VERTICES CONTAINED IN ALL OR IN NO MINIMUM SEMITOTAL DOMINATING SET OF A TREE

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Abstract

Let $G$ be a graph with no isolated vertex. In this paper, we study a parameter that is squeezed between arguably the two most important domination parameters; namely, the domination number, $\gamma(G)$, and the total domination number, $\gamma_t(G)$. A set $S$ of vertices in a graph $G$ is a semitotal dominating set of $G$ if it is a dominating set of $G$ and every vertex in $S$ is within distance 2 of another vertex of $S$. The semitotal domination number, $\gamma_{t2}(G)$, is the minimum cardinality of a semitotal dominating set of $G$. We observe that $\gamma(G) \leq \gamma_{t2}(G) \leq \gamma_t(G)$. We characterize the set of vertices that are contained in all, or in no minimum semitotal dominating set of a tree.

Keywords: domination, semitotal domination, trees.

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1. Introduction

In this paper, we continue the study of a parameter, called the semitotal domination number, that is squeezed between arguably the two most important domination parameters; namely, the domination number and the total domination

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number. A dominating set in a graph $G$ is a set $S$ of vertices of $G$ such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. A total dominating set, abbreviated a TD-set, of a graph $G$ with no isolated vertex is a set $S$ of vertices of $G$ such that every vertex in $V(G)$ is adjacent to at least one vertex in $S$. The total domination number of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of $G$. The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the so-called domination book [4]. Total domination is now well studied in graph theory. For a recent book on the topic, see [9]. A survey of total domination in graphs can also be found in [5].

The concept of semitotal domination in graphs was introduced and studied by Goddard, Henning and McPillan [3], and studied further in [6, 7] and elsewhere. A set $S$ of vertices in a graph $G$ with no isolated vertices is a semitotal dominating set, abbreviated semi-TD-set, of $G$ if it is a dominating set of $G$ and every vertex in $S$ is within distance 2 of another vertex of $S$. The semitotal domination number, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semi-TD-set of $G$. A semi-TD-set of $G$ of cardinality $\gamma_{t2}(G)$ is called a $\gamma_{t2}(G)$-set. Since every TD-set is a semi-TD-set, and since every semi-TD-set is a dominating set, we have the following observation first observed in [3]. For every graph $G$ with no isolated vertex, $\gamma(G) \leq \gamma_{t2}(G) \leq \gamma_t(G)$.

Mynhardt [10] characterized all the vertices that are in all, or in no minimum dominating set. Moreover, the same type of results were established by Cockayne, Henning and Mynhardt in [2] for total domination, Henning and Plummer [8] for paired domination and Blidia, Chellali and Khelifi [1] for double domination. Motivated by these results, we aim to characterize all the vertices that are in all, or in no minimum semitotal dominating set in a rooted tree $T$.

1.1. Terminology and Notation

For notation and graph theory terminology that are not defined herein, we refer the reader to [9]. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ of order $n = |V|$ and edge set $E = E(G)$ of size $m = |E|$, and let $v$ be a vertex in $V$. We denote the degree of $v$ in $G$ by $d_G(v)$. A leaf of $G$ is a vertex of degree 1, while a support vertex of $G$ is a vertex adjacent to a leaf. A strong support vertex is a support vertex with at least two leaf-neighbors. We define a branch vertex as a vertex of degree at least 3. A star is a tree with at most one vertex that is not a leaf.

For a set $S \subseteq V$, the subgraph induced by $S$ is denoted by $G[S]$. A cycle and path on $n$ vertices are denoted by $C_n$ and $P_n$, respectively. For two vertices $u$ and $v$ in a connected graph $G$, the distance $d_G(u, v)$ between $u$ and $v$ is the length of a shortest $(u, v)$-path in $G$. The distance $d_G(v, S)$ between a vertex
v and a set S of vertices in a graph G is the minimum distance from v to a vertex of S in G. The maximum distance among all pairs of vertices of G is the diameter of a graph G which is denoted by diam(G). The open neighborhood of a vertex v is the set \( N_G(v) = \{ u \in V \mid uv \in E \} \) and the closed neighborhood of v is \( N_G[v] = \{ v \} \cup N_G(v) \). For a set \( S \subseteq V \), its open neighborhood is the set

\[
N_G(S) = \bigcup_{v \in S} N_G(v),
\]

and its closed neighborhood is the set \( N_G[S] = N_G(S) \cup S \). If the graph G is clear from the context, we omit it in the above expressions. For example, we write \( d(u), d(u, v), N(v) \) and \( N[v] \) rather than \( d_G(u), d_G(u, v), N_G(v) \) and \( N_G[v] \), respectively.

Let X and Y be subsets of vertices in G. If \( Y \subseteq N[X] \), then we say the set X dominates the set Y in G and that the set Y is dominated by X. Furthermore, if \( Y = \{ y \} \), then we simply say that y is dominated by X rather than \( \{ y \} \) is dominated by X. Thus, if a vertex v is dominated by X, then \( N[v] \cap X \neq \emptyset \). We note that if X dominates V, then X is a dominating set in G. Hence, if X is a dominating set in G, then \( N[X] = V \). Additionally, we say that X semitotally dominates the set Y in G if each vertex in X lies within distance 2 of another vertex in X, and in turn the set Y is said to be semitotally dominated by X.

For a graph G, we define the sets \( A_{12}(G) \) and \( N_{12}(G) \) as follows:

\[
A_{12}(G) = \{ v \in V(G) \mid v \text{ is in every } \gamma_{12}(G)-\text{set} \},
\]

and

\[
N_{12}(G) = \{ v \in V(G) \mid v \text{ is in no } \gamma_{12}(G)-\text{set} \}.
\]

A rooted tree T distinguishes one vertex r called the root. For each vertex \( v \neq r \) of T, the parent of v is the neighbor of v on the unique \( (r, v) \)-path, while a child of v is any other neighbor of v. We denote all the children of a vertex v by C(v). A descendant of v is a vertex \( u \neq v \) such that the unique \( (r, u) \)-path contains v. Thus, every child of v is a descendant of v. A grandchild of v is a descendant of v at distance 2 from v. We let \( D(v) \) denote the set of descendants of v, and we define \( D[v] = D(v) \cup \{ v \} \). The set of leaves in T is denoted by \( L(T) \) and the set of support vertices is denoted by \( S(T) \). The maximal subtree at v is the subtree of T induced by \( D[v] \), and is denoted by \( T_v \). The set of leaves in \( T_v \) distinct from v we denote by \( L(v) \); that is, \( L(v) = D(v) \cap L(T) \). The set of branch vertices of T is denoted by \( B(T) \). For \( j \in \{ 0, 1, 2, 3, 4 \} \), we define

\[
L^j(v) = \{ u \in L(v) \mid d(u, v) \equiv j \pmod 5 \}.
\]

Furthermore, let

\[
L^1_1(v) = \{ x \in L^1(v) \mid d(v, x) = 1 \} \quad \text{and} \quad L^1_2(v) = L^1(v) \setminus L^1_1(v).
\]
We sometimes write $L^j_T(v)$ to emphasize the tree (or subtree) concerned. Additionally, we define the path from $v$ to a leaf in $L^j_T(v)$ to be a $L^j_T(v)$-path. Given a vertex $x$ of a tree $T$, we say we attach a path of length $q$ to $x$ if we add a vertex-disjoint path $P_q$ on $q$ vertices and join $x$ to a leaf of the path $P_q$. In this case, we simply write that we attach $P_q$ to $x$. We next define an essential support vertex in a tree.

**Definition 1.** A vertex $v$ in a tree $T$ is an essential support vertex in $T$ if and only if $v$ has exactly one leaf-neighbor, $v \in A_{t2}(T)$ and $N(v) \subseteq N_{t2}(T)$.

We note that if $v$ is an essential support vertex in a tree $T$, then $v$ has exactly one leaf-neighbor and $N[v] \cap D = \{v\}$ for every $\gamma_{t2}(T)$-set $D$.

### 2. Tree Pruning

In this paper, we use a method called tree pruning to characterize the sets $A_{t2}(T)$ and $N_{t2}(T)$ for an arbitrary tree $T$. Let $T$ be a tree rooted at a vertex $v$. Suppose that $T$ is not a star. We let $C(v)$ denote the set of children of $v$ that belong to $P_t$’s that are attached to $v$. Furthermore, we let the descendants at distance 2 from $v$ along $P_3$’s that are attached to $v$ be denoted by $Gr(v)$ and we call them special grandchildren of $v$. The pruning of $T$ is performed with respect to its root, $v$. If $d(u) \leq 2$ for each $u \in V(T_v \setminus \{v\}$, then let $T_v = T$. Otherwise, let $u$ be a branch vertex at maximum distance from $v$ (we note that $|C(u)| \geq 2$ and $d(x) \leq 2$ for each $x \in D(u)$). We identify the following types of branch vertices:

- **(T.1)** $|L^3(u)| \geq 1$.
- **(T.2)** $L^3(u) = \emptyset$, $|L^1(u)| \geq 1$ and $|L^0(u) \cup L^2(u) \cup L^4(u)| \geq 1$.
- **(T.3)** $L^3(u) = L^0(u) = L^2(u) = L^4(u) = \emptyset$ and $|L^1(u)| \geq 2$.
- **(T.4)** $L^3(u) = L^1(u) = \emptyset$ and $|L^4(u)| \geq 1$.
- **(T.5)** $L^3(u) = L^1(u) = L^4(u) = \emptyset$, $|L^2(u)| = 1$ and $|L^0(u)| \geq 1$.
- **(T.6)** $L^3(u) = L^1(u) = L^4(u) = \emptyset$ and $|L^2(u)| \geq 2$.
- **(T.7)** $L^3(u) = L^1(u) = L^4(u) = L^2(u) = \emptyset$.

We now apply the following pruning process.

(a) If $u$ is type (T.1) or (T.2), then delete $D(u)$ and attach a $P_3$ to $u$.
(b) If $u$ is type (T.3), then delete $D(u)$ and attach a $P_4$ to $u$.
(c) If $u$ is type (T.4) or (T.6), then delete $D(u)$ and attach a $P_4$ to $u$.
(d) If $u$ is type (T.5), then delete $D(u)$ and attach a $P_2$ to $u$.
(e) If $u$ is type (T.7), then delete $D(u)$ and attach a $P_5$ to $u$.

