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ON THE MUTUALLY NON ISOMORPHIC $\ell_p(\ell_q)$ SPACES, A SURVEY

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Abstract

In this note we survey the partial results needed to show the following general theorem: $\{\ell_p(\ell_q) : 1 \le p, q \le +\infty\}$ *is a family of mutually non isomorphic Banach spaces.* We also comment some related facts and open problems.

Keywords: Banach spaces, isomorphic spaces, complemented subspace.

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1. Introduction

The aim of this note is to provide an account of our work [5, 6, 7] on the family of Banach spaces $\{\ell_p(\ell_q) : 1 \le p, q \le +\infty\}$. Summing up, we give a careful look at the following result: $\{\ell_p(\ell_q) : 1 \le p, q \le +\infty\}$ *is a family of mutually non isomorphic Banach spaces.* This result may be mainly attributed to Pełczyński but, as we will explain below, we believe that some parts of it are beyond his initial point of view. In fact, it may be misleading to call it *a result*, it would be much more enlightening to speak about *a collection of results.* So, looking at all the pieces of this collection we will see that a lot of people is behind the proofs.

In this survey we wish to explain the several cases we need to consider and the several steps we take. We also try to give an idea of the tools we use to complete the whole work.

Our approach will yield also a knowledge of the subspaces and complemented subspaces of the members of the family.

Besides, we include some comments on the motivation of our work, we present some further extension of the main result, and we propose some open problems. Finally, we take advantage of this opportunity to fill a gap in one of our proofs in [7]. This is done in the last section.

We use standard notation in Banach space theory as in [2, 8] or [11]. In particular, for two Banach spaces X and Y, the notations $X \approx Y, X \supset Y, X \supset_{(c)} Y$ will mean: X is isomorphic to Y, X has a subspace isomorphic to Y, X has a complemented subspace isomorphic to Y.

2. The starting point

On January 2007 Joaquín Motos posed us the following:

Question (J. Motos). *Are the Banach spaces $\ell_\infty(\ell_1)$ and $\ell_1(\ell_\infty)$ isomorphic?*

We believe that he posed us this question because it was in the same spirit of a work we had done some years earlier [4]. The question had arisen in the study of certain function spaces [12, 13] and it had its roots in [15, footnote of page 242], where Triebel said he had learnt from Pełczyński the following result: *Let $1 \leq p_0, p_1 \leq \infty$ and $1 < q_0, q_1 < \infty$. Then the spaces $\ell_{p_0}(\ell_{q_0})$ and $\ell_{p_1}(\ell_{q_1})$ are isomorphic if and only if $p_0 = p_1$ and $q_0 = q_1$.* In fact, Moto's question refers to Pełczyński's statement but for one of the cases in which the statement does not apply.

We answered the question in the negative in [5], and some time later, on December 2009, E.M. Galego posed us another question in a similar spirit:

Question (E.M. Galego). *Is it true that $\ell_1(c_0) \not\supset c_0(\ell_1)$?*

We got a negative answer (see below) and at that moment we thought that Pełczyński's statement deserved a careful study. This is what we have done in [7]. Firstly, we wished to know if the statement is still true when we add the extreme values of the q's. As we have already said, it is true. In other words, we have

Theorem 1 [7]. *Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. Then the spaces $\ell_{p_0}(\ell_{q_0})$ and $\ell_{p_1}(\ell_{q_1})$ are isomorphic if and only if $p_0 = p_1$ and $q_0 = q_1$.*

On the other hand, we wished to have a complete and detailed proof of the whole theorem, not only of the cases non included in Pełczyński's original statement. The reason of this is clear: in the footnote of [15] only a very short outline of the proof was given. Of course, at the beginning we hoped the outline could be completed without any problem, but we were not able to do this. We finally could prove the original Pełczyński's statement but we had to go much further than just the ideas given in [15]. To be precise, we have needed further ideas to prove the following two assertions:

(a) If $\ell_p(\ell_q)$ and $\ell_q(\ell_p)$ are isomorphic then $p = q$ (see Proposition 2.5 of [7]), and

(b) If $\ell_\infty(\ell_{q_0})$ and $\ell_\infty(\ell_{q_1})$ are isomorphic then $q_0 = q_1$ (see Section 4, especially Theorem 4.4, of [7]).

