimator $\widetilde{\boldsymbol{\Gamma}}$ for $\boldsymbol{\Gamma}$, the following normal equation has to be solved

$$\left(\begin{bmatrix} nvu-1 & -(u-1) & -(v-1)u \\ -(u-1) & [(nv-1)u+1](u-1) & -(v-1)u(u-1) \\ -(v-1)u & -(v-1)u(u-1) & [(n-1)v+1](v-1)u^2 \end{bmatrix} \otimes \boldsymbol{I}_{\frac{m(m+1)}{2}}\right)\begin{bmatrix}\boldsymbol{\sigma}^{(0)} \\ \boldsymbol{\sigma}^{(1)} \\ \boldsymbol{\sigma}^{(2)}\end{bmatrix} = \begin{bmatrix}\boldsymbol{r}^{(0)} \\ \boldsymbol{r}^{(1)} \\ \boldsymbol{r}^{(2)}\end{bmatrix},$$

$$\begin{bmatrix}\boldsymbol{\sigma}^{(0)} \\ \boldsymbol{\sigma}^{(1)} \\ \boldsymbol{\sigma}^{(2)}\end{bmatrix} = \left(\begin{bmatrix} \frac{(n-1)vu+1}{n(n-1)v^2u^2} & \frac{1}{n(n-1)v^2u^2} & \frac{1}{n(n-1)v^2u^2} \\ \frac{1}{n(n-1)v^2u^2} & \frac{(n-1)vu+u-1}{n(n-1)v^2u^2(u-1)} & \frac{1}{n(n-1)v^2u^2} \\ \frac{1}{n(n-1)v^2u^2} & \frac{1}{n(n-1)v^2u^2} & \frac{nv-1}{n(n-1)v^2(v-1)u^2} \end{bmatrix} \otimes \boldsymbol{I}_{\frac{m(m+1)}{2}}\right)\begin{bmatrix}\boldsymbol{r}^{(0)} \\ \boldsymbol{r}^{(1)} \\ \boldsymbol{r}^{(2)}\end{bmatrix}. \tag{3.14}$$

The right hand side of this equation can be expressed by $\boldsymbol{C}_0$, $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ defined in the following way

$$\boldsymbol{C}_0 = \sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{r=1}^{n}\left(\boldsymbol{y}_{r,ts} - \widetilde{\boldsymbol{\mu}}\right)\left(\boldsymbol{y}_{r,ts} - \widetilde{\boldsymbol{\mu}}\right)', \tag{3.15}$$

$$\boldsymbol{C}_1 = \sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{\substack{s^*=1\\ s\neq s^*}}^{u}\sum_{r=1}^{n}\left(\boldsymbol{y}_{r,ts^*} - \widetilde{\boldsymbol{\mu}}\right)\left(\boldsymbol{y}_{r,ts} - \widetilde{\boldsymbol{\mu}}\right)', \tag{3.16}$$

$$\boldsymbol{C}_2 = \sum_{t=1}^{v}\sum_{\substack{t^*=1\\ t\neq t^*}}^{v}\sum_{s=1}^{u}\sum_{s^*=1}^{u}\sum_{r=1}^{n}\left(\boldsymbol{y}_{r,t^*s^*} - \widetilde{\boldsymbol{\mu}}\right)\left(\boldsymbol{y}_{r,ts} - \widetilde{\boldsymbol{\mu}}\right)', \tag{3.17}$$

where $\widetilde{\boldsymbol{\mu}} = \frac{1}{nvu}\sum_{r=1}^{n}\sum_{t=1}^{v}\sum_{s=1}^{u}\boldsymbol{y}_{r,ts}$ and then we have

$$\left(\begin{bmatrix} nvu-1 & -(u-1) & -(v-1)u \\ -(u-1) & [(nv-1)u+1](u-1) & -(v-1)u(u-1) \\ -(v-1)u & -(v-1)u(u-1) & [(n-1)v+1](v-1)u^2 \end{bmatrix} \otimes \boldsymbol{I}_m\right)\begin{bmatrix}\boldsymbol{\Gamma}_0\\ \boldsymbol{\Gamma}_1\\ \boldsymbol{\Gamma}_2\end{bmatrix} = \begin{bmatrix}\boldsymbol{C}_0\\ \boldsymbol{C}_1\\ \boldsymbol{C}_2\end{bmatrix}.$$

