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ON SOME PROPERTIES OF QUOTIENTS OF HOMOGENEOUS $C(K)$ SPACES

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Abstract

We say that an infinite, zero dimensional, compact Hausdorff space K has property $(*)$ if for every nonempty open subset U of K there exists an open and closed subset V of U which is homeomorphic to K. We show that if K is a compact Hausdorff space with property $(*)$ and X is a Banach space which contains a subspace isomorphic to the space $C(K)$ of all scalar (real or complex) continuous functions on K and Y is a closed linear subspace of X which does not contain any subspace isomorphic to the space $C([0,1])$, then the quotient space X/Y contains a subspace isomorphic to the space $C(K)$.

Keywords: nonseparable $C(K)$ spaces, quotients of $C(K)$ spaces.

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1. Introduction

The Banach space of all scalar (real or complex) continuous functions on a compact Hausdorff space K will be denoted by $C(K)$. For a closed subset U of K by $C_0(K||U)$ will be denoted the subspace of $C(K)$ of all functions vanishing on U. All Banach spaces considered in the paper are over the same scalar field.

Lindenstrauss and Pełczyński in [5, Theorem 2.1] showed that if a Banach space X contains a subspace isomorphic to the space $C([0,1])$ and Y is a closed linear subspace of X which does not contain any subspace isomorphic to the space $C([0,1])$, then the quotient space X/Y contains an isomorphic copy of the space $C([0,1])$. The paper is devoted to show the following generalization of the

Lindenstrauss and Pełczyński result if K is a compact Hausdorff space and Y is a closed linear subspace of a Banach space X such that for every nonempty open subset U of K there exists a nonempty open subset V of U such that the space $C_0(K||K \setminus V)$ is isomorphic to the space $C(K)$ and X contains a subspace isomorphic to the space $C(K)$ and Y does not contain any subspace isomorphic to the space $C([0,1])$, then the quotient space X/Y contains a subspace isomorphic to the space $C(K)$. We will say that an infinite, zero dimensional, compact Hausdorff space K has property $(*)$ if for every nonempty open subset U of K there exists an open and closed subset V of U which is homeomorphic to K. It is clear that Hausdorff compact spaces with property $(*)$ satisfy the assumptions of the theorem above. There are quite many compact Hausdorff spaces with property $(*)$, for example: (1) Cantor cubes $\{0,1\}^\Gamma$ for every infinite set Γ, (2) products of the two arrows space $\mathbb{L}^\Gamma$ for every nonempty set Γ, (3) $\beta\mathbb{N} \setminus \mathbb{N}$. Our result for $K = \{0,1\}^{\mathbb{N}}$ provides the Lindenstrauss and Pełczyński theorem.

The proof of our theorem is a modification of the idea of Lindenstrauss and Pełczyński. There are some differences between the situation we consider and the situation considered in [5]. The space K may not have enough many open and closed subsets and we can not apply in an easy way the Haar system of functions in the proof. Moreover in [5] only real scalars are considered. For complex scalars Lemma 2.1 in [5] is not valid (it may be modified but it needs an explaination). Our approach is independent from Lindenstrauss and Pełczyński results and it is self-contained. Instead of the Haar system we work with binary trees. First we present a general method of constructions of subspaces isomorphic to the space $C([0,1])$ in $C(K)$ spaces. This part of our considerations contains technicalities we need in the proof of our main result. It is of course possible to apply in the proof of Theorem 4 sequences that are close to the Haar system of functions, but then the technicalities that should be presented would be similar to these in the proof of Theorem 3.

The paper is divided into two sections. The second section contains main results and its consequences.

2. Main results

We will construct isomorphic embeddings of the space $C([0,1])$ applying the following fact (see [6, Fact 3]).

Proposition 1. *If (e_n) is a sequence in a Banach space X such that*

(1) $e_n = e_{2n} + e_{2n+1}$ *for every n,*

(2) *there exist constants $0 < c \leqslant C$ such that*

$$c \max_{2^n \leqslant k < 2^{n+1}} |a_k| \leqslant \Big\| \sum_{k=2^n}^{2^{n+1}-1} a_k e_k \Big\| \leqslant C \max_{2^n \leqslant k < 2^{n+1}} |a_k|$$

for every n and for every scalars $a_{2^n}, \ldots, a_{2^{n+1}-1}$, then the closed linear hull of the set $\{e_n : n \in \mathbb{N}\}$ in X is isomorphic to $C([0,1])$. Moreover, if $c = C$, the subspace is isometric to $C(\{0,1\}^{\mathbb{N}})$.