This step of the pruning process, where all the descendants of $u$ are deleted and a path of length 1, 2, 3, 4 or 5 is attached to $u$ to give a tree in which $u$ has degree 2, is called a pruning of $T_v$ at $u$. Repeat the above process until a tree...
$T_v$ is obtained with $d(u) \leq 2$ for each $u \in V(T_v) \setminus \{v\}$. The tree $T_v$ is called the pruning of $T_v$. To simplify notation, we write $L^j_v(v)$ instead of $L^j_{T_v}(v)$.

3. Main Results

In this paper, we aim to establish a characterization of the set of vertices contained in all or none of the minimum semi-TD-sets in a tree $T$ of order $n \geq 2$.

In the trivial case when $T = P_2$, we note that $A_{i2}(T) = V(T)$, while if $T = P_3$, then $A_{i2}(T) = N_{i2}(T) = \emptyset$. If $T$ is a star $K_{1,n-1}$ with central vertex $v$ and $n \geq 4$, then $A_{i2}(T) = \{v\}$ and $N_{i2}(T) = \emptyset$. Hence in what follows we restrict our attention to the more interesting case when $n \geq 4$ and $T$ is not a star. We shall prove the following main results.\(^3\)

**Theorem 1.** Let $T$ be a tree with order at least 4 that is not a star and is rooted at a vertex $v$ such that $d(u) \leq 2$ for each $u \in V(T) \setminus \{v\}$. Then,

(a) $v \in A_{i2}(T)$ if and only if one of the following hold:

(i) $|L^3(v)| \geq 1$ and $|L^1(v) \cup L^3(v)| \geq 2$.

(ii) $L^3(v) = \emptyset$ and $|L^1(v)| \geq 3$.

(iii) $L^3(v) = \emptyset$ and $|L^1(v)| = 2$.

(iv) $L^3(v) = \emptyset$, $|L^1(v)| \leq 1$, $|L^1(v)| = 2$ and $|L^0(v) \cup L^2(v) \cup L^4(v)| \geq 1$.

(v) $L^2(v) = L^3(v) = L^4(v) = \emptyset$, $|L^1(v)| = |L^1(v)| = 1$ and $|L^0(v)| \geq 1$.

(b) $v \in N_{i2}(T)$ if and only if one of the following hold:

(i) $L^1(v) = L^3(v) = \emptyset$ and $|L^3(v)| \geq 1$, or

(ii) $L^1(v) = L^3(v) = L^4(v) = \emptyset$ and $|L^2(v)| \geq 2$.

**Theorem 2.** Let $v$ be a vertex of a tree $T$ with order at least 4 that is not a star. Then,

(a) $v \in A_{i2}(T)$ if and only if one of the following hold:

(i) $|L^3(v)| \geq 1$ and $|L^1(v) \cup L^3(v)| \geq 2$.

(ii) $L^3(v) = \emptyset$ and $|L^1(v)| \geq 3$.

(iii) $L^3(v) = \emptyset$ and $|L^1(v)| = 2$.

(iv) $L^3(v) = \emptyset$, $|L^1(v)| \leq 1$, $|L^1(v)| = 2$ and $|L^0(v) \cup L^2(v) \cup L^4(v)| \geq 1$.

(v) $L^2(v) = L^3(v) = L^4(v) = \emptyset$, $|L^1(v)| = |L^1(v)| = 1$ and $|L^0(v)| \geq 1$.

(b) $v \in N_{i2}(T)$ if and only if one of the following hold:

(i) $L^1(v) = L^3(v) = L^4(v) = \emptyset$ and $|L^3(v)| \geq 1$, or

(ii) $L^1(v) = L^3(v) = L^4(v) = \emptyset$ and $|L^2(v)| \geq 2$.

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\(^3\)An example to illustrate Theorem 2 is presented in the appendix.
4. Preliminary Results

The semitotal domination number of a path and a cycle is determined in [3].

Lemma 3 [3]. For \( n \geq 3 \), \( \gammaTD(P_n) = \gammaTD(C_n) = \left\lceil \frac{2n}{5} \right\rceil \).

Lemma 3 immediately infers that every path \( P_n \) where \( n \equiv 0 \pmod{5} \) has a unique \( \gammaTD(P_n) \)-set. That is, if we number the vertices in \( V(P_n) \) consecutively starting at 1, then the \( \gammaTD(P_n) \)-set is the set of all vertices with numbers congruent to 2 (mod 5) and 4 (mod 5). Additionally, the paths \( P_2 \) and \( P_7 \) also have unique minimum semi-TD-sets. We state this formally as follows.

Observation 4. The paths \( P_2, P_7 \) and \( P_n \), where \( n \equiv 0 \pmod{5} \), all have unique minimum semi-TD-sets.

We shall need the following result first observed in [6].

Observation 5. If \( G \) is a connected graph that is not a star, then there is a \( \gammaTD(G) \)-set that contains no leaf of \( G \).

We proceed with the following two lemmas that will be useful when proving our main results. We use the standard notation \([k] = \{1, 2, \ldots, k\}\).

Lemma 6. Let \( T \) be a tree of order at least 3. Let \( t \) be a support vertex in \( T \) and let \( u' \) be a leaf-neighbor of \( t \). If \( T' \) is the tree obtained from \( T \) by attaching a path of length 5 to \( u' \), then \( \gammaTD(T') = \gammaTD(T) + 2 \).

Proof. Suppose \( T' \) is obtained from \( T \) by adding to \( u' \) the path \( uwxyz \) together with the edge \( uu' \). Every \( \gammaTD(T) \)-set can be extended to a semi-TD-set of \( T' \) by adding to it the vertices \( w \) and \( y \), and so \( \gammaTD(T') \leq \gammaTD(T) + 2 \). Let \( D' \) be a \( \gammaTD(T') \)-set. If \( z \in D' \), then we can replace \( z \) in \( D' \) by \( y \). Hence we may choose \( D' \) so that \( D' \cap \{y, z\} = \{y\} \). In order to semitotally dominate the vertex \( y \), we note that \( x \) or \( w \) belong to \( D' \). If \( x \in D' \), then we can replace \( x \) in \( D' \) by \( w \). Hence we may choose \( D' \) so that \( D' \cap \{x, w\} = \{w\} \). If \( u \in D' \), then we can replace \( u \) in \( D' \) by \( u' \). Hence we may choose \( D' \) so that \( u \notin D' \). If \( t \in D' \), then we can replace \( u' \) in \( D' \) with a neighbor of \( t \) different from \( u' \). If \( t \notin D' \) and \( |D' \cap N(t)| \geq 2 \), then we can replace \( u' \) in \( D' \) with the vertex \( t \). If \( t \notin D' \) and \( D' \cap N[t] = \{u'\} \), then in order to dominate the neighbors of \( t \) different from \( u' \), the set \( D' \) contains at least one vertex at distance 2 from \( t \) in \( T \), implying once again that we can replace \( u' \) in \( D' \) with the vertex \( t \). Hence, we may choose \( D' \) so that \( u' \notin D' \). In order to dominate the vertex \( u' \), we note that \( t \in D' \). Since \( D' \) is a semi-TD-set of \( T' \), the set \( D' \setminus \{w, y\} \) is necessarily a semi-TD-set of \( T \), implying that \( \gammaTD(T) \leq |D'| - 2 = \gammaTD(T') - 2 \). Consequently, \( \gammaTD(T') = \gammaTD(T) + 2 \).
Let $u$ imply that $v$ is an essential support vertex in $T$, let $v \in V(T) \setminus \{u', t\}$. If $t$ is not an essential support vertex in $T$, let $v \in V(T)$. Then the following hold.

(a) $v \in \mathcal{A}_2(T)$ if and only if $v \in \mathcal{A}_2(T')$.

(b) $v \in \mathcal{N}_2(T)$ if and only if $v \in \mathcal{N}_2(T')$.

Proof. Suppose $T'$ is obtained from $T$ by adding to $u'$ the path $uwxyz$ together with the edge $uu'$.

(a) Suppose that $v \notin \mathcal{A}_2(T)$. Let $D$ be a $\gamma_2(T)$-set that does not contain $v$. Then, $D \cup \{w, y\}$ is a semi-TD-set of $T'$ of cardinality $|D| + 2 = \gamma_2(T) + 2 = \gamma_2(T')$ by Lemma 6. Consequently, $D \cup \{w, y\}$ is a $\gamma_2(T')$-set that does not contain $v$, implying that $v \notin \mathcal{A}_2(T')$. Therefore, by contraposition, if $v \in \mathcal{A}_2(T')$, then $v \in \mathcal{A}_2(T)$.

Conversely, suppose that $v \in \mathcal{A}_2(T)$. Suppose to the contrary that $v \notin \mathcal{A}_2(T')$. Let $D'$ be a $\gamma_2(T')$-set that does not contain the vertex $v$, and let $D = D' \cap V(T)$. If $v = u'$, then by Observation 5, there exists a $\gamma_2(T)$-set that does not contain $v$, contradicting our assumption that $v \in \mathcal{A}_2(T)$. Hence, $v \neq u'$. Proceeding as in the proof of Lemma 6, we can choose $D'$ so that $D' \cap \{w, x, y, z, u\} = \{w, y\}$. Thus, $D = D' \setminus \{w, y\}$ and, by Lemma 6, $|D| = |D'| - 2 = \gamma_2(T') - 2 = \gamma_2(T)$. If $v \neq t$, then proceeding as in the proof of Lemma 6, we can additionally choose $D'$ so that $D' \cap \{u', t\} = \{t\}$, implying that the set $D$ is a $\gamma_2(T)$-set that does not contain $v$, a contradiction. Hence, $v = t$. By supposition, $v \notin D'$, and so neither neighbor of $u'$ in $T'$ belongs to $D'$, implying that $u' \notin D'$.