Solving this equation we get

$$\begin{bmatrix}\boldsymbol{\Gamma}_0\\ \boldsymbol{\Gamma}_1\\ \boldsymbol{\Gamma}_2\end{bmatrix} = \left(\begin{bmatrix} nvu-1 & -(u-1) & -(v-1)u \\ -(u-1) & [(nv-1)u+1](u-1) & -(v-1)u(u-1) \\ -(v-1)u & -(v-1)u(u-1) & [(n-1)v+1](v-1)u^2 \end{bmatrix} \otimes \boldsymbol{I}_m\right)^{-1}\begin{bmatrix}\boldsymbol{C}_0\\ \boldsymbol{C}_1\\ \boldsymbol{C}_2\end{bmatrix},$$

$$\begin{bmatrix}\boldsymbol{\Gamma}_0\\ \boldsymbol{\Gamma}_1\\ \boldsymbol{\Gamma}_2\end{bmatrix} = \left(\begin{bmatrix} \frac{(n-1)vu+1}{n(n-1)v^2u^2} & \frac{1}{n(n-1)v^2u^2} & \frac{1}{n(n-1)v^2u^2} \\ \frac{1}{n(n-1)v^2u^2} & \frac{(n-1)vu+u-1}{n(n-1)v^2u^2(u-1)} & \frac{1}{n(n-1)v^2u^2} \\ \frac{1}{n(n-1)v^2u^2} & \frac{1}{n(n-1)v^2u^2} & \frac{nv-1}{n(n-1)v^2(v-1)u^2} \end{bmatrix} \otimes \boldsymbol{I}_m\right)\begin{bmatrix}\boldsymbol{C}_0\\ \boldsymbol{C}_1\\ \boldsymbol{C}_2\end{bmatrix}.$$

It is worth noting that all elements off diagonal of matrix on the right hand side in the above equation are equal. Estimators for $\boldsymbol{\Gamma}_0$, $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ are

$$\begin{aligned}
\widetilde{\boldsymbol{\Gamma}}_0 &= \frac{(n-1)vu+1}{n(n-1)v^2u^2}\boldsymbol{C}_0 + \frac{1}{n(n-1)v^2u^2}\boldsymbol{C}_1 + \frac{1}{n(n-1)v^2u^2}\boldsymbol{C}_2,\\
\widetilde{\boldsymbol{\Gamma}}_1 &= \frac{1}{n(n-1)v^2u^2}\boldsymbol{C}_0 + \frac{(n-1)vu+u-1}{n(n-1)v^2u^2(u-1)}\boldsymbol{C}_1 + \frac{1}{n(n-1)v^2u^2}\boldsymbol{C}_2,\\
\widetilde{\boldsymbol{\Gamma}}_2 &= \frac{1}{n(n-1)v^2u^2}\boldsymbol{C}_0 + \frac{1}{n(n-1)v^2u^2}\boldsymbol{C}_1 + \frac{nv-1}{n(n-1)v^2(v-1)u^2}\boldsymbol{C}_2.
\end{aligned}$$

Now using Result 2 we are ready to formulate the following theorem.

Theorem 2. *Assume that* $\boldsymbol{y}_{nvum\times 1} \sim N\big((\mathbf{1}_{nvu}\otimes \boldsymbol{I}_m)\boldsymbol{\mu}, \boldsymbol{I}_n\otimes\boldsymbol{\Gamma}\big)$ *with doubly exchangeable covariance structure on* $\boldsymbol{\Gamma}$*, i.e.,*

$$\boldsymbol{\Gamma} = \boldsymbol{I}_v\otimes\boldsymbol{I}_u\otimes(\boldsymbol{\Gamma}_0-\boldsymbol{\Gamma}_1)+\boldsymbol{I}_v\otimes\boldsymbol{J}_u\otimes(\boldsymbol{\Gamma}_1-\boldsymbol{\Gamma}_2)+\boldsymbol{J}_v\otimes\boldsymbol{J}_u\otimes\boldsymbol{\Gamma}_2,$$