In particular, to prove this last assertion we have needed a deep theorem of Bourgain, Casazza, Lindenstrauss and Tzafriri [3] which was unknown when [15] was written.

3. Subspaces of $\ell_p(\ell_q)$ for $1 \le p, q < \infty$

At the first stage we study the separable members of our family, that is, those whose indices p and q are finite. We are interested in knowing when a separable $\ell_p(\ell_q)$ space contains a copy of other member of the family. We wish to answer precisely the following question:

Question 1. *For which p_0, p_1, q_0, q_1, with $1 \le p_0, p_1, q_0, q_1 < \infty$, does one have*

$$\ell_{p_0}(\ell_{q_0}) \supset \ell_{p_1}(\ell_{q_1})?$$

It is obvious that the preceding containment holds in the following three situations: (a) $p_1 = q_1 = p_0$, (b) $p_1 = q_1 = q_0$, and (c) $p_1 = p_0$ and $q_1 = q_0$. The point is that it does hold *only* in these trivial situations. Thus the answer to Question 1 is

Theorem 2. *Let $1 \le p_0, p_1, q_0, q_1 < \infty$, then $\ell_{p_0}(\ell_{q_0}) \supset \ell_{p_1}(\ell_{q_1})$ if and only if one of the following conditions holds:*

(a) *$p_1 = q_1 = p_0$,*

(b) *$p_1 = q_1 = q_0$, or*

(c) *$p_1 = p_0$ and $q_1 = q_0$.*

The proof of this Theorem relies in two previous steps: The study of when ℓ_r embeds into $\ell_p(\ell_q)$ and the study of when $\ell_p(\ell_q) \supset \ell_q(\ell_p)$. We get the following answers:

(1) *Let $1 \le p, q, r < \infty$, then $\ell_p(\ell_q) \supset \ell_r$ if and only if $r = p$ or $r = q$.*

(2) *Let $1 \le p, q < \infty$, then $\ell_p(\ell_q) \supset \ell_q(\ell_p)$ if and only if $p = q$.*

It is in the proof of (1) where we use the hint given by Pełczyński in [15].

Note that point (2) says that for $p \ne q$ one has $\ell_p(\ell_q) \not\supset \ell_q(\ell_p)$. One can follow an analogous way to the proof of point (2) to show that

(3) *If* $1 \leq p < \infty$*, then* $\ell_p(c_0) \not\supset c_0(\ell_p)$ *and* $c_0(\ell_p) \not\supset \ell_p(c_0)$

Thus, in particular, we have an affirmative answer to E.M. Galego's question.

On the other hand, observe that Theorem 2 implies Theorem 1 for finite parameters.

The next step is the study of the non separable members of our family, that is, those which have some of their indices p or q equal to infinity.

Since ℓ_∞ contains a copy of each separable Banach space and also a copy of each dual of a separable Banach space, in this case the study of subspaces of $\ell_p(\ell_q)$ does not provide an answer to our question. In fact, we have

- *If either* $p = \infty$ *or* $q = \infty$*, then* $\ell_p(\ell_q) \supset \ell_\infty \supset \ell_{p_1}(\ell_{q_1})$ *for all indices* $p_1, q_1 \in [1, +\infty]$

For this reason we are going to look at *complemented* subspaces of $\ell_p(\ell_q)$.

First, let us recall that ℓ_∞ does not contain a complemented copy of any separable Banach space. Hence if both indices p and q are equal to infinity, then the space $\ell_p(\ell_q) = \ell_\infty(\ell_\infty) \approx \ell_\infty$ contains a complemented copy of $\ell_{p_1}(\ell_{q_1})$ only if $p_1 = q_1 = \infty$. For this reason only remains to know which are the complemented subspaces of $\ell_p(\ell_\infty)$ and $\ell_\infty(\ell_q)$ for $1 \leq p, q < \infty$. This is what we will do in the following two sections.