If $D$ is a semi-TD-set in $T$, then $D$ is a $\gamma_2(T)$-set that does not contain the vertex $v$, contradicting our assumption that $v \in \mathcal{A}_2(T)$. Hence, $D$ is not a semi-TD-set in $T$, implying that no vertex in $D$ is at distance 1 or 2 from $u'$. Thus, $D \cap N[v] = \{u'\}$. In particular, we note that $u'$ is the only leaf-neighbor of $v$ in $T$.

We show next that for every $\gamma_2(T)$-set $S$, $N[v] \cap S = \{v\}$. For notational convenience, let $T$ be rooted at the vertex $v$ and let $N(v) \setminus \{u'\} = \{v_1, \ldots, v_k\}$. For $i \in [k]$, let $T_i$ denote the maximal subtree of $T$ rooted at $v_i$ (and so, $T_i = T_{v_i}$) and let $D_i = D \cap V(T_i)$. We note that $v_i \notin D_i$ and that the set $D_i$ is a semi-TD-set in $T_i$ for all $i \in [k]$. Suppose that there exists a $\gamma_2(T)$-set, $S$, such that $|N[v] \cap S| \geq 2$. Since $v \in \mathcal{A}_2(T)$, we note that $v \in S$. If $u' \in S$, we can simply replace $u'$ in $S$ with a neighbor of $v$ that is not a leaf. Renaming the children of $v$ if necessary, we may therefore assume that $v_1 \in S$. Let $S_1 = S \cap V(T_1)$. Since the set $D_1$ contains a vertex at distance 2 from $v$ in $T$, we note that the set $(S \setminus S_1) \cup D_1$ is a semi-TD-set of $T$, implying that $|S| = \gamma_2(T) \leq |S| - |S_1| + |D_1|$,
or, equivalently, $|S_1| \leq |D_1|$. We now consider the set $S^* = (D\setminus D_1) \cup S_1$. Since $v'$ and $v_1$ are at distance 2 apart in $T$, the set $S^*$ is a semi-TD-set of $T$, implying that $\gamma_{12}(T) \leq |S^*| \leq |D| - |D_1| + |S_1| \leq |D| = \gamma_{12}(T)$. Consequently, $|S^*| = \gamma_{12}(T)$ and $S^*$ is a $\gamma_{12}(T)$-set that does not contain the vertex $v$, a contradiction. Therefore, for every $\gamma_{12}(T)$-set $S$, we have $N[v] \cap S = \{v\}$. Moreover, this result together with our earlier observation that $u'$ is the only leaf-neighbor of $v$ in $T$ imply that $v$ is an essential support vertex in $T$, a contradiction (recalling that here $v = t$). Hence, $v \in A_{t_2}(T')$. This completes the proof of part (a).

(b) Suppose that $v \in \hat{N}_{t_2}(T')$. We show that $v \in \hat{N}_{t_2}(T)$. Suppose to the contrary that there exists a $\gamma_{12}(T)$-set, $D$, that contains the vertex $v$. Then, $D \cup \{w, y\}$ is a semi-TD-set of $T'$ of cardinality $|D| + 2 = \gamma_{12}(T) + 2 = \gamma_{12}(T')$. Consequently, $D \cup \{w, y\}$ is a $\gamma_{12}(T')$-set that contains $v$, a contradiction. Therefore, $v \in \hat{N}_{t_2}(T)$.

Conversely, suppose that $v \in \hat{N}_{t_2}(T)$. We show that $v \in \hat{N}_{t_2}(T')$. Suppose to the contrary that there exists a $\gamma_{12}(T')$-set, $D'$, that contains the vertex $v$. Let $D = D' \cap V(T)$. Proceeding as in the proof of Lemma 6, we can choose $D'$ so that $D' \cap \{w, x, y, z, u\} = \{w, y\}$. Thus, $D = D' \setminus \{w, y\}$. If $v \neq u'$, then proceeding as in the proof of Lemma 6, we can further choose $D'$ so that $D' \cap \{u', t\} = \{t\}$, implying that the set $D$ is a $\gamma_{12}(T)$-set containing $v$, a contradiction. Hence, $v = u'$. If $D$ is a semi-TD-set in $T$, then the set $D$ is a $\gamma_{12}(T)$-set containing $v$, a contradiction. Hence, $D$ is not a semi-TD-set in $T$, implying that no vertex in $D$ is at distance 1 or 2 from $u'$. Thus, $D \cap N[t] = \{u'\}$. In particular, this implies that $u'$ is the only leaf-neighbor of $t$ in $T$. An analogous proof to that employed in the proof of part (a) shows the vertex $t$ is an essential support vertex in $T$, contradicting the fact that in this case $v = u'$. Therefore, $v \in \hat{N}_{t_2}(T')$.

5. Proof of Theorem 1

Proof. Let $T$ be a tree with order at least 4 that is not a star and is rooted at a vertex $v$ such that $d(u) \leq 2$ for each $u \in V(T) \setminus \{v\}$. For each $w \in L(v)$ such that $d_T(v, w) \geq 6$, let $T'$ be the tree obtained by replacing the $(v, w)$-path in $T$ with a $(v, w)$-path of length $j$, $j \in \{5, 6, 2, 3, 4\}$ if $w \in L'(v), i \in \{0, 1, 2, 3, 4\}$, respectively. By repeated applications of Lemma 7, $v \in A_{t_2}(T) (\hat{N}_{t_2}(T), respectively) if and only if $v \in A_{t_2}(T') (\hat{N}_{t_2}(T'), respectively). Hence, in what follows, we assume $T = T'$. If $v$ is a leaf of $T$, then by our earlier assumptions, $T$ is a path $P_n$ where $n \in \{4, 5, 6, 7\}$. If $n \in \{4, 6\}$, then $v \notin A_{t_2}(T) \cup \hat{N}_{t_2}(T)$. If $n \in \{5, 7\}$, then by Observation 4, $v \in \hat{N}_{t_2}(T)$. Hence, we may assume that $v$ is not a leaf in $T$. Let $D$ be an arbitrary $\gamma_{12}(T)$-set and let $W$ be the set of vertices at distance 3 from a leaf of some $L_3^1(v)$-path. We proceed further with a series of claims.
Claim A. If $|L_1^1(v)| \geq 2$, then $v \in A_{l2}(T)$.

Proof. Suppose $|L_1^1(v)| \geq 2$. Thus, $v$ is a strong support vertex in $T$ and therefore has at least two leaf-neighbors. Moreover, $|L_0^0(v) \cup L_1^1(v) \cup L_2^2(v) \cup L_3^3(v) \cup L_4^4(v)| \geq 1$ since $T$ is not a star. Let $w$ be a neighbor of $v$ that is not a leaf. Suppose, to the contrary, that $v \notin A_{l2}(T)$. Let $S$ be a $\gamma_{l2}(T)$-set that does not contain the vertex $v$. The set $S$ contains all leaf-neighbors of $v$. Since $N[w] \cap S \neq \emptyset$, we note that $v$ is within distance 2 from at least one vertex in $N[w] \cap S$. Further, no vertex in $N[w] \cap S$ is a leaf-neighbor of $v$. Replacing the leaf-neighbors of $v$ in $S$ with the vertex $v$ produces a semi-TD-set in $T$ of cardinality less than $|S| = \gamma_{l2}(T)$, a contradiction. Hence, $v \in A_{l2}(T)$.

By Claim A, we may assume that $|L_1^1(v)| \leq 1$.

Claim B. If $L(v) = L_0^0(v)$, then $v \notin A_{l2}(T) \cup \mathcal{N}_{l2}(T)$.

Proof. Suppose $L(v) = L_0^0(v)$. Then, $L_1^1(v) \cup L_2^2(v) \cup L_3^3(v) \cup L_4^4(v) = \emptyset$. Let $S = \text{Gr}(v) \cup S(T) \cup \{v\}$. The set $S$ is a semi-TD-set of $T$, and so $\gamma_{l2}(T) \leq |S| = 2|L_0^0(v)| + 1$. Recall that $D$ is an arbitrary $\gamma_{l2}(T)$-set. If $vv_1v_2v_3v_4v_5$ is a path emanating from $v$ in $T$, then $v_5$ is a leaf in $T$ and $|D \cap \{v_2, v_3, v_4, v_5\}| \geq 2$, implying that the set $D$ contains at least two vertices from each path of order 5 attached to $v$ and at least one vertex in $N[v]$. Thus, $\gamma_{l2}(T) = |D| \geq 2|L_0^0(v)| + 1 = |S| \geq \gamma_{l2}(T)$. Consequently, we must have equality throughout this inequality chain. In particular, $|S| = \gamma_{l2}(T) = 2|L_0^0(v)| + 1$ and $S$ is a $\gamma_{l2}(T)$-set. Replacing $v$ in $S$ with an arbitrary neighbor of $v$ produces a $\gamma_{l2}(T)$-set not containing $v$. Hence, $v \notin A_{l2}(T) \cup \mathcal{N}_{l2}(T)$.

By Claim B, we may assume that $L(v) \neq L_0^0(v)$.

Claim C. If $L(v) = L_0^0(v) \cup L_1^1(v)$ where $|L_1^1(v)| = 1$ and $|L_0^0(v)| \geq 1$, then $v$ is an essential support vertex in $T$. In particular, $v \in A_{l2}(T)$.