where $\boldsymbol{\Gamma}_0$, $\boldsymbol{\Gamma}_1$ *and* $\boldsymbol{\Gamma}_2$ *are* $m\times m$ *unknown symmetric matrices such that* $\boldsymbol{\Gamma}$ *is positive definite. Then*

$$\widetilde{\boldsymbol{\mu}} = \frac{1}{nvu}\sum_{r=1}^{n}\sum_{t=1}^{v}\sum_{s=1}^{u}\boldsymbol{y}_{r,ts}, \tag{3.18}$$

where $\underset{nvum\times 1}{\boldsymbol{y}} = (\boldsymbol{y}_1', \boldsymbol{y}_2', \ldots, \boldsymbol{y}_n')'$ *with* $\boldsymbol{y}_r = (\boldsymbol{y}_{r,11}', \ldots, \boldsymbol{y}_{r,vu}')'$ *with* $\boldsymbol{y}_{r,ts} = (\boldsymbol{y}_{r,ts1}', \ldots, \boldsymbol{y}_{r,tsm}')'$ *and*

$$\widetilde{\boldsymbol{\Gamma}} = \boldsymbol{I}_v\otimes\boldsymbol{I}_u\otimes(\widetilde{\boldsymbol{\Gamma}}_0-\widetilde{\boldsymbol{\Gamma}}_1)+\boldsymbol{I}_v\otimes\boldsymbol{J}_u\otimes(\widetilde{\boldsymbol{\Gamma}}_1-\widetilde{\boldsymbol{\Gamma}}_2)+\boldsymbol{J}_v\otimes\boldsymbol{J}_u\otimes\widetilde{\boldsymbol{\Gamma}}_2, \tag{3.19}$$

where $\widetilde{\boldsymbol{\Gamma}}_0 = \frac{(n-1)vu+1}{n(n-1)v^2u^2}\boldsymbol{C}_0 + \frac{1}{n(n-1)v^2u^2}(\boldsymbol{C}_1+\boldsymbol{C}_2)$, $\widetilde{\boldsymbol{\Gamma}}_1 = \frac{(n-1)vu+u-1}{n(n-1)v^2u^2(u-1)}\boldsymbol{C}_1 + \frac{1}{n(n-1)v^2u^2}(\boldsymbol{C}_0+\boldsymbol{C}_2)$ *and* $\widetilde{\boldsymbol{\Gamma}}_2 = \frac{nv-1}{n(n-1)v^2(v-1)u^2}\boldsymbol{C}_2 + \frac{1}{n(n-1)v^2u^2}(\boldsymbol{C}_0+\boldsymbol{C}_1)$ *are the best unbiased estimators* (BUE) *for* $\boldsymbol{\mu}$ *and* $\boldsymbol{\Gamma}$*, respectively. Here* $\boldsymbol{C}_0$, $\boldsymbol{C}_1$ *and* $\boldsymbol{C}_2$ *are defined in* (3.15), (3.16) *and* (3.17)*, respectively.*

Proof. These estimators for $\boldsymbol{\mu}$ and $\boldsymbol{\Gamma}$ are BLUE and BQUE, respectively. Now, because they are function of complete statistics from Result 2 it follows that they are BUE. ∎

Now we are able to make a statement that estimators presented in Theorem 2 are consistent and obviously the family of distribution of above estimators is complete.