Our constructions of sequences with properties (1) and (2) above will apply the following well known fact.

Proposition 2. *If (f_n) is a sequence in a Banach space X such that*

$$f_{2n+1} = f_n - f_{2n}$$

for every n, then

(a) *for every n, there exists a unique pair (l, p) such that $l \in 2\mathbb{N} \cup \{1\}$, $p \in \mathbb{N}$, $2n + 2 = 2^p(l + 1)$ and*

$$f_{2n+1} = f_l - \sum_{j=1}^{p} f_{2^j l + 2^j - 2},$$

(b) *if we gather together for each n all representations above of all members of the set $\{f_{2^n}, \ldots, f_{2^{n+1}-1}\}$, then every element of the set $\{f_j : j = 2, 4, \ldots, 2^{n+1} - 2\}$ appears exactly twice and the function f_1 appears only once.*

The following result shows what is needed to construct an isomorphic copy of the space $C([0,1])$ in a $C(K)$ space.

Theorem 3. *Let K be a compact Hausdorff space. If there exist $0 < \delta < 1$ and a sequence $\{f_n : n \in 2\mathbb{N} \cup \{1\}\} \subset C(K)$ and sequences (V_n) and (U_n) of open nonempty subsets of K with the following properties:*

(1) $U_{2n} \cup U_{2n+1} \subset V_n \subset \overline{V_n} \subset U_n$ *and* $U_{2n} \cap U_{2n+1} = \emptyset$ *for every* $n \in \mathbb{N}$,

(2) $|f_n(t)| \leqslant \frac{\delta}{4^n}$ *for every* $t \in K \setminus U_n$ *and* $n \in 2\mathbb{N} \cup \{1\}$,

(3) $|1 - f_n(t)| \leqslant \frac{\delta}{4^n}$ *for every* $t \in V_n$ *and* $n \in 2\mathbb{N} \cup \{1\}$,

(4) $\|f_n\| = 1$ *for every* $n \in 2\mathbb{N} \cup \{1\}$,

then the closed linear hull of the set $\{f_n : n \in 2\mathbb{N} \cup \{1\}\}$ in $C(K)$ is isomorphic to the space $C([0,1])$. Moreover, if for every n we put

$$f_{2n+1} = f_l - \sum_{j=1}^{p} f_{2^j l + 2^j - 2}\,,$$

where $2n+2 = 2^p(l+1)$, $l \in 2\mathbb{N} \cup \{1\}$, $p \in \mathbb{N}$, *then the following inequalities*

$$\Big(1-\frac{\delta}{2}\Big) \max_{2^n \leqslant k < 2^{n+1}} |a_k| \leqslant \Big\| \sum_{k=2^n}^{2^{n+1}-1} a_k f_k \Big\| \leqslant \Big(3+\frac{2\delta}{5}\Big) \max_{2^n \leqslant k < 2^{n+1}} |a_k|$$

hold for every n *and for every scalars* $a_{2^n}, \ldots, a_{2^{n+1}-1}$.

Proof. According to Proposition 2 we have equality $f_n = f_{2n} + f_{2n+1}$ for every n. Let $h_n = \sum_{k=2^n}^{2^{n+1}-1} |f_k|$. It is easy to check that

$$h_1(t) = |f_2(t)| + |f_1(t) - f_2(t)| \leqslant \begin{cases} 1 + \frac{\delta}{4} + \frac{\delta}{4^2} & \text{if } t \in V_2 \cup V_3 \\ 3 & \text{if } t \in K. \end{cases}$$

We will show the following inequality

$$h_n(t) \leqslant \begin{cases} 1 + \frac{\delta}{4} + \sum_{k=1}^{2^n-1} \frac{2\delta}{4^{2k}} & \text{if } t \in \bigcup_{k=2^n}^{2^{n+1}-1} V_k \\ 3 + \frac{\delta}{4} + \sum_{k=1}^{2^n-1} \frac{2\delta}{4^{2k}} & \text{if } t \in K. \end{cases}$$

It is clear that if $r, s \in \mathbb{N}$ and $r < s$, then $U_s \subset U_r$ if and only if $s \in [2^m r, 2^m r + 2^m - 1]$ where m is the integer part of $\log_2 \frac{s}{r}$. For every $k \in \mathbb{N}$, let $l_k \in 2\mathbb{N} \cup \{1\}$ and $p_k \in \mathbb{N}$ be such that $(l_k+1)2^{p_k} = 2k+2$.