Proof. Suppose $L(v) = L_0^0(v) \cup L_1^1(v)$ where $|L_1^1(v)| = 1$ and $|L_0^0(v)| \geq k \geq 1$. In this case, $L_2^2(v) \cup L_3^3(v) \cup L_4^4(v) = \emptyset$. Let $L_1^1(v) = \{u\}$. We note that $u$ is the only leaf-neighbor of $v$ in $T$. We show that $v \in A_{l2}(T)$ and $N(v) \subseteq \mathcal{N}_{l2}(T)$, implying that $v$ is an essential support vertex of $T$. Let $S = \text{Gr}(v) \cup S(T) \cup \{v\}$. The set $S$ is a semi-TD-set of $T$, and so $\gamma_{l2}(T) \leq |S| = 2k + 1$. If $vv_1v_2v_3v_4v_5$ is a path emanating from $v$ in $T$, then $v_5$ is a leaf in $T$ and $|D \cap \{v_2, v_3, v_4, v_5\}| \geq 2$. In particular, the set $D$ contains at least two vertices from each path of order 5 attached to $v$. Further, $D$ contains at least one of $u$ and $v$. Thus, $\gamma_{l2}(T) = |D| \geq 2k + 1 = |S| \geq \gamma_{l2}(T)$. Consequently, we must have equality throughout this inequality chain. In particular, $|S| = \gamma_{l2}(T) = 2k + 1$, implying that $S$ is a $\gamma_{l2}(T)$-set.
Suppose that there exists a $\gamma_{\ell 2}(T)$-set, $D'$, that does not contain $v$. In this case, $v \in D'$. Further, in order to semitotally dominate $u$, we note that $|(D' \setminus \{u\}) \cap N(v)| \geq 1$. This, however, implies that along one of $P_v$’s attached to $v$ in $T$, at least three of its vertices belong to $D'$, which in turn implies that $|D'| \geq 2k + 2 > |S|$, a contradiction. Hence, $v \in \mathcal{A}_{\ell 2}(T)$. As observed earlier, if $vv_1v_2v_3v_4v_5$ is a path emanating from $v$ in $T$, then $|D \cap \{v_2, v_3, v_4, v_5\}| \geq 2$. Further, since $v \in \mathcal{A}_{\ell 2}(T)$, we note that $v \in D$. Thus if $|D \cap N(v)| \geq 1$, then $\gamma_{\ell 2}(T) = |D| \geq 2k + 2$, a contradiction. Therefore, $N(v) \cap D = \emptyset$, implying that $N(v) \subseteq \mathcal{N}_{\ell 2}(T)$. Thus, $v$ is an essential support vertex in $T$. 

By our earlier assumptions, $|L_1^1(v)| \leq 1$ and $L(v) \neq L_0^0(v)$. By Claim C, we may assume that $L(v) \neq L_0^0(v) \cup L_1^1(v)$.

**Claim D.** Suppose $|L^3(v)| \geq 1$. Then the following hold.

(a) If $|L^3(v)| \geq 2$, then $v \in \mathcal{A}_{\ell 2}(T)$.

(b) If $|L^3(v)| = 1$ and $|L^1(v)| \geq 1$, then $v \in \mathcal{A}_{\ell 2}(T)$.

(c) If $|L^3(v)| = 1$, $L^1(v) = \emptyset$ and $|L_0^0(v) \cup L^2(v) \cup L^4(v)| \geq 1$, then $v \notin \mathcal{A}_{\ell 2}(T) \cup \mathcal{N}_{\ell 2}(T)$.

**Proof.** (a) Suppose $|L^3(v)| \geq 2$. Let $\{u_3, v_3\} \subseteq L^3(v)$ and let $vu_1u_2u_3$ and $vv_1v_2v_3$ be the $(v, u_3)$-path and the $(v, v_3)$-path. By our earlier assumptions, the vertex $v$ has at most one leaf-neighbor. Further, we remark that there may exist leaves at distance 2, 4, 5 and 6 from $v$ in $T$. The set $S(T) \cup C^{(4)}(v) \cup \text{Gr}(v) \cup W \cup \{v\}$ is a semi-TD-set of cardinality $2(|L_0^0(v)| + |L_2^2(v)| + |L^4(v)|) + |L^2(v)| + |L^3(v)| + 1$, and so $\gamma_{\ell 2}(T) \leq 2(|L_0^0(v)| + |L_2^2(v)| + |L^4(v)|) + |L^2(v)| + |L^3(v)| + 1$.

Suppose $D$ does not contain $v$. Then, $D$ will contain at least two vertices from each $L_0^0(v)$-path, at least three vertices from each $L_3^3(v)$-path, at least one vertex from each $L_2^2(v)$-path, at least two vertices from each $L_4^4(v)$-path, and at least two vertices from each $L_4^4(v)$-path. Further, if $|L_1^1(v)| = 1$, then $D$ contains the leaf-neighbor of $v$. If $u_3 \in D$, we can replace $u_3$ in $D$ with $u_2$. Hence, we may choose $D$ so that $D \cap \{u_1, u_2, u_3\} = \{u_1, u_2\}$. This implies that $\gamma_{\ell 2}(T) = |D| \geq 2(|L_0^0(v)| + |L_3^3(v)| + |L_2^2(v)| + |L^4(v)| + 2L^3(v) + |L_1^1(v)| \geq 2(|L_0^0(v)| + L_2^2(v) + L^3(v) + |L^3(v)| + |L^2(v)| + |L^4(v)|) + 2L^3(v) + 2L^2(v) > 2(|L_0^0(v)| + |L_2^2(v)| + |L^4(v)|) + |L^2(v)| + |L^3(v)| + 1$, a contradiction. Hence, $v \in D$. Since $D$ is an arbitrary $\gamma_{\ell 2}(T)$-set, we deduce that $v \in \mathcal{A}_{\ell 2}(T)$.

(b) Suppose that $|L^3(v)| = 1$ and $|L^1(v)| \geq 1$. Let $L^3(v) = \{u_3\}$ and let $vu_1u_2u_3$ be the $(v, u_3)$-path. Suppose firstly that $L_1^1(v) = \emptyset$, and so $L^1(v) = L_2^2(v)$. In this case, the set $S(T) \cup C^{(4)}(v) \cup \text{Gr}(v) \cup W \cup \{v\}$ is a semi-TD-set of cardinality $2(|L_0^0(v)| + |L^1(v)| + |L^4(v)|) + |L^2(v)| + 2$, and so $\gamma_{\ell 2}(T) \leq 2(|L_0^0(v)| + |L^1(v)| + |L^4(v)|) + |L^2(v)| + 2$. Suppose $D$ does not contain $v$. Then, $D$ contains at least two vertices on the path $u_1u_2u_3$ and at least three vertices from
each \(L_2^1(v)\)-path. Further, \(D\) contains at least two vertices from each \(L^0(v)\)-path, two vertices from each \(L^4(v)\)-path and one vertex from each \(L^2(v)\)-path. However, this implies that \(\gamma_2(T) = |D| \geq 2(|L^0(v)| + |L^4(v)|) + 3|L_1^1(v)| + |L^2(v)| + 2 > 2(|L^0(v)| + |L^1(v)| + |L^4(v)|) + |L^2(v)| + 2\), a contradiction. Hence, \(v \in D\), and since \(D\) is an arbitrary \(\gamma_2(T)\)-set, \(v \in A_2(T)\).

Suppose secondly that \(|L_1^1(v)| = 1\). Let \(L_1^1(v) = \{u\}\). In this case, the set \(S(T) \cup C^4(v) \cup \text{Gr}(v) \cup W \cup \{v\}\) is a semi-TD-set of cardinality \(2(|L^0(v)| + |L_1^1(v)| + |L^4(v)|) + |L^2(v)| + 2\), and so \(\gamma_2(T) \leq 2(|L^0(v)| + |L_1^1(v)| + |L^4(v)|) + |L^2(v)| + 2\). Suppose \(D\) does not contain \(v\). Then, \(u \in D\) and \(D\) contains at least two vertices on the path \(u_1u_2u_3\) and at least three vertices from each \(L_2^1(v)\)-path. The number of vertices needed from each \(L^0(v)\)-path, \(L^2(v)\)-path and \(L^4(v)\)-path remains unchanged. However, this implies that \(\gamma_2(T) = |D| \geq 2(|L^0(v)| + |L^4(v)|) + 3L_2^1(v) + |L^2(v)| + 2L_4^4(v) + |L_1^1(v)| = 2(|L^0(v)| + |L^4(v)|) + 3L_2^1(v) + |L^2(v)| + 3 > 2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 2\), a contradiction. Hence, \(v \in D\), and since \(D\) is an arbitrary \(\gamma_2(T)\)-set, \(v \in A_2(T)\).

(c) Suppose that \(|L^3(v)| = 1\), \(L^1(v) = \emptyset\) and \(|L^0(v) \cup L^2(v) \cup L^4(v)| \geq 1\). Let \(L^3(v) = \{u_3\}\) and let \(v u_1u_2u_3\) be the \((v, u_3)\)-path. Every leaf of \(T\), different from \(u_3\), is at distance 2, 4 or 5 from \(v\), and so \(L(v) \setminus \{u_3\} = L^0(v) \cup L^2(v) \cup L^4(v)\). By Observation 5, there is a \(\gamma_2(T)\)-set, say \(D'\), that contains no leaf of \(T\), implying that \(S(T) \subseteq D'\). The set \(D'\) contains at least two vertices from each \(L^0(v)\)-path and at least two vertices from each \(L^4(v)\)-path. Further, \(D'\) contains at least one vertex from each \(L^2(v)\)-path and at least two vertices from the \((v, u_3)\)-path. This implies that \(\gamma_2(T) \geq 2(|L^0(v)| + |L^4(v)|) + |L^2(v)| + 2\). On the other hand, the set of children of \(v\) that do not belong to any \(L^0(v)\)-path, together with the set \(S(T) \cup \text{Gr}(v)\) form a semi-TD-set, say \(S\), of \(T\) of cardinality \(2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 1\), and so \(\gamma_2(T) \leq 2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 1\). Consequently, \(\gamma_2(T) = 2(|L^0(v)| + |L^4(v)|) + |L^2(v)| + 2\). Moreover, \(S\) and \((S \setminus \{u_1\}) \cup \{v\}\) are \(\gamma_2(T)\)-sets, implying that \(v \notin A_2(T) \cap N_2(T)\).

By Claim D, we may assume that \(L^3(v) = \emptyset\).