Theorem 3. *Estimators given in* (3.18) *and* (3.19) *are consistent. Moreover, the family of distributions of these estimators is complete.*

Proof. Note that the variance of the quadratic forms $\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}$, where $\boldsymbol{y}\sim N(\boldsymbol{\mu}, \boldsymbol{V})$, is given by the following formula

$$\operatorname{var}(\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}) = 2\operatorname{tr}\big\{(\boldsymbol{AVAV}) + (\boldsymbol{AVA}\boldsymbol{\mu}\boldsymbol{\mu}' + \boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{AVA})\big\}. \tag{3.20}$$

In a special case, if $\boldsymbol{A} = \boldsymbol{MAM}$, and if $\boldsymbol{MV} = \boldsymbol{VM}$ then $\boldsymbol{A}\boldsymbol{\mu}\boldsymbol{\mu}' = \mathbf{0}$, and (3.20) reduces to the following form

$$\mathrm{var}(\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}) = 2\mathrm{tr}(\boldsymbol{MAVAV}). \tag{3.21}$$

Now making an use of (3.21) of doubly exchangeable covariance structure of the covariance matrix of $\boldsymbol{y}$ and from (3.8) it follows that for any fixed $\boldsymbol{\Gamma}$ if $n \to \infty$ then

$$\begin{aligned} var(\widetilde{\sigma}_{ij}^{(0)}) &= \frac{2((n-1)uv+1)}{(n-1)nu^2v^2}\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0) + \frac{4(u-1)}{(n-1)nu^2v^2}\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) \\ &+ \frac{4(v-1)}{(n-1)nuv^2}\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \\ &+ \frac{2(u-1)((n-1)uv+u-1)}{(n-1)nu^2v^2}\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) \\ &+ \frac{4(u-1)(v-1)}{(n-1)nuv^2}\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \\ &+ \frac{2(v-1)(nv-1)}{(n-1)nv^2}\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \to 0. \end{aligned}$$

Similarly it follows that we get for each fixed $\boldsymbol{\Gamma}$ if n tends to ∞ then

$$\begin{aligned} var(\widetilde{\sigma}_{ij}^{(1)}) &= \frac{2((n-1)uv+u-1)}{(n-1)n(u-1)u^2v^2}\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0) \\ &+ \frac{4((u-2)u((n-1)v+1)+1)}{(n-1)n(u-1)u^2v^2}\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) \\ &+ \frac{4(v-1)}{(n-1)nuv^2}\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \\ &+ \frac{2(u((u-3)u+3)((n-1)v+1)-1)}{(n-1)n(u-1)u^2v^2}\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) \\ &+ \frac{4(u-1)(v-1)}{(n-1)nuv^2}\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \\ &+ \frac{2(v-1)(nv-1)}{(n-1)nv^2}\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \to 0 \end{aligned}$$

and also we get that for each fixed $\boldsymbol{\Gamma}$ if $n \to \infty$ then

$$
\begin{aligned}
var(\widetilde{\sigma}_{ij}^{(2)}) &= \frac{2(nv-1)}{(n-1)nu^2(v-1)v^2}\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0)\\
&\quad + \frac{4(u-1)(nv-1)}{(n-1)nu^2(v-1)v^2}\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1)\\
&\quad + \frac{4(n(v-2)v+1)}{(n-1)nu(v-1)v^2}\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2)\\
&\quad + \frac{2(u-1)^2(nv-1)}{(n-1)nu^2(v-1)v^2}\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1)\\
&\quad + \frac{4(u-1)(n(v-2)v+1)}{(n-1)nu(v-1)v^2}\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2)\\
&\quad + \frac{2(nv((v-3)v+3)-1)}{(n-1)n(v-1)v^2}\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \to 0.
\end{aligned}
$$

To finish the proof, note that estimators for $\boldsymbol{\mu}$ and estimators for elements of covariance matrix are one-to-one functions of minimal sufficient statistic given by (3.11) and (3.12). ■