We start with the first inequality. Let $2k+1 \in [2^n, 2^{n+1}-1]$. It is easy to see that

$$\begin{aligned} &[2^{p_k} l_k, 2^{p_k} l_k + 2^{p_k} - 1] \cap \mathbb{N} \\ &= \{2k+1\} \cup \bigcup_{j=1}^{p_k} [2^{p_k-j}(2^j l_k + 2^j - 2), 2^{p_k-j}(2^j l_k + 2^j - 2) + 2^{p_k-j} - 1] \cap \mathbb{N}. \end{aligned}$$

Hence $U_{2k+1} \subset U_{l_k}$ and $U_{2k+1} \cap U_{2^j l_k + 2^j - 2} = \emptyset$ for each $j = 1, \ldots, p_k$ and

$$|f_{2k+1}(t)| \leqslant 1 + \sum_{j=1}^{p_k} \frac{\delta}{4^{2^j l_k + 2^j - 2}}$$

for every $t \in V_{2k+1}$. If $j \in [2^{p_k-r}(2^r l_k + 2^r - 2), 2^{p_k-r}(2^r l_k + 2^r - 2) + 2^{p_k-r} - 1]$ for some $r \in \{1, \ldots, p_k\}$, then $V_j \subset V_{2^r l_k + 2^r - 2} \subset V_{l_k}$ and $U_{2^r l_k + 2^r - 2} \cap U_{2^m l_k + 2^m - 2} = \emptyset$ for every $m \in \{1, 2, \ldots, p_k\}$, $m \neq r$ and

$$\begin{aligned} |f_{2k+1}(t)| &\leqslant |(f_{l_k} - f_{2^r l_k + 2^r - 2})(t)| + \sum_{m=1, m\neq r}^{p_k} \frac{\delta}{4^{2^m l_k + 2^m - 2}} \\ &\leqslant \frac{\delta}{4^{l_k}} + \sum_{m=1}^{p_k} \frac{\delta}{4^{2^m l_k + 2^m - 2}} \end{aligned}$$

for every $t \in V_j$. If $j \in [2^n, 2^{n+1}-1] \setminus [2^{p_k} l_k, 2^{p_k} l_k + 2^{p_k} - 1]$, then we have

$$|f_{2k+1}(t)| \leqslant \frac{\delta}{4^{l_k}} + \sum_{m=1}^{p_k} \frac{\delta}{4^{2^m l_k + 2^m - 2}}$$

for every $t \in V_j$. Moreover, for every k we have

$$|f_{2k}(t)| \leqslant \begin{cases} 1 & \text{if } t \in V_{2k} \\ \frac{\delta}{4^{2k}} & \text{if } t \in \bigcup_{j=2^{n-1}, j \neq 2k}^{2^n - 1} V_j. \end{cases}$$

According to Proposition 2 (b) we obtain the following inequality

$$h_n(t) \leqslant 1 + \frac{\delta}{4} + \sum_{k=1}^{2^n - 1} \frac{2\delta}{4^{2k}} < 1 + \frac{2\delta}{5}$$

for every $t \in \bigcup_{j=2^n}^{2^{n+1}-1} V_j$ by summing all inequalities above for all $2k, 2k+1$ in $[2^n, 2^{n+1}-1]$. The consideration above shows also that

$$h_n(t) - |f_k(t)| \leqslant \frac{\delta}{4} + \sum_{k=1}^{2^n - 1} \frac{2\delta}{4^{2k}} < \frac{2\delta}{5}$$

for every $k \in [2^n, 2^{n+1}-1]$ and $t \in V_k$.