4. Complemented subspaces of $\ell_p(\ell_\infty)$ for $1 \leq p < \infty$

Our question now is

Question 2. *Let* $1 \leq p < \infty$*. For which* p_1*,* q_1 *with* $1 \leq p_1, q_1 \leq \infty$ *does one have*

$$\ell_p(\ell_\infty) \supset_{(c)} \ell_{p_1}(\ell_{q_1})?$$

The answer will be again "only in trivial situations":

Theorem 3. *Let* $1 \leq p, p_1, q_1 \leq \infty$*. Then* $\ell_p(\ell_\infty) \supset_{(c)} \ell_{p_1}(\ell_{q_1})$ *if and only if one of the following three conditions holds:*

(a) $p_1 = q_1 = p$,

(b) $p_1 = q_1 = \infty$, *or*

(c) $p_1 = p$ *and* $q_1 = \infty$.

As in the previous section, Theorem 3 is a consequence of the study of particular cases:

(1) *If* $1 \leq p, r < \infty$*, then* $\ell_p(\ell_\infty) \supset_{(c)} \ell_r$ *if and only if* $r = p$.

(2) *If* $1 \leq p < \infty$*, then* $\ell_p(\ell_\infty) \not\supset_{(c)} \ell_\infty(\ell_p)$.

We have statements very similar to those of Section 2, however the proofs now are much more involved.

For some values of the indices p and q the results in (1) can be found in previous works: [4, Theorem 4.1.2] and [3].

The proof of point (2) for $p = 1$ is in [5].

To show point (2) for $1 < p < +\infty$, it is necessary to know something about complemented subspaces of $\ell_\infty(\ell_q)$. We will do this in the next section.

5. Complemented subspaces of $\ell_\infty(\ell_q)$ for $1 \leq q < \infty$

Following our approach, the question in this situation is

Question 3. *Let $1 \leq q < \infty$. For which p_1, q_1, with $1 \leq p_1, q_1 \leq \infty$, does one have*

$$\ell_\infty(\ell_q) \supset_{(c)} \ell_{p_1}(\ell_{q_1})?$$

In contrast with Questions 1 and 2, this time we do not know a complete answer. Also, in contrast with previous sections, the answer "only in trivial situations" does not work. This is clear in the next result, which is an immediate consequence of old results of Johnson [9] and Pełczyński [14]:

(1) *For all q, $1 < q < \infty$, one has $\ell_\infty(\ell_q) \supset_{(c)} \ell_2$.*

Thus, taking $p_1 = q_1 = 2$, and any $q \neq 2$, we have $\ell_\infty(\ell_q) \supset_{(c)} \ell_{p_1}(\ell_{q_1}) = \ell_2(\ell_2) \approx \ell_2$, although p_1 and q_1 are not equal neither to q nor ∞.

Now, what about Question 3 in the particular case $p_1 = q_1$? That is

Question 4. *Let $1 \leq q < \infty$. For which r, with $1 \leq r < \infty$, does one have*

$$\ell_\infty(\ell_q) \supset_{(c)} \ell_r?$$

Unfortunately, a complete answer is not known. Partial answers tell us that the solution is: "it depends on the values of q and r". In fact, point (1) says us that the answer for $1 < q < \infty$ and for $r = 2$ is "always". Nevertheless, we prove that for $q = 1$ the answer is "only in the trivial situation". This is also the case for $1 < q < \infty$ and $r = 1$. This last assertion is an immediate consequence of a result of Díaz and Kalton (see [4, Theorem 5.2.3.]). Summing up we have:

(2) *Let $1 \leq r < \infty$, then $\ell_\infty(\ell_1) \supset_{(c)} \ell_r$ if and only if $r = 1$.*

(3) *Let $1 < q < \infty$, then $\ell_\infty(\ell_q) \supset_{(c)} \ell_1$ if and only if $q = 1$.*

Points (1), (2) and (3) are the answers we know to Question 4.

According to our partial answers to Question 3 and 4, to prove Theorem 1 it suffices to know when two spaces of the type $\ell_\infty(\ell_q)$ are isomorphic. Using a deep result of Bourgain, Casazza, Lindensatruss and Tzafriri [3], we can show that

(4) *Let* $1 < q, r < \infty$. *If* $\ell_\infty(\ell_q) \supset_{(c)} \ell_r$ *then* r *must be between* 2 *and* q.