4. Comparison of BUE in two models

In this paragraph author compares variances of covariance parameters in two models Mo1 and Mo2. Both with a doubly exchangeable covariance structure, in which the first one has the unstructured mean vector $\mathbf{1}_n \otimes \boldsymbol{\mu}$ where $\boldsymbol{\mu}$ has vum components and the second one has the structured mean vector $\mathbf{1}_{nvu} \otimes \boldsymbol{\mu}$ where $\boldsymbol{\mu}$ has m components. We compare variances of estimators under model Mo1, $\widetilde{\sigma}_{[1]ij}^{(0)}$, $\widetilde{\sigma}_{[1]ij}^{(1)}$ and $\widetilde{\sigma}_{[1]ij}^{(2)}$, with model Mo2, $\widetilde{\sigma}_{[2]ij}^{(0)}$, $\widetilde{\sigma}_{[2]ij}^{(1)}$ and $\widetilde{\sigma}_{[2]ij}^{(2)}$. It is clear that the expectation of $\widetilde{\sigma}_{[1]ij}^{(0)}$, $\widetilde{\sigma}_{[1]ij}^{(1)}$ and $\widetilde{\sigma}_{[1]ij}^{(2)}$ calculated for Mo1 are unbiased under Mo2. From the Lehmann-Scheffé theorem (see [7]) it follows that the variance of all estimators both covariance and expectation parameters have smaller variances in Mo2. The inverse conclusion is not true because estimators for covariance parameters have different expectation under Mo1.

In model Mo1, estimators of mean vector $\boldsymbol{\mu}_{[1]}$ and components of variance-covariance matrix $\boldsymbol{\Gamma}_{[1]}$, i.e., $\boldsymbol{\Gamma}_{[1]0}$, $\boldsymbol{\Gamma}_{[1]1}$, $\boldsymbol{\Gamma}_{[1]2}$ are, respectively (see Theorem 2 in [11]).

$$\widetilde{\boldsymbol{\mu}}_{[1]} = \frac{1}{n}\sum_{r=1}^{n}\boldsymbol{y}_r,$$
$$\widetilde{\boldsymbol{\Gamma}}_{[1]0} = \frac{1}{(n-1)vu}\boldsymbol{C}_{[1]0},$$
$$\widetilde{\boldsymbol{\Gamma}}_{[1]1} = \frac{1}{(n-1)vu(u-1)}\boldsymbol{C}_{[1]1},$$
$$\widetilde{\boldsymbol{\Gamma}}_{[1]2} = \frac{1}{(n-1)v(v-1)u^2}\boldsymbol{C}_{[1]2},$$

where $\boldsymbol{C}_{[1]0}, \boldsymbol{C}_{[1]1}$ and $\boldsymbol{C}_{[1]2}$ are the following matrices

$$\boldsymbol{C}_{[1]0} = \sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{r=1}^{n}\left(\boldsymbol{y}_{r,ts} - \overline{\boldsymbol{y}}_{\bullet,ts}\right)\left(\boldsymbol{y}_{r,ts} - \overline{\boldsymbol{y}}_{\bullet,ts}\right)',$$
$$\boldsymbol{C}_{[1]1} = \sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{s\neq s^*=1}^{u}\sum_{r=1}^{n}\left(\boldsymbol{y}_{r,ts^*} - \overline{\boldsymbol{y}}_{\bullet,ts^*}\right)\left(\boldsymbol{y}_{r,ts} - \overline{\boldsymbol{y}}_{\bullet,ts}\right)',$$
$$\boldsymbol{C}_{[1]2} = \sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{t\neq t^*=1}^{v}\sum_{s^*=1}^{u}\sum_{r=1}^{n}\left(\boldsymbol{y}_{r,t^*s^*} - \overline{\boldsymbol{y}}_{\bullet,t^*s^*}\right)\left(\boldsymbol{y}_{r,ts} - \overline{\boldsymbol{y}}_{\bullet,ts}\right)',$$

where $\overline{\boldsymbol{y}}_{\bullet,ts} = \frac{1}{n}\sum_{r=1}^{n}\boldsymbol{y}_{r,ts}$, for $t = 1,\ldots,v$ and $s = 1,\ldots,u$.