To show the second inequality we apply the mathematical induction. Applying the property (2) for every $t \notin \bigcup_{k=2^{n+1}}^{2^{n+2}-1} U_k$ we obtain

$$\begin{aligned} h_{n+1}(t) &= \sum_{k=2^n}^{2^{n+1}-1} \big(|f_{2k}(t)| + |f_k(t) - f_{2k}(t)|\big) \leqslant h_n(t) + 2 \sum_{k=2^n}^{2^{n+1}-1} |f_{2k}(t)| \\ &\leqslant 3 + \frac{\delta}{4} + \sum_{k=1}^{2^n - 1} \frac{2\delta}{4^{2k}} + 2 \sum_{k=2^n}^{2^{n+1}-1} \frac{\delta}{4^{2k}} \leqslant 3 + \frac{\delta}{4} + \sum_{k=1}^{2^{n+1}-1} \frac{2\delta}{4^{2k}} \leqslant 3 + \frac{2\delta}{5}. \end{aligned}$$

For every $t \in U_{2j} \cup U_{2j+1} \subset V_j$ for some $j \in [2^n, 2^{n+1}-1]$ we have

$$\begin{aligned} h_{n+1}(t) &\leqslant h_n(t) + 2 \sum_{k=2^n}^{2^{n+1}-1} |f_{2k}(t)| \leqslant 1 + \frac{\delta}{4} + \sum_{k=1}^{2^n - 1} \frac{2\delta}{4^{2k}} + 2 + 2 \sum_{k=2^n, k \neq j}^{2^{n+1}-1} \frac{\delta}{4^{2k}} \\ &\leqslant 3 + \frac{\delta}{4} + \sum_{k=1}^{2^{n+1}-1} \frac{2\delta}{4^{2k}} \leqslant 3 + \frac{2\delta}{5}. \end{aligned}$$

Let $a_{2^n}, \ldots, a_{2^{n+1}-1}$ be arbitrary scalars. Let $|a_j| = \max_{2^n \leqslant k < 2^{n+1}} |a_k|$. Then

$$\Big\| \sum_{k=2^n}^{2^{n+1}-1} a_k f_k \Big\| \leqslant \Big\| \sum_{k=2^n}^{2^{n+1}-1} |a_k||f_k| \Big\| \leqslant \Big\| |a_j| h_n \Big\| \leqslant \Big(3 + \frac{2\delta}{5}\Big) \max_{2^n \leqslant k < 2^{n+1}} |a_k|$$

and for every $t \in V_j$ we have

$$\left\| \sum_{k=2^n}^{2^{n+1}-1} a_k f_k \right\| \geqslant |a_j||f_j(t)| - \sum_{k=2^n, k\neq j}^{2^{n+1}-1} |a_k| f_k(t)|$$
$$\geqslant |a_j|\Big(1 - \frac{\delta}{4^j}\Big) - |a_j|(|h_n(t)| - |f_j(t)|)$$
$$\geqslant \Big(1 - \frac{\delta}{2}\Big) \max_{2^n \leqslant k < 2^{n+1}} |a_k|.$$

Thus we have shown that the sequence (f_n) has properties (1) and (2) of Proposition 1. Therefore the closed linear hull of the set $\{f_n : n \in 2\mathbb{N} \cup \{1\}\}$ in $C(K)$ is isomorphic to the space $C([0,1])$. ∎

A point x in a topological space K is called isolated if the set $\{x\}$ is open in K.

Theorem 4. *Let K be a compact Hausdorff space and let Z be an infinite dimensional Banach space with the following property: for every nonempty open subset U of K there exists a nonempty open subset V of U such that the space $C_0(K||K \setminus V)$ contains a subspace isomorphic to the space Z. If X is a Banach space containing a subspace isomorphic to the space $C(K)$ and Y is a closed linear subspace of X which does not contain any subspace isomorphic to the space $C([0,1])$, then the quotient space X/Y contains a subspace isomorphic to the space Z.*

Proof. Since Z is infinite dimensional, K has infinite many points and it does not have isolated points. Moreover every nonempty open subset of K contains infinite many points. Let $I : C(K) \to X$ be an isomorphic embedding. Then there exist constants $0 < c \leqslant C$ such that $c\|f\| \leqslant \|I(f)\| \leqslant C\|f\|$ for every $f \in C(K)$.