**Claim E.** If \(|L^1(v)| \geq 3\), then \(v \in A_2(T)\).

**Proof.** Suppose, firstly, that \(|L^0(v) \cup L^2(v) \cup L^4(v)| \neq \emptyset\). The vertex set \(S(T) \cup C^4(v) \cup \text{Gr}(v) \cup W \cup \{v\}\) is a semi-TD-set of cardinality \(2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 1\), and so \(\gamma_2(T) \leq 2(|L^0(v)| + |L_2^1(v)| + |L^4(v)|) + |L^2(v)| + 1\). Suppose \(D\) does not contain \(v\). If \(L_1^1(v) = \emptyset\), then every leaf is at distance 2, 4, 5 or 6 from \(v\) in \(T\) and \(L^4(v) = L_2^1(v)\). In this case, \(D\) contains at least three vertices from each \(L_2^1(v)\)-path, two vertices from each \(L^0(v)\)-path, two vertices from each \(L^4(v)\)-path and one vertex from each \(L^2(v)\)-path. Hence, \(\gamma_2(T) = |D| > 3|L^1(v)| + 2(|L^0(v) + L^4(v)|) + |L^2(v)| > 2(|L^0(v)| + |L^1(v)| + |L^4(v)|) + |L^2(v)| + 1\), a contradiction. Therefore, \(L_1^1(v) \neq \emptyset\). Let \(L_1^1(v) = \{u\}\). Every leaf
is at distance 1, 2, 4, 5 or 6 from \( v \) in \( T \). In this case, \( D \) contains the leaf \( u \), implying that \( \gamma_2(T) = |D| > 3|L_1^0(v)| + 2(|L_0^0(v)| + |L_1^1(v)| + |L_1^2(v)| + |L_1^4(v)| > 2(|L_0^0(v)| + |L_1^1(v)| + |L_1^2(v)| + |L_1^4(v)| + 1, a contradiction. Hence, \( v \in D \), and since \( D \) is an arbitrary \( \gamma_2(T) \)-set, \( v \in A_{\gamma_2(T)} \).

Suppose, secondly, that \( L_0^0(v) \cup L_1^2(v) \cup L_4^0(v) = \emptyset \). Thus, \( L(v) = L_1^1(v) \). Let \( u_6 \in L_1^1(v) \) and let \( vu_1u_2u_3u_4u_5u_6 \) be the \((v, u_6)\)-path. The vertex set \( S(T) \cup W \cup \{u_1, v\} \) is a semi-TD-set of cardinality \( 2|L_2^1(v)| + 2 \), and so \( \gamma_2(T) \leq 2|L_2^1(v)| + 2 \). Suppose \( D \) does not contain \( v \). If \( L_1^1(v) = \emptyset \), then every leaf is at distance 6 from \( v \) in \( T \) and \( L(v) = L_1^1(v) = L_2^1(v) \). In this case, \( D \) contains at least three vertices from each \( L_1^1(v) \)-path. Hence, \( \gamma_2(T) = |D| \geq 3|L_1^1(v)| > 2|L_1^1(v)| + 2 \), a contradiction. If \( L_1^1(v) \neq \emptyset \), then letting \( L_1^1(v) = \{u\} \), every leaf in \( L_1^1(v) \) is at distance 6 from \( v \) in \( T \). In this case, \( D \) contains at least three vertices from each \( L_1^1(v) \)-path and the leaf \( u \). Hence, \( \gamma_2(T) = |D| \geq 3|L_2^1(v)| + 1 > 2|L_1^1(v)| + 2 \), a contradiction. Hence, \( v \in D \), and since \( D \) is an arbitrary \( \gamma_2(T) \)-set, \( v \in A_{\gamma_2(T)} \). 

By Claim E, we may assume that \( |L_1^1(v)| \leq 2 \).

**Claim F.** Suppose \( |L_1^1(v)| = 2 \). Then the following hold.

(a) If \( |L_0^0(v) \cup L_2^1(v) \cup L_4^0(v)| \geq 1 \), then \( v \in A_{\gamma_2(T)} \).

(b) If \( L_0^0(v) = L_2^1(v) = L_4^0(v) = \emptyset \), then \( v \notin A_{\gamma_2(T)} \cup N_{\gamma_2(T)} \).

**Proof.** (a) Suppose \( |L_0^0(v) \cup L_2^1(v) \cup L_4^0(v)| \geq 1 \). The vertex set \( S(T) \cup C(v) \cup Gr(v) \cup W \cup \{v\} \) is a semi-TD-set of cardinality \( 2(|L_0^0(v)| + |L_2^1(v)| + |L_4^0(v)|) + |L_2^2(v)| + 1 \), and so \( \gamma_2(T) \leq 2(|L_0^0(v)| + |L_2^1(v)| + |L_4^0(v)| + |L_2^2(v)| + 1 \). Suppose \( D \) does not contain \( v \). If \( L_1^1(v) = \emptyset \), then \( L^1(v) = L_2^1(v) \) and \( |L_2^1(v)| = 2 \). In this case, \( D \) contains at least three vertices from each \( L_2^1(v) \)-path, two vertices from each \( L_0^0(v) \)-path, two vertices from each \( L_2^1(v) \)-path and one vertex from each \( L_2^1(v) \)-path. Hence, \( \gamma_2(T) = |D| \geq 2(|L_0^0(v) + |L_2^1(v)|) + |L_2^2(v)| + 6 > 2(|L_0^0(v) + |L_2^1(v)|) + |L_2^2(v)| + 1 = 2(|L_0^0(v)| + |L_2^1(v)| + |L_4^0(v)| + |L_2^2(v)| + 1 \), a contradiction. Therefore, \( L_1^1(v) \neq \emptyset \). Let \( L_1^1(v) = \{u\} \) and let \( L_2^1(v) = \{u_6\} \). Additionally, let \( vu_1u_2u_3u_4u_5u_6 \) be the \((v, u_6)\)-path. In this case, \( D \) contains the leaf \( u \) and at least three vertices from the \( \{u_1, u_6\} \)-path, at least one vertex from each \( L_2^1(v) \)-path and at least two vertices from each \( L_0^0(v) \)-path and \( L_2^1(v) \)-path, implying that \( \gamma_2(T) = |D| \geq 2(|L_0^0(v) + |L_4^0(v)|) + |L_2^2(v)| + 4 > 2(|L_0^0(v)| + |L_2^1(v)| + |L_4^0(v)| + |L_2^1(v)| + |L_2^2(v)| + 1 \), a contradiction. Hence, \( v \in D \), and since \( D \) is an arbitrary \( \gamma_2(T) \)-set, \( v \in A_{\gamma_2(T)} \).

(b) Suppose \( L_0^0(v) = L_2^1(v) = L_4^0(v) = \emptyset \). Let \( u_6 \in L_2^1(v) \) and let the path \( vu_1u_2u_3u_4u_5u_6 \) be the \((v, u_6)\)-path. Suppose firstly that \( L_1^1(v) = \emptyset \). Then, \( L^1(v) = L_2^1(v) \). Let \( v_6 \in L_2^1(v) \) \( \{u_6\} \) and let \( vu_1u_2u_3u_4u_5v_6 \) be the \((v, v_6)\)-path. In this case, \( T = P_{13} \) and \( \gamma_2(T) = 6 \). Further, the set \( S = \{u_1, u_3, u_5, v_1, v_3, v_5\} \)
is a $\gamma_2(T)$-set not containing $v$, while $(S \setminus \{u_1\}) \cup \{v\}$ is a $\gamma_2(T)$-set containing $v$. Hence, $v \notin A_2(T) \cup N_2(T)$. Suppose secondly that $L_1^1(v) \neq \emptyset$ and let $L_1^1(v) = \{u\}$. In this case, $T = P_8$ and $\gamma_2(T) = 4$. Further, the set $S = \{u, u_1, u_3, u_5\}$ is a $\gamma_2(T)$-set not containing $v$. Moreover, $(S \setminus \{u_1\}) \cup \{v\}$ is a $\gamma_2(T)$-set containing $v$. Hence, once again $v \notin A_2(T) \cup N_2(T)$. 

By Claim F, we may assume that $|L_1^1(v)| \leq 1$.

Claim G. If $|L_1^1(v)| = 1$, then $v \notin A_2(T) \cup N_2(T)$.

Proof. Suppose firstly that $L_2^2(v) = L_4^4(v) = \emptyset$. By our earlier assumptions, the vertex $v$ is not a leaf in $T$, $L_3^3(v) = \emptyset$ and $L_2(v) \neq L_0^0(v) \cup L_1^1(v)$, implying that $|L_0^0(v)| \geq 1$ and $L_2(v) = L_2^2(v)$. Let $L_2^1(v) = \{u_6\}$ and let $vu_1u_2u_3u_4u_5u_6$ be the $(v, u_6)$-path. Every semi-TD-set of $T$ contains at least two vertices from each $L_0^0(v)$-path and at least three vertices from the $(v, u_6)$-path, and so $\gamma_2(T) \geq 2|L_0^0(v)| + 3$. However, the set $S = S(T) \cup Gr(v) \cup \{v, u_3\}$ is a semi-TD-set of $T$ of cardinality $2|L_0^0(v)| + 3$, and so $\gamma_2(T) \leq |S| = 2|L_0^0(v)| + 3$. Consequently, $\gamma_2(T) = 2|L_0^0(v)| + 3$ and $S$ is a $\gamma_2(T)$-set containing $v$. Moreover, $S' = (S \setminus \{v\}) \cup \{u_1\}$ is a $\gamma_2(T)$-set containing $v$. Hence, $v \notin A_2(T) \cup N_2(T)$.