Alternatively, it can also calculate and present graphically the difference of variances for both models.

$$\begin{aligned} var(\widetilde{\sigma}^{(0)}_{[2]ij}) - var(\widetilde{\sigma}^{(0)}_{[1]ij}) = & -\frac{2}{(n-1)nu^2v^2}((vu-1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0) \\ & - 2(u-1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) \\ & - 2(v-1)utr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \\ & + (u-1)((v-1)u+1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) \\ & - 2(v-1)u(u-1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \\ & + (v-1)u^2tr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2)). \end{aligned}$$

After simple calculations we get

$$\begin{aligned} var(\widetilde{\sigma}^{(0)}_{[2]ij}) - var(\widetilde{\sigma}^{(0)}_{[1]ij}) \;=\; & -\frac{2}{(n-1)nu^2v^2}((u-1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0) \\ & -2(u-1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) \\ & +(u-1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) \\ & +(v-1)utr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0) \\ & -2(v-1)utr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \\ & +(v-1)utr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \\ & +(v-1)u(u-1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) \\ & -2(v-1)u(u-1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \\ & +(v-1)u(u-1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2)), \end{aligned}$$

thus $var(\widetilde{\sigma}^{(0)}_{[2]ij}) - var(\widetilde{\sigma}^{(0)}_{[1]ij}) < 0$ if

$$\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0) + \mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) > 2tr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1),$$

$$\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0) + \mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) > 2tr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2),$$

$$\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) + \mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) > 2tr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2)$$

which holds for any fixed $\boldsymbol{\Gamma}_0, \boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2$.

Similarly, it is easy to see that

$$\begin{aligned} var(\widetilde{\sigma}^{(1)}_{[2]ij}) - var(\widetilde{\sigma}^{(1)}_{[1]ij}) \;=\; & -\frac{2}{(n-1)n(u-1)u^2v^2} \\ & ((u(v-1)+1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0) \\ & -2(1-u(u-2)(v-1))\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) \\ & -2(v-1)u(u-1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \\ & +(u((u-3)u+3)(v-1)+1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) \\ & -2(v-1)u(u-1)^2tr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \\ & +(v-1)u^2(u-1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2)). \end{aligned}$$

After simple calculations we get

$$\begin{aligned} var(\widetilde{\sigma}^{(1)}_{[2]ij}) - var(\widetilde{\sigma}^{(1)}_{[1]ij}) \;=\; & -\frac{2}{(n-1)n(u-1)u^2v^2} \\ & ((1-(u-2)u(v-1))\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0) \\ & - 2(1-(u-2)u(v-1))\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) \\ & + (1-(u-2)u(v-1))\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) \\ & + (v-1)u(u-1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0) \\ & - 2(v-1)u(u-1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \\ & + (v-1)u(u-1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \\ & + (v-1)u(u-1)^2 tr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) \\ & - 2(v-1)u(u-1)^2 tr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \\ & + (v-1)u(u-1)^2 tr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2)), \end{aligned}$$

thus $var(\widetilde{\sigma}^{(1)}_{[2]ij}) - var(\widetilde{\sigma}^{(1)}_{[1]ij}) < 0$ if

$$\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0) + \mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) > 2tr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1),$$

$$\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0) + \mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) > 2tr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2),$$

$$\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) + \mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) > 2tr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2)$$

which holds for any fixed $\boldsymbol{\Gamma}_0, \boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2$.

$$\begin{aligned} var(\widetilde{\sigma}^{(2)}_{[2]ij}) - var(\widetilde{\sigma}^{(2)}_{[1]ij}) \;=\; & -\frac{2}{(n-1)nu^2(v-1)v^2} \\ & (\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0) - 2(1-u)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) \\ & - 2u\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) + (u-1)^2\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) \\ & - 2u(u-1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) + u^2\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2)). \end{aligned}$$

After simple calculations we get

$$\begin{aligned} var(\widetilde{\sigma}^{(2)}_{[2]ij}) - var(\widetilde{\sigma}^{(2)}_{[1]ij}) \;=\; & -\frac{2}{(n-1)nu^2(v-1)v^2}((1-u)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0) \\ & - 2(1-u)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) \\ & + (1-u)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) + utr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0) \\ & - 2utr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) + utr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \\ & + u(u-1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) \\ & - 2u(u-1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) \\ & + u(u-1)\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2)), \end{aligned}$$

thus $var(\widetilde{\sigma}^{(2)}_{[2]ij}) - var(\widetilde{\sigma}^{(2)}_{[1]ij}) < 0$ if

$$\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0) + \mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) > 2tr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1),$$

$$\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0) + \mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) > 2tr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_0\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2),$$

$$\mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1) + \mathrm{tr}(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2) > 2tr(\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_1\boldsymbol{A}_{ij}\boldsymbol{\Gamma}_2)$$

which holds for any fixed $\boldsymbol{\Gamma}_0, \boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2$.