We will show that *for every nonempty open subset U of K there exist a nonempty open subset V of U and $\varepsilon > 0$ such that*

$$\|I(f) + Y\| \geqslant \varepsilon\|f\|$$

for every $f \in C_0(K||K \setminus V)$.

Suppose that this does not hold. Then there exists a nonempty open subset U_0 of K such that for every nonempty open subset U of U_0 and for every $\varepsilon > 0$ there exists $f \in C_0(K||K \setminus U)$ such that

$$\|I(f) + Y\| < \varepsilon\|f\|.$$

Fix $0 < \delta < 1$.

First Step. We find a nonempty open subset U_1 of U_0 and $g_1 \in C_0(K||K \setminus U_1)$ such that $\overline{U_1} \subset U_0$ and

$$\|g_1\| = 1 \quad \text{and} \quad \|I(g_1) + Y\| < \frac{c\delta}{4}.$$

Let $t_1 \in U_1$ be such that $|g_1(t_1)| = 1$. Let $f_1 = \overline{g_1(t_1)} g_1$. Let $V_1 = \{t \in K : |1 - f_1(t)| < \frac{\delta}{4}\}$.

Second Step. Since V_1 contains at least two points, we find nonempty open subsets U_2, U_3 of V_1 such that $U_2 \cap U_3 = \emptyset$. According to our assumption we find $g_2 \in C_0(K||K \setminus U_2)$ such that

$$\|g_2\| = 1, \quad \text{and} \quad \|I(g_2) + Y\| < \frac{c\delta}{4^2}.$$

Let $t_2 \in U_2$ be such that $|g_2(t_2)| = 1$. Let $f_2 = \overline{g_2(t_2)} g_2$. Let $V_2 = \{t \in K : |1 - f_2(t)| < \frac{\delta}{4^2}\}$, and V_3 be any nonempty open subset of U_3 such that $\overline{V_3} \subset U_3$.

Next Steps. Continuing the procedure we are able to find sequences (V_n) and (U_n) of nonempty open subsets of U_0 and a sequence $\{f_n : n \in 2\mathbb{N} \cup \{1\}\}$ in $C(K)$ with the following properties:

(1) $U_{2n} \cup U_{2n+1} \subset V_n \subset \overline{V_n} \subset U_n \subset U_0$ and $U_{2n} \cap U_{2n+1} = \emptyset$ for every $n \in \mathbb{N}$,
(2) $|f_n(t)| = 0$ for every $t \in K \setminus U_n$ and $n \in 2\mathbb{N} \cup \{1\}$,
(3) $|1 - f_n(t)| \leqslant \frac{\delta}{4^n}$ for every $t \in V_n$ and $n \in 2\mathbb{N} \cup \{1\}$,
(4) $\|f_n\| = 1$ for every $n \in 2\mathbb{N} \cup \{1\}$,
(5) $\|I(f_n) + Y\| < \frac{c\delta}{4^n}$ for every $n \in 2\mathbb{N} \cup \{1\}$.

For every $n \in 2\mathbb{N} \cup \{1\}$, let $h_n \in Y$ be such that $\|I(f_n) - h_n\| \leqslant \frac{c\delta}{4^n}$. For every n we define

$$f_{2n+1} = f_l - \sum_{j=1}^{p} f_{2^j l + 2^j - 2} \quad \text{and} \quad h_{2n+1} = h_l - \sum_{j=1}^{p} h_{2^j l + 2^j - 2}\,,$$

where $2n + 2 = 2^p(l + 1)$, $l \in 2\mathbb{N} \cup \{1\}$, $p \in \mathbb{N}$. Then $f_n = f_{2n} + f_{2n+1}$ and $h_n = h_{2n} + h_{2n+1}$ for every $n \in \mathbb{N}$. In view of Proposition 2 we have

$$\Big\| \sum_{k=2^n}^{2^{n+1}-1} I(f_k) - h_k \Big\| \leqslant \frac{c\delta}{4} + \sum_{k=1}^{2^n-1} \frac{2c\delta}{4^{2k}} < \frac{2c\delta}{5}.$$