Suppose secondly that $|L_2^2(v) \cup L_4^4(v)| \geq 1$. The vertex set $S = S(T) \cup C_4^4(v) \cup Gr(v) \cup W \cup \{v\}$ is a semi-TD-set of $T$ of cardinality $2(|L_0^0(v)| + |L_1^1(v)| + |L_4^4(v)| + |L_2^2(v)|) + 1$, and so $\gamma_2(T) \leq 2(|L_0^0(v)| + |L_1^1(v)| + |L_4^4(v)|) + |L_2^2(v)| + 1$. Suppose $L_1^1(v) = \emptyset$, and so $L_1^1(v) = L_2^2(v)$ and $L_1^1(v) = 1$. In this case, let $L_1^1(v) = \{u_6\}$ and let $vu_1u_2u_3u_4u_5u_6$ be the $(v, u_6)$-path. The set $D$ contains at least three vertices from the $(v, u_6)$-path, at least one vertex from each $L_2^2(v)$-path and at least two vertices from each $L_0^0(v)$-path and $L_4^4(v)$-path, implying that $\gamma_2(T) = |D| \geq 2(|L_0^0(v)| + |L_4^4(v)|) + |L_2^2(v)| + 1 = 2(|L_0^0(v)| + |L_2^2(v)| + |L_4^4(v)|) + |L_2^2(v)| + 1$. Consequently, $\gamma_2(T) = 2(|L_0^0(v)| + |L_4^4(v)|) + |L_2^2(v)| + 3$ and $S$ is a $\gamma_2(T)$-set containing $v$. Moreover, the set $(S \setminus \{v\}) \cup \{u_1\}$ is a $\gamma_2(T)$-set that does not contain $v$. Hence, $v \notin A_2(T) \cup N_2(T)$. Suppose next that $L_1^1(v) = L_2^2(v) = \{u\}$. In this case, $L_2^2(v) = \emptyset$ and the set $D$ contains at least one of $u$ and $v$, at least one vertex from each $L_2^2(v)$-path and at least two vertices from each $L_0^0(v)$-path and $L_4^4(v)$-path, implying that $\gamma_2(T) = |D| \geq 2(|L_0^0(v)| + |L_4^4(v)|) + |L_2^2(v)| + 1 = 2(|L_0^0(v)| + |L_2^2(v)| + |L_4^4(v)|) + |L_2^2(v)| + 1$. Consequently, $\gamma_2(T) = 2(|L_0^0(v)| + |L_4^4(v)|) + |L_2^2(v)| + 1$ and $S$ is a $\gamma_2(T)$-set containing $v$. Moreover, the set $(S \setminus \{v\}) \cup \{u\}$ is a $\gamma_2(T)$-set that does not contain $v$. Hence, once again $v \notin A_2(T) \cup N_2(T)$.

By Claim G, we may assume that $L_1^1(v) = \emptyset$.

Claim H. Suppose $L_1^1(v) = \emptyset$. Then the following hold.

(a) If $|L_4^4(v)| \geq 1$, then $v \in N_2(T)$.

(b) If $|L_2^2(v)| = 1$ and $L_4^4(v) = \emptyset$, then $v \notin A_2(T) \cup N_2(T)$. 


(c) If $|L^2(v)| \geq 2$ and $L^4(v) = \emptyset$, then $v \in N_{t2}(T)$.

**Proof.** (a) Suppose $|L^4(v)| \geq 1$. Every leaf is at distance 2, 4 or 5 from $v$ in $T$. The set $D$ contains at least one vertex from each $L^2(v)$-path and at least two vertices from each $L^0(v)$-path and each $L^4(v)$-path. Thus, $\gamma_{t2}(T) = |D| \geq 2(|L^0(v)| + |L^4(v)|) + |L^2(v)|$ with strict inequality if $v \in D$. The set $C^{(4)}(v) \cup S(T) \cup \text{Gr}(v)$ is a semi-TD-set of $T$ of cardinality $2(|L^0(v)| + |L^4(v)|) + |L^2(v)|$, and so $\gamma_{t2}(T) \leq 2(|L^0(v)| + |L^4(v)|) + |L^2(v)|$. Consequently, $\gamma_{t2}(T) = |D| = 2(|L^0(v)| + |L^4(v)|) + |L^2(v)|$ and $v \notin D$. Since $D$ is an arbitrary $\gamma_{t2}(T)$-set, $v \in N_{t2}(T)$.

(b) Suppose $|L^2(v)| = 1$ and $L^4(v) = \emptyset$. Let $L^2(v) = \{u_2\}$ and let $uu_1u_2$ be the $(v, u_2)$-path. Then, $L^0(v) = L(v) \setminus \{u_2\}$ and $S = S(T) \cup \text{Gr}(v) \cup \{v, u_1\}$ is a semi-TD-set of cardinality $2|L^0(v)| + 2$, and so $\gamma_{t2}(T) \leq |S| = 2|L^0(v)| + 2$. The set $D$ contains at least two vertices from the set $N[v] \setminus \{u_2\}$ and at least two vertices not in $N[v]$ from each $L^0(v)$-path. Thus, $\gamma_{t2}(T) = |D| \geq 2|L^0(v)| + 2$. Consequently, $\gamma_{t2}(T) = 2|L^0(v)| + 2$ and $S$ is a $\gamma_{t2}(T)$-set that contains the vertex $v$. Moreover, the set $(S \setminus \{v\}) \cup \{u_2\}$ is a $\gamma_{t2}(T)$-set that does not contain $v$. Hence, $v \notin A_{t2}(T) \cup N_{t2}(T)$.

(c) Suppose that $|L^2(v)| \geq 2$ and $L^4(v) = \emptyset$. Every leaf is at distance 2 or 5 from $v$ in $T$. The set $D$ contains at least one vertex from each $L^2(v)$-path and at least two vertices from each $L^0(v)$-path. Thus, $\gamma_{t2}(T) = |D| \geq 2|L^0(v)| + |L^2(v)|$ with strict inequality if $v \in D$. The set $S(T) \cup \text{Gr}(v)$ is a semi-TD-set of cardinality $2|L^0(v)| + |L^2(v)|$, and so $\gamma_{t2}(T) \leq 2|L^0(v)| + |L^2(v)|$. Consequently, $\gamma_{t2}(T) = |D| = 2|L^0(v)| + |L^2(v)|$ and $v \notin D$. Since $D$ is an arbitrary $\gamma_{t2}(T)$-set, $v \in N_{t2}(T)$.

Theorem 1 now follows from Claims A, B, C, D, E, F, G and H.

6. **Proof of Theorem 2**

Let $T$ be a rooted tree that is not a star with root $v$ that contains at least one branch vertex different from $v$. We shall adopt the following notation. Let $u$ be a branch vertex at maximum distance from $v$ and let $k_0 = |L^0(u)|$, $k_1 = |L^1(u)|$, $k_2 = |L^2(u)|$, $k_3 = |L^3(u)|$ and $k_4 = |L^4(u)|$. Let $w$ be the parent of $u$ (possibly, $v = w$). Let $T'$ be the tree obtained from $T$ by applying the following operations.

$O_1$: For $k_3 \geq 1$, let $T'$ be the tree obtained from $T$ by deleting $D(u)$ and attaching a path $P_3$ to $u$.

$O_2$: For $k_3 = 0$, $k_1 \geq 1$ and $k_0 + k_2 + k_4 \geq 1$, let $T'$ be the tree obtained from $T$ by deleting $D(u)$ and attaching a path $P_3$ to $u$. 

$O_3$: For $k_0 = k_2 = k_3 = k_4 = 0$ and $k_1 \geq 2$, let $T'$ be the tree obtained from $T$ by deleting $D(u)$ and attaching a path $P_1$ to $u$.

$O_4$: For $k_1 = k_3 = 0$ and $k_4 \geq 1$, let $T'$ be the tree obtained from $T$ by deleting $D(u)$ and attaching a path $P_4$ to $u$.

$O_5$: For $k_1 = k_3 = k_4 = 0$, $k_2 = 1$ and $k_0 \geq 1$, let $T'$ be the tree obtained from $T$ by deleting $D(u)$ and attaching a path $P_5$ to $u$.

$O_6$: For $k_1 = k_3 = k_4 = 0$ and $k_2 \geq 2$, let $T'$ be the tree obtained from $T$ by deleting $D(u)$ and attaching a path $P_6$ to $u$.

$O_7$: For $k_1 = k_2 = k_3 = k_4 = 0$, let $T'$ be the tree obtained from $T$ by deleting $D(u)$ and attaching a path $P_7$ to $u$.

Our next result, namely Theorem 2, establishes a key result relating the semitotal domination numbers of the trees $T$ and $T'$. Theorem 2 follows immediately from Theorem 1 and Theorem 8. We use the standard notation $[k] = \{1, 2, \ldots, k\}$ once again.

**Theorem 8.** Let $T$ be a tree with order at least 4 that is not a star and is rooted at a vertex $v$ such that $T$ contains at least one branch vertex $u$ different from $v$ and let $T'$ be the tree defined immediately before the statement of the theorem. Let $w$ be the parent of $u$ (possibly, $w = v$). Suppose that $T'$ is obtained from $T$ by applying operation $O_i$ for some $i \in [7]$. Then,

$$\gamma_{12}(T') = \begin{cases} 
\gamma_{12}(T) - 2k_0 - k_2 - k_3 - 2k_4 + 1 & \text{for } i = 1, \\
\gamma_{12}(T) - 2k_0 - k_2 - 2k_4 + 1 & \text{for } i = 2, \\
\gamma_{12}(T) & \text{for } i = 3, \\
\gamma_{12}(T) - 2k_0 - k_2 - 2k_4 + 2 & \text{for } i = 4, \\
\gamma_{12}(T) - 2k_0 & \text{for } i = 5, \\
\gamma_{12}(T) - 2k_0 - k_2 + 2 & \text{for } i = 6, \\
\gamma_{12}(T) - 2k_0 + 2 & \text{for } i = 7.
\end{cases}$$

Further, in all cases, the following properties $P_A$ and $P_N$ hold:

$P_A$: $v \in A_{12}(T)$ if and only if $v \in A_{12}(T')$.

$P_N$: $v \in N_{12}(T)$ if and only if $v \in N_{12}(T')$.

**Proof.** For each vertex $x \in L(u)$ replace the $(u, x)$-path in $T$ with a $(u, x)$-path of length $j$, where $j \in \{5, 1, 2, 3, 4\}$ if $x \in L^i(u)$ when $i \in \{0, 1, 2, 3, 4\}$, respectively. Let $T''$ denote the resulting tree. By repeated applications of Lemma 7, we deduce that $v \in A_{12}(T)$ ($N_{12}(T)$, respectively) if and only if $v \in A_{12}(T'')$ ($N_{12}(T'')$, respectively). Hence, we assume $T = T''$. With this assumption, every leaf of $T$ that is a descendant of $u$ is within distance 5 from $u$. We proceed further with a series of five claims.
Claim I. Suppose \( k_3 \geq 1 \). Then, \( T' \) is obtained from \( T \) by operation \( O_1 \) and \( \gamma_{L_2}(T') = \gamma_{L_2}(T) - 2k_0 - k_2 - k_3 - 2k_4 + 1 \) and properties \( P_A \) and \( P_N \) hold.