For graphical illustration of these differences, author fixed $\boldsymbol{\Gamma}_0 = \boldsymbol{I}$ and $\boldsymbol{\Gamma}_1 = \boldsymbol{\Gamma}_2 = \boldsymbol{0}$. For each figure, values for n are chosen from 3 to 25 and for u and v from 2 to 10. For the plot of n and u, v is treated as constant and $v = 2$. Similarly, For the plot of n and v, u is treated as constant and $u = 2$.

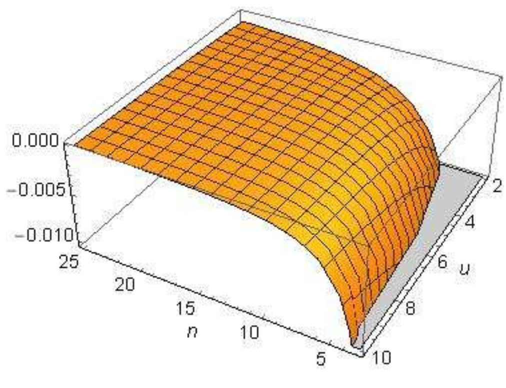

Figure 1. $var(\widetilde{\sigma}^{(0)}_{[2]ij}) - var(\widetilde{\sigma}^{(0)}_{[1]ij})$ for n and u, plotting separate figure for parameters n and v is redundant because difference is symmetric with respect to u and v.

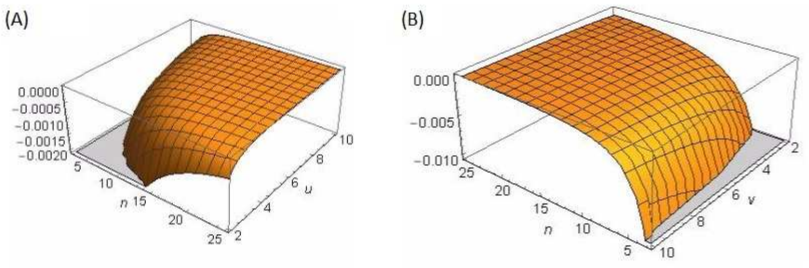

Figure 2. (A) $var(\widetilde{\sigma}^{(1)}_{[2]ij}) - var(\widetilde{\sigma}^{(1)}_{[1]ij})$ for n and u, and (B) $var(\widetilde{\sigma}^{(1)}_{[2]ij}) - var(\widetilde{\sigma}^{(1)}_{[1]ij})$ for n and v.

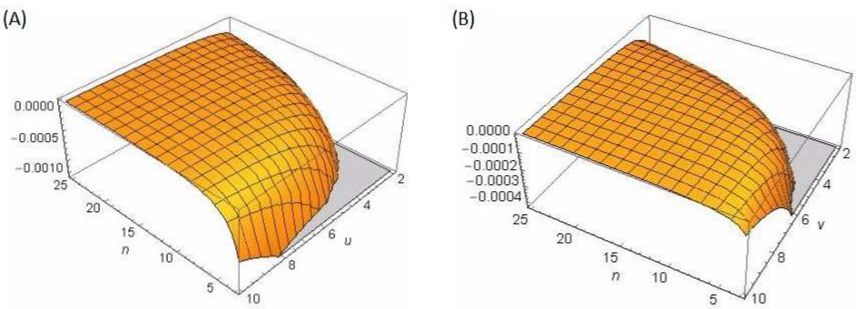

Figure 3. (A) $var(\widetilde{\sigma}^{(2)}_{[2]ij}) - var(\widetilde{\sigma}^{(2)}_{[1]ij})$ for n and u, and (B) $var(\widetilde{\sigma}^{(2)}_{[2]ij}) - var(\widetilde{\sigma}^{(2)}_{[1]ij})$ for n and v.