According to Theorem 3 for every n and every scalars $a_{2^n}, \dots, a_{2^{n+1}-1}$ we have

$$\begin{aligned}\Big\|\sum_{k=2^n}^{2^{n+1}-1} a_k h_k\Big\| &\leqslant \Big\|\sum_{k=2^n}^{2^{n+1}-1} a_k I(f_k)\Big\| + \Big\|\sum_{k=2^n}^{2^{n+1}-1} a_k (I(f_k) - h_k)\Big\| \\ &\leqslant \Big(C\Big(3+\frac{2\delta}{5}\Big) + \sum_{k=2^n}^{2^{n+1}-1} \|I(f_k)-h_k\|\Big) \max_{2^n \leqslant k < 2^{n+1}} |a_k| \\ &\leqslant C\Big(3+\frac{4\delta}{5}\Big) \max_{2^n \leqslant k < 2^{n+1}} |a_k|\end{aligned}$$

and

$$\begin{aligned}\Big\|\sum_{k=2^n}^{2^{n+1}-1} a_k h_k\Big\| &\geqslant \Big\|\sum_{k=2^n}^{2^{n+1}-1} a_k I(f_k)\Big\| - \Big\|\sum_{k=2^n}^{2^{n+1}-1} a_k (I(f_k) - h_k)\Big\| \\ &\geqslant \Big(c\Big(1-\frac{\delta}{2}\Big) - \sum_{k=2^n}^{2^{n+1}-1} \|I(f_k)-h_k\|\Big) \max_{2^n \leqslant k < 2^{n+1}} |a_k| \\ &\geqslant c\Big(1-\frac{9\delta}{10}\Big) \max_{2^n \leqslant k < 2^{n+1}} |a_k|.\end{aligned}$$

According to Proposition 1 the space Y contains a subspace isomorphic to the space $C([0,1])$. This contradicts our assumption. Therefore, for every nonempty open subset U of K there exists a nonempty open subset V of U and $\varepsilon > 0$ such that

$$\|I(f) + Y\| \geqslant \varepsilon \|f\|$$

for every $f \in C_0(K||K \setminus V)$. This shows that the space $C_0(K||K\setminus V)$ is isomorphic to a subspace of X/Y. ∎

An inspection of arguments used in the proof above shows that without the homogenity assumption there holds the following version of Theorem 4.

Theorem 5. *Let K be a compact Hausdorff space without isolated points. If X is a Banach space containing a subspace isomorphic to the space $C(K)$ and Y is a closed linear subspace of X which does not contain any subspace isomorphic to the space $C([0,1])$, then the quotient space X/Y contains a subspace isomorphic to a space $C_0(K||K \setminus U)$ for some nonempty open subset U of K.*

It is easy to find a compact Hausdorff space K such that for every open subset U of K there exists an open nonempty subset V of U for which spaces $C_0(K||K \setminus V)$ and $C(K)$ are not isomorphic. This property has for example the lexicographic square (see [4]). If U is an open and closed subset of K, then we have equality $C_0(K||K \setminus U) = C(U)$. A Hausdorff compact space K is called zero dimensional if it has a base of topology consisting of open and closed sets. We will say that an infinite, zero dimensional, compact Hausdorff space K has property $(*)$ if for every nonempty open subset U of K there exists a nonempty open and closed subset V of U which is homeomorphic to K. As a straightforward consequence of Theorem 4 we obtain the following corollary.

Corollary 6. *Let K be a compact Hausdorff space with property $(*)$. If a Banach space X contains a subspace isomorphic to the space $C(K)$ and Y is a closed linear subspace of X which does not contain any subspace isomorphic to the space $C([0,1])$, then the quotient space X/Y contains a subspace isomorphic to the space $C(K)$.*

The two-arrows space is the following topological space:

$$\mathbb{L} = \{(t,0) : 0 < t \leqslant 1\} \cup \{(t,1) : 0 \leqslant t < 1\}$$

equipped with the base of topology: at a point $(t,0)$ of the form

$$\big\{\{(s,r) : t - \tfrac{1}{n} < s < t, 0 < s, r \in \{0,1\}\} \cup \{(t,0)\} : n \in \mathbb{N}\big\}$$

and at a point $(t,1)$ of the form

$$\big\{\{(s,r) : t < s < t + \tfrac{1}{n}, s < 1, r \in \{0,1\}\} \cup \{(t,1)\} : n \in \mathbb{N}\big\}.$$

The space $\mathbb{L}$ is Hausdorff, compact, sequentially compact, hereditarily separable and hereditarily Lindelöf (see [2, p. 270]).