**Proof.** Suppose \( k_3 \geq 1 \). Thus, \( T' \) is obtained from \( T \) by operation \( O_1 \). Let \( u_3 \in L^3(u) \) and let \( u_1u_2u_3 \) be the \((u, u_3)\)-path. Renaming vertices, if necessary, we may assume that \( T' = T - (D(u) \setminus \{u_1, u_2, u_3\}) \). Let \( H = T[D(u) \setminus \{u_1, u_2, u_3\}] \) and let \( X_H = (S(T) \cup \text{Gr}(u) \cup C(4)(u)) \cap V(H) \). We note that \( |X_H| = 2k_0 + k_2 + k_3 + 2k_4 + 1 \). By Observation 5, there exists a \( \gamma_{L_2}(T') \)-set \( S \) that contains the vertex \( u_2 \). If \( u_1 \in S \), then we can replace \( u_1 \) in \( S \) with \( u \). Thus, we may assume \( S \cap \{u, u_1, u_2, u_3\} = \{u, u_2\} \). The set \( S \) can be extended to a semi-TD-set of \( T \) by adding to it the set \( X_H \), implying that \( \gamma_{L_2}(T) \leq |S| + |X_H| = \gamma_{L_2}(T') + |X_H| \).

Conversely, let \( D \) be a \( \gamma_{L_2}(T) \)-set and let \( D_u = D \cap D(u) \). The set \( D \) contains at least two vertices from each \( L^0(u) \)-path and \( L^1(u) \)-path, and at least one vertex from each \( L^2(u) \)-path and \( L^3(u) \)-path, implying that \( |D_u| \geq 2k_0 + k_2 + k_3 + 2k_4 = |X_H| + 1 \). By Observation 5, we can choose \( D \) so that \( S(T) \subseteq D \). In particular, \( u_2 \in D \). If \( u_1 \in D \), then we can replace \( u_1 \) in \( D \) with \( u \). Hence, we may assume that \( D \cap \{u, u_1, u_2, u_3\} = \{u, u_2\} \), implying that \( D \cap V(T') = (D \setminus D_u) \cup \{u_2\} \) is a semi-TD-set of \( T \). Therefore, \( \gamma_{L_2}(T) \leq |D| - |D_u| + 1 \leq |D| - (|X_H| + 1) + 1 = |D| - |X_H| = \gamma_{L_2}(T) - |X_H| \). Consequently, \( \gamma_{L_2}(T) = \gamma_{L_2}(T') + |X_H| = \gamma_{L_2}(T') + 2k_0 + k_2 + k_3 + 2k_4 + 1 \).

Suppose \( v \notin A_{L_2}(T') \) and let \( S' \) be a \( \gamma_{L_2}(T') \)-set that does not contain the vertex \( v \). If \( u_3 \in S' \), then we can replace \( u_3 \) in \( S' \) by \( u_2 \). Hence, we may assume that \( u_2 \in S' \). If \( u_1 \in S' \), then we can replace \( u_1 \) in \( S' \) by \( u \). Hence, we may assume that \( S' \cap \{u, u_1, u_2, u_3\} = \{u, u_2\} \). With these assumptions, the set \( S' \cup X_H \) is a semi-TD-set of \( T \) of cardinality \( |S'| + |X_H| = \gamma_{L_2}(T') + |X_H| = \gamma_{L_2}(T) \). Hence, \( S' \cup X_H \) is a \( \gamma_{L_2}(T) \)-set not containing \( v \), implying that \( v \notin A_{L_2}(T) \). Therefore, by contraposition, if \( v \in A_{L_2}(T) \), then \( v \in A_{L_2}(T') \).

Conversely, suppose \( v \in A_{L_2}(T') \). Suppose to the contrary that \( v \notin A_{L_2}(T) \). Let \( D \) be a \( \gamma_{L_2}(T) \)-set that does not contain \( v \). Analogous to our earlier arguments, we can choose such a set \( D \) so that \( D \cap D(u) = X_H \cup \{u, u_2\} \). Therefore, \( D \cap V(T') \) is a \( \gamma_{L_2}(T') \)-set that does not contain \( v \), a contradiction. Hence, if \( v \in A_{L_2}(T') \), then \( v \in A_{L_2}(T) \). Thus, property \( P_A \) holds. Analogous arguments show that property \( P_N \) holds.

By Claim I, we may assume that \( k_3 = 0 \), for otherwise the desired result follows.

Claim J. Suppose \( k_1 \geq 1 \). Then, \( T' \) is obtained from \( T \) by operation \( O_1 \) for some \( i \in \{2, 3\} \) and

\[
\gamma_{L_2}(T') = \begin{cases} 
\gamma_{L_2}(T) - 2k_0 - k_2 - 2k_4 + 1 & \text{for } i = 2, \\
\gamma_{L_2}(T) & \text{for } i = 3.
\end{cases}
\]

Further, the properties \( P_A \) and \( P_N \) hold in both cases.
Proof. Suppose \( k_1 \geq 1 \). Let \( u' \) be a leaf-neighbor of \( u \). We proceed further with a series of two subclaims.

Claim J.1. If \( k_0 + k_2 + k_4 \geq 1 \), then \( \gamma_{12}(T') = \gamma_{12}(T) - 2k_0 - k_2 - 2k_4 + 1 \) and properties \( P_A \) and \( P_{N'} \) hold.

Proof. Suppose \( k_0 + k_2 + k_4 \geq 1 \). Thus, \( T' \) is obtained from \( T \) by operation \( O_2 \). Let \( P : u_1 u_2 u_3 \) be the path \( P_3 \) added to \( T - D(u) \) when constructing \( T' \), where \( u \) is adjacent to \( u_1 \). Let \( H = T[D(u)] \) and let \( X_H = (S(T) \cup \text{Gr}(u) \cup \mathcal{C}(u)) \cap V(H) \).

We note that \( |X_H| = 2k_0 + k_2 + 2k_4 \). By Observation 5 there exists a \( \gamma_{12}(T') \)-set, \( S \), such that \( u_2 \in S \). If \( u_1 \in S \), then we can replace \( u_1 \) in \( D \) with \( u \). Hence, we may assume that \( S \cap \{u, u_1, u_2, u_3\} = \{u, u_2\} \). Since \( k_0 + k_2 + k_4 \geq 1 \), the set \( S \setminus \{u_2\} \) can be extended to a semi-TD-set of \( T \) by adding to it the set \( X_H \), implying that \( \gamma_{12}(T) \leq |S \setminus \{u_2\}| + |X_H| = \gamma_{12}(T') + |X_H| - 1 \).

Conversely, let \( D \) be a \( \gamma_{12}(T) \)-set and let \( D_u = D \cap D(u) \). The set \( D \) contains at least two vertices from each \( L^0(u) \)-path and \( L^4(u) \)-path, and at least one vertex from each \( L^2(u) \)-path, implying that \( |D_u| \geq 2k_0 + k_2 + 2k_4 = |X_H| \). By Observation 5, we can choose \( D \) so that \( S(T) \subseteq D \). In particular, \( u \in D \), implying that \( (D \setminus D_u) \cup \{u_2\} \) is a semi-TD-set of \( T' \), and so \( \gamma_{12}(T') \leq |D| - |D_u| + 1 \). If \( |D_u| > |X_H| \), then \( (D \setminus D_u) \cup X_H \) is a semi-TD-set of \( T \) of cardinality less than \( |D| \), a contradiction. Hence, \( |D_u| = |X_H| \) and \( \gamma_{12}(T') \leq |D| - |D_u| + 1 = \gamma_{12}(T) - |X_H| + 1 \). Consequently, \( \gamma_{12}(T) = \gamma_{12}(T') + |X_H| - 1 = \gamma_{12}(T') + 2k_0 + k_2 + 2k_4 - 1 \).

Suppose \( v \notin A_{12}(T') \) and let \( S' \) be a \( \gamma_{12}(T') \)-set that does not contain the vertex \( v \). If \( u_3 \in S' \), then we can replace \( u_3 \) in \( S' \) by \( u_2 \). Hence, we may assume that \( u_2 \in S' \). If \( u_1 \in S' \), then we can replace \( u_1 \) in \( S' \) by \( u_2 \). Hence, we may assume that \( S' \cap \{u, u_1, u_2, u_3\} = \{u, u_2\} \). With these assumptions, the set \( S = (S' \setminus \{u_2\}) \cup X_H \) is a semi-TD-set of \( T \) of cardinality \( |S'| + |X_H| - 1 = \gamma_{12}(T') + |X_H| - 1 = \gamma_{12}(T) \). Hence, \( S \) is a \( \gamma_{12}(T) \)-set not containing \( v \), implying that \( v \notin A_{12}(T) \). Therefore, by contraposition, if \( v \in A_{12}(T) \), then \( v \in A_{12}(T') \).

Conversely, suppose \( v \in A_{12}(T') \). Suppose to the contrary that \( v \notin A_{12}(T) \). Let \( D \) be a \( \gamma_{12}(T) \)-set that does not contain \( v \). Analogous to our earlier arguments, we can choose such a set \( D \) so that \( D \cap D(u) = X_H \cup \{u\} \). Therefore, \( (D \cap V(T')) \cup \{u_2\} \) is a \( \gamma_{12}(T') \)-set that does not contain \( v \), a contradiction. Hence, if \( v \in A_{12}(T') \), then \( v \in A_{12}(T) \). Thus, property \( P_A \) holds. Analogous arguments show that property \( P_{N'} \) holds.

Claim J.2. If \( k_0 + k_2 + k_4 = 0 \), then \( \gamma_{12}(T') = \gamma_{12}(T) \) and properties \( P_A \) and \( P_{N'} \) hold.

Proof. Since \( k_0 + k_2 + k_4 = 0 \), we have \( k_1 \geq 2 \). Thus, \( T' \) is obtained from \( T \) by operation \( O_3 \). Renaming vertices if necessary, \( T' = T - (D(u) \setminus \{u'\}) \). By assumption, the tree \( T \) is not a star, implying that the tree \( T' \) is not a star. By Observation 5, there exists a \( \gamma_{12}(T') \)-set \( S \) that contains the vertex \( u \) and
Suppose there exists a \( \gamma \)-TD-set \( S \) of \( T \). Let \( v \) be a vertex in \( S \). Hence, we may choose the set \( S \) to be a \( \gamma \)-TD-set not containing \( v \). By Observation 5, we can choose \( S \) to be a \( \gamma \)-TD-set that does not contain \( v \). Consequently, \( \gamma(T) = \gamma_2(T) \).