Therefore, it follows from (3) and (4) the following

Theorem 4. *Let* $1 \leq q_0, q_1 \leq \infty$. *Then* $\ell_\infty(\ell_{q_0}) \approx \ell_\infty(\ell_{q_1})$ *if and only if* $q_0 = q_1$.

Finally, Theorem 1 is an immediate consequence of Theorems 2, 3 and 4.

6. Extensions of the main result

Now we wish to mention two extensions of Theorem 1.

The first one was motivated by a question that F. Cabello asked us on June 2009:

- *Are the Banach spaces* $\ell_\infty(c_0)$ *and* $c_0(\ell_\infty)$ *isomorphic*?

We did not find an answer in the literature and worked on the problem. We could prove that the answer is "no" (see [6]). However, on 2013, E.M. Galego called our attention about an old work by T.E. Khmyleva [10] which had already given the answer many years before. We would like to remark that we have follow quite a different approach: she got the answer by showing that $\ell_\infty(c_0) \not\supset_{(c)} c_0(\ell_\infty)$, while we get it showing that $\ell_\infty(c_0)$ is not isomorphic to a quotient of $c_0(\ell_\infty)$.

With this in mind we have the following extension of Theorem 1 [6, Theorem 2].

Theorem 5. *Let* $p_0, p_1, q_0, q_1 \in \{0\} \cup [1, +\infty]$ *and let us denote* c_0 *by* ℓ_0, *then the Banach spaces* $\ell_{p_0}(\ell_{q_0})$ *and* $\ell_{p_1}(\ell_{q_1})$ *are isomorphic if and only if* $p_0 = p_1$ *and* $q_0 = q_1$.

Recently, Albiac and Ansorena [1] have proved another extension of Theorem 1, showing that it also holds in the non convex setting:

Theorem 6 [1]**.** *Let* $0 < p_0, p_1, q_0, q_1 \leq +\infty$, *then the spaces* $\ell_{p_0}(\ell_{q_0})$ *and* $\ell_{p_1}(\ell_{q_1})$ *are isomorphic if and only if* $p_0 = p_1$ *and* $q_0 = q_1$.

7. Open problems

We have mentioned that complemented subspaces of $\ell_\infty(\ell_q)$ are not well known. Then, according with Questions 3 and 4, and statement (4) in Section 5 we propose the following open problems [7]:

Open Problem 1. *Let $1 < q < \infty$. Is it true that for $1 < r < \infty$ one has $\ell_\infty(\ell_q) \supset_{(c)} \ell_r$ if and only if $r \in [q, 2]$ or $r \in [2, q]$?*

If the answer is "no":

Open Problem 2. *Let $1 < q < \infty$. For which r, with $1 < r < \infty$, does one have $\ell_\infty(\ell_q) \supset_{(c)} \ell_r$?*

Or more in general

Open Problem 3. *Let $1 < q < \infty$. For which p_1, q_1, with $1 < p_1, q_1 \leq \infty$, does one have $\ell_\infty(\ell_q) \supset_{(c)} \ell_{p_1}(\ell_{q_1})$?*

8. FILLING A GAP

On April 2013 Eloi Medina Galego kindly pointed out to us a gap in our proof of Proposition 4.3 of [7]. One can see in the proof that the study of the case $q < 2 < p$ is missing. We take advantage of this note to fill the gap. Actually we only need to use the same ideas already used there. For the sake of completeness we give a complete proof of the proposition. It consists on a small variation of the proof of Lemma 4.2, which now is not necessary.