Proposition 7. *The following compact Hausdorff spaces have property $(*)$:*

(1) *Cantor cubes $\{0,1\}^\Gamma$ for every infinite set Γ,*

(2) *products of the two-arrows space $\mathbb{L}^\Gamma$ for every set Γ,*

(3) $\beta\mathbb{N} \setminus \mathbb{N}$.

Moreover, we have the following simple fact.

Proposition 8. *Let Γ be a nonempty set. If $\{K_\gamma : \gamma \in \Gamma\}$ is a family of compact Hausdorff spaces with property $(*)$, then the product space $\mathcal{P}_{\gamma\in\Gamma} K_\gamma$ has also property $(*)$.*

Proof of Proposition 7. The property $(*)$ for Cantor cubes is obvious. It is easy to see that for every $0 \leqslant s < t \leqslant 1$ the set

$$\{(u,0) : s < u \leqslant t\} \cup \{(u,1) : s \leqslant u < t\}$$

is open and closed in $\mathbb{L}$. Moreover the set is homeomorphic to $\mathbb{L}$.

The fact that $\beta\mathbb{N} \setminus \mathbb{N}$ has property $(*)$ is well known (see [3, p. 98–99]). ∎

The space $C(\mathbb{L})$ is isometrically isomorphic to the Banach space $D(0,1)$ of all scalar left continuous functions on the interval $[0,1]$ with a right-hand limit at each point of $[0,1)$ equipped with the sup norm. Spaces $C(\mathbb{L}^{\Gamma})$ have many interesting properties (see [1, 6, 7]). A Banach space X has the separable complementation property if for every separable subspace Y of X there exists a separable and complemented subspace Z of X which contains Y. As a straightforward consequence of Corollary 6 and Proposition 7 and [6, Corollary 8] we obtain the following result.

Corollary 9. *If a Banach space X contains an isomorphic copy of the space $C(\mathbb{L})$ and Y is a closed linear subspace of X which does not contain any subspace isomorphic to the space $C([0,1])$, then the quotient space X/Y does not have the separable complementation property.*

It is well known that for every infinite set Γ the space $C([0,1]^{\Gamma})$ contains a subspace isometric to $C(\{0,1\}^{\Gamma})$ and the space $C(\{0,1\}^{\Gamma})$ contains a subspace isometric to $C([0,1]^{\Gamma})$. As a straightforward consequence of Theorem 4 and facts above we obtain also

Corollary 10. *Let Γ be a nonempty set. If X is a Banach space containing an isomorphic copy of the space $C([0,1]^{\Gamma})$ and Y is a closed linear subspace of X that does not contain any subspace isomorphic to the space $C([0,1])$, then the quotient space X/Y contains a subspace isomorphic to the space $C([0,1]^{\Gamma})$.*

The characteristic function of a set W is denoted by χ_W. The next corollary follows from the proof of Theorem 4.

Corollary 11. *Let K be a compact Hausdorff space with property $(*)$. If Y is a closed linear subspace of $C(K)$ which does not contain any subspace isomorphic to the space $C([0,1])$, then for every nonempty open subset U of K there exists a nonempty open and closed subset V of U such that the operator $J : Y \to C(K)$ given by the formula $J(f) = f\chi_{K\setminus V}$ is an isomorphic embedding.*

Proof. Let U be a nonempty open subset of K. We showed in the proof of Theorem 4 that there exists a nonempty open and closed subset V of U and $0 < \varepsilon < 1$ such that $\|f\chi_V + Y\| \geqslant \varepsilon\|f\chi_V\|$ for every $f \in C(K)$. Consequently for every $f \in Y$ we have

$$\|f\chi_{K\setminus V}\| = \|f\chi_V - f\| \geqslant \varepsilon\|f\chi_V\|.$$

and

$$\varepsilon\|f\| = \max\{\varepsilon\|f\chi_{K\setminus V}\|, \varepsilon\|f\chi_V\|\} \leqslant \|f\chi_{K\setminus V}\|.$$

■

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