All three figures reveal the fact that differences between variances of estimators for $\sigma^{(0)}, \sigma^{(1)}$ and $\sigma^{(2)}$ in models Mo2 and Mo1 are negative and if $n \to \infty$ then tend to 0, thus variances of estimators for sigmas in Mo2 are smaller than corresponding variances of estimators for sigmas in Mo1.

5. Conclusions

Under multivariate normality, the free-coordinate approach was used to obtain linear and quadratic estimates for parameters that are sufficient, complete, unbiased and consistent in the model with doubly exchangeable covariance structure with structured mean vector. Comparison with the model with the same covariance structure but unstructured mean vector shows that estimators of covariance parameters in model with structured mean vector have smaller variances.

References

[1] M. Fonseca, J.T. Mexia and R. Zmyślony, *Least squares and generalized least squares in models with orthogonal block structure*, J. Statistical Planning and Inference **140** (5) (2010) 1346–1352. doi:10.1016/j.jspi.2009.12.007

[2] H. Drygas, The Coordinate-Free Approach to Gauss-Markov Estimation (Berlin, Heidelberg, Springer, 1970).

[3] S. Gnot, W. Klonecki and R. Zmyślony, *Uniformly minimum variance unbiased estimation in various classes of estimators*, Statistics **8** (2) (1977) 199–210. doi:10.1080/02331887708801366

[4] S. Gnot, W. Klonecki and R. Zmyślony, *Best unbiased estimation: a coordinate free-approach*, Probab. and Statis. **1** (1) (1980) 1–13.

[5] P. Jordan, J. von Neumann and E. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, Ann. Math. **35** (1) (1934) 29–64. doi:10.2307/1968117

[6] W. Kruskal, *When are Gauss-Markov and Least Squares Estimators Identical ? A Coordinate-Free Approach*, Ann. Math. Stat. **39** (1) (1968) 70–75. doi:10.1214/aoms/1177698505

[7] E.L. Lehmann and G. Casella, Theory of Point Estimation (Second Edition, Springer, 1998).

[8] A. Roy and R. Leiva, *Estimating and testing a structured covariance matrix for three-level multivariate data*, Comm. Statist. Theory Methods **40** (11) (2011) 1945–1963. doi:10.1080/03610921003672212

[9] A. Roy and M. Fonseca, *Linear models with doubly exchangeable distributed errors*, Comm. Statist. Theory Methods **41** (2012) 2545–2569. doi:10.1080/03610926.2011.609953

[10] A. Roy, R. Zmyślony, M. Fonseca and R. Leiva, *Optimal estimation for doubly multivariate data in blocked compound symmetric covariance structure*, J. Multivariate Analysis **144** (2016) 81–90. doi:10.1016/j.jmva.2015.11.001

[11] A. Roy, A. Kozioł, R. Zmyślony, M. Fonseca and R. Leiva, *Best unbiased estimates for parameters of three-level multivariate data with doubly exchangeable covariance structure*, submitted for publication.

[12] J.F. Seely, *Quadratic subspaces and completeness*, Ann. Math. Statist. **42** (2) (1971) 710–721. doi:10.1214/aoms/1177693420

[13] J.F. Seely, *Completeness for a family of multivariate normal distributions*, Ann. Math. Statist. **43** (1972) 1644–1647. doi:10.1214/aoms/1177692396

[14] J.F. Seely, *Minimal sufficient statistics and completeness for multivariate normal families*, Sankhya (Statistics), Indian J. Statist. Ser. A **39** (2) (1977) 170–185.

[15] R. Zmyślony, *On estimation of parameters in linear models*, Appl. Math. **XV** (1976) 271–276.

[16] R. Zmyślony, *A characterization of best linear unbiased estimators in the general linear model*, Lecture Notes in Statistics **2** (1978) 365–373. doi:10.1007/978-1-4615-7397-5_27

[17] R. Zmyślony, *Completeness for a family of normal distributions*, Math. Stat. Banach Center Publications **6** (1980) 355–357.

Received 15 September 2016