Proposition 4.3 [7]**.** *Let $1 < p, q < \infty$. If $\ell_\infty(\ell_q) \supset_{(c)} \ell_p$ then either $q \leq p \leq 2$ or $2 \leq p \leq q$.*

Proof. If $\ell_\infty(\ell_q) \supset_{(c)} \ell_p$, Proposition 4.1 implies $(\sum_{n=1}^\infty \oplus \ell_q^n)_{c_0} \supset_{(c)} (\sum_{n=1}^\infty \oplus \ell_p^n)_{c_0}$. But

$$c_0(\ell_q) \supset_{(c)} \left(\sum_{n=1}^\infty \oplus \ell_q^n \right)_{c_0}.$$

Therefore we deduce that $c_0(\ell_q)$ contains ℓ_p^n uniformly complemented. That is, there exists $K > 0$ such that for each $n \in \mathbb{N}$ we can find $\{z_1, \ldots, z_n\}$ in $c_0(\ell_q)$ in such a way that they generate an n-dimensional K-complemented subspace and such that

$$\frac{1}{K}(|\lambda_1|^p + \cdots + |\lambda_n|^p)^{1/p} \leq \|\lambda_1 z_1 + \cdots + \lambda_n z_n\| \leq K(|\lambda_1|^p + \cdots + |\lambda_n|^p)^{1/p}$$

for all scalars $\lambda_1, \ldots, \lambda_n$. Then, putting $\alpha = type(\ell_q)$ and $\beta = cotype(\ell_q)$, by [3, Proposition 2.1], there exists $M > 0$ such that for each n there exists a partition $\{\sigma_1^n, \ldots, \sigma_{r_n}^n\}$ of $\{1, \ldots, n\}$ satisfying

$$\frac{1}{M} \max_{1 \leq s \leq r_n} \left(\sum_{i \in \sigma_s^n} |\lambda_i|^\beta \right)^{1/\beta} \leq \|\lambda_1 z_1 + \cdots + \lambda_n z_n\| \leq M \max_{1 \leq s \leq r_n} \left(\sum_{i \in \sigma_s^n} |\lambda_i|^\alpha \right)^{1/\alpha}$$

for all scalars $\lambda_1, \dots, \lambda_n$. For each n let us denote

$$k_n = \max_{1 \le s \le r_n} |\sigma_s^n|,$$

where $|\sigma_s^n|$ is the cardinal of the finite set σ_s^n. With no loss of generality we can suppose

$$k_n = |\sigma_1^n|.$$

Given $n \in \mathbb{N}$, take $\lambda_1, \dots, \lambda_n$ in the following way:

$$\lambda_i = \begin{cases} 1 & \text{if } i \in \sigma_1^n \\ 0 & \text{otherwise.} \end{cases}$$

Using the preceding inequalities we get

$$\frac{1}{K}(k_n)^{1/p} \le \left\| \sum_{i \in \sigma_1^n} z_i \right\| \le K(k_n)^{1/p}$$

and

$$\frac{1}{M}(k_n)^{1/\beta} \le \left\| \sum_{i \in \sigma_1^n} z_i \right\| \le M(k_n)^{1/\alpha}.$$

Therefore, for each $n \in \mathbb{N}$ we have

$$\frac{1}{K}(k_n)^{1/p} \le M(k_n)^{1/\alpha} \quad \text{and} \quad \frac{1}{M}(k_n)^{1/\beta} \le K(k_n)^{1/p}.$$

Thus, either $\frac{1}{\beta} \le \frac{1}{p} \le \frac{1}{\alpha}$ or $\sup_n k_n < +\infty$. In the first case, taking account that $\alpha = \min\{q, 2\}$ and $\beta = \max\{q, 2\}$ [8, Ch. 11], we obtain the inequality $\min\{q, 2\} \le p \le \max\{q, 2\}$, which means that either $q \le p \le 2$ or $2 \le p \le q$. Let us see that the second case is impossible. It follows from our inequalities that

$$\frac{1}{K}(|\lambda_1|^p + \cdots + |\lambda_n|^p)^{1/p} \le \|\lambda_1 z_1 + \cdots + \lambda_n z_n\| \le M \max_{1 \le s \le r_n} \left(\sum_{i \in \sigma_s^n} |\lambda_i|^\alpha \right)^{1/\alpha}$$

for all n and all scalars λ_i. Let us denote $l = \sup_n k_n = \max_n k_n$ and let us take $\lambda_i = 1$ for all i. We get

$$\frac{1}{K} n^{1/p} \le M \max_{1 \le s \le r_n} (|\sigma_s^n|)^{1/\alpha} \le M l^{1/\alpha}$$

for all n. This is a contradiction. ∎

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