Suppose \( v \notin A_2(T) \) and let \( S' \) be a \( \gamma_2(T') \)-set that does not contain the vertex \( v \). If \( u' \in S' \) and \( u \in S \), then if \( u \in S \) we replace \( u' \) in \( S \) with a vertex \( x \in N[u] \setminus \{u\} \) such that \( x \neq v \). Hence we may assume that \( u' \notin S' \) (which is possible since \( T' \) is not a star). Thus the set \( S' \) is a \( \gamma_2(T) \)-set not containing \( v \), implying that \( v \notin A_2(T) \). Therefore, by contraposition, if \( v \in A_2(T) \), then \( v \notin A_2(T') \).

Conversely, suppose \( v \in A_2(T') \). Suppose to the contrary that \( v \notin A_2(T) \). Let \( D \) be a \( \gamma_2(T) \)-set that does not contain \( v \). If \( D \) contains a leaf-neighbor \( z \) of \( u \), then if \( u \in D \) we can replace \( z \) in \( D \) with a vertex \( x \in N[u] \setminus \{u\} \) such that \( x \neq v \). Hence, we may choose the set \( D \) so that \( D \cap D[u] = \{u\} \). Therefore, \( D \) is a \( \gamma_2(T) \)-set that does not contain \( v \), a contradiction. Hence, if \( v \in A_2(T') \), then \( v \in A_2(T) \). Thus, property \( P_A \) holds. Analogous arguments show that property \( P_{N'} \) holds.

Claim J follows immediately from Claim J.1 and Claim J.2. □

By Claim J, we may assume that \( k_1 = 0 \), for otherwise the desired result follows.

Claim K. Suppose \( k_4 \geq 1 \). Then, \( T' \) is obtained from \( T \) by operation \( O_4 \) and \( \gamma_2(T') = \gamma_2(T) - 2k_0 - k_2 + 2k_4 + 2 \) and properties \( P_A \) and \( P_{N'} \) hold.

Proof. Suppose \( k_4 \geq 1 \). Thus, \( T' \) is obtained from \( T \) by operation \( O_4 \). By our earlier assumptions, \( k_1 = k_3 = 0 \). Let \( u_4 \in L^4(u) \) and let \( uu_4u_2u_3u_4 \) be the \((u, u_4)\)-path. Renaming vertices if necessary, we may assume that \( T' = T - (D(u) \setminus \{u_1, u_2, u_3, u_4\}) \). Let \( H = T[D(u) \setminus \{u_1, u_2, u_3, u_4\}] \) and let \( X_H = (S(T) \cup C^{(4)}(u) \cup \text{Gr}(u)) \cap V(H) \). We note that \( |X_H| = 2k_0 + k_2 + 2(k_4 - 1) \). By Observation 5, there exists a \( \gamma_2(T') \)-set \( S \) that contains the vertex \( u_3 \). If \( u_2 \in S \), then we can replace \( u_2 \) in \( S \) with \( u_1 \). Thus, we may assume \( S \cap \{u_1, u_2, u_3, u_4\} = \{u_1, u_3\} \). Then the set \( S \) can therefore be extended to a semi-TD-set of \( T \) by adding to it the set \( X_H \), implying that \( \gamma_2(T) \leq |S| + |X_H| = \gamma_2(T') + |X_H| \).

Conversely, let \( D \) be a \( \gamma_2(T) \)-set and let \( D_u = D \cap D(u) \). The set \( D \) contains at least two vertices from each \( L^0(u) \)-path and \( L^4(u) \)-path, and at least one vertex from each \( L^2(u) \)-path, implying that \( |D_u| \geq 2k_0 + k_2 + 2k_4 = |X_H| + 2 \). On the other hand, the set \( (D \setminus D_u) \cup \{u_1, u_3\} \) is a semi-TD-set of \( T' \), and so
Suppose \( S \) can be extended to a semi-TD-set of \( X \). Let \( T \) be obtained from \( S \) by adding to it the set \( \gamma \), that is, \( T = S \cup \gamma \).

Thus, \( S' \) is a semi-TD-set of \( T \). Consequently, \( \gamma \) is a semi-TD-set of \( T \). Hence, \( \gamma \) is a semi-TD-set of \( T \).

Suppose \( v \notin A_2(T') \) and let \( S' \) be a semi-TD-set of \( T \) that does not contain the vertex \( v \). If \( u_4 \in S' \), then we can replace \( u_4 \) in \( S' \) with \( u_3 \). Hence we may assume \( u_4 \notin S' \). If \( u_2 \in S' \), then we can replace \( u_2 \) in \( S' \) with \( u_1 \). Thus, we may assume \( S' \cap \{u_1, u_2, u_3, u_4\} = \{u_1, u_3\} \).

Further, the properties \( P_4 \) and \( P_N \) hold in both cases.

Claim L. Suppose \( k_2 \geq 1 \). Then, \( T' \) is obtained from \( T \) by operation \( O_i \) for some \( i \in \{5, 6\} \) and

\[
\gamma_2(T') = \begin{cases} 
\gamma_2(T) - 2k_0 & \text{for } i = 5, \\
\gamma_2(T) - 2k_0 - k_2 + 2 & \text{for } i = 6.
\end{cases}
\]

Further, the properties \( P_A \) and \( P_N \) hold in both cases.

Proof. Suppose \( k_2 \geq 1 \). Let \( u_2 \in L^2(u) \) and let \( uu_1u_2 \) be the \((u, u_2)\)-path in \( T \). By our earlier assumptions, \( k_1 = k_3 = k_4 = 0 \). We proceed further with a series of two subclaims.

Claim L.1. If \( k_2 = 1 \), then \( \gamma_2(T') = \gamma_2(T) - 2k_0 \) and properties \( P_A \) and \( P_N \) hold.

Proof. Suppose that \( k_2 = 1 \) and hence, \( k_0 \geq 1 \) and \( L^2(u) = \{u_2\} \). Thus, \( T' \) is obtained from \( T \) by operation \( O_5 \). Let \( uu_1u_2 \) be the \((u, u_2)\)-path. Renaming vertices if necessary, \( T' = T - (D(u) \setminus \{u_1, u_2\}) \). Let \( H = T[D(u) \setminus \{u_1, u_2\}] \) and let \( X_H = (S(T) \cup \text{Gr}(u)) \cap V(H) \). We note that \( |X_H| = 2k_0 \). Every \( \gamma_2(T') \)-set \( S \) can be extended to a semi-TD-set of \( T \) by adding to it the set \( X_H \), implying that \( \gamma_2(T) \leq |S| + |X_H| = \gamma_2(T') + |X_H| \).
Conversely, let $D$ be an $\gamma_{t2}(T)$-set and let $D_u = D \cap D(u)$. The set $D_u$ contains at least two vertices from each $L^0(u)$-path and one of the vertices $u_1$ or $u_2$, implying that $|D_u| \geq 2k_0 + 1 = |X_H| + 1$. The set $(D \setminus D_u) \cup \{u_1\}$ is a semi-TD-set of $T'$, and so $\gamma_{t2}(T') \leq |D| - |D_u| + 1 \leq \gamma_{t2}(T) - |X_H|$. Consequently, $\gamma_{t2}(T) = \gamma_{t2}(T') + |X_H| = \gamma_{t2}(T') + 2k_0$.

Suppose $v \notin A_{t2}(T')$ and let $S'$ be a $\gamma_{t2}(T')$-set that does not contain the vertex $v$. Then, the set $S' \cup X_H$ is a $\gamma_{t2}(T)$-set not containing $v$, implying that $v \notin A_{t2}(T)$. Therefore, by contraposition, if $v \in A_{t2}(T)$, then $v \in A_{t2}(T')$.

Conversely, suppose $v \in A_{t2}(T')$. Suppose to the contrary that $v \notin A_{t2}(T)$. Let $D$ be a $\gamma_{t2}(T)$-set that does not contain $v$. Analogous to our earlier arguments, we can choose such a set $D$ so that $D \cap D(u) = X_H \cup \{u_1\}$. Thus, $D \setminus X_H$ is a semi-TD-set of $T'$ of cardinality $|D| - |X_H| = \gamma_{t2}(T) - |X_H| = \gamma_{t2}(T')$. The set $D \setminus X_H$ is therefore a $\gamma_{t2}(T')$-set that does not contain $v$, a contradiction. Hence, if $v \in A_{t2}(T')$, then $v \in A_{t2}(T)$. Thus, property $P_A$ holds. Analogous arguments show that property $P'_A$ holds.

**Claim L.2.** If $k_2 \geq 2$, then $\gamma_{t2}(T') = \gamma_{t2}(T) - 2k_0 - k_2 + 2$ and properties $P_A$ and $P'_A$ hold.

**Proof.** Suppose $k_2 \geq 2$. Thus, $T'$ is obtained from $T$ by operation $O_6$. Let $P : u_1u_2u_3u_4$ be the path $P_2$ added to $T - D(u)$ when constructing $T'$, where $u$ is adjacent to $u_1$. Let $H = T[D(u)]$ and let $X_H = (S(T) \cup Gr(u)) \cap V(H)$. We note that $|X_H| = 2k_0 + k_2$. By Observation 5, there exists a $\gamma_{t2}(T')$-set $S$ that contains the vertex $u_3$. If $u_2 \in S$, then we may replace $u_2$ in $S$ with $u_1$. Hence we may choose $S$ so that $S \setminus \{u_1, u_2, u_3, u_4\} = \{u_1, u_3\}$. The set $S \setminus \{u_1, u_3\}$ can therefore be extended to a semi-TD-set of $T$ by adding to it the set $X_H$, implying that $\gamma_{t2}(T) \leq |S \setminus \{u_1, u_3\}| + |X_H| = \gamma_{t2}(T') + |X_H| - 2$.

Conversely, let $D$ be a $\gamma_{t2}(T)$-set and let $D_u = D \cap D(u)$. The set $D_u$ contains at least two vertices from each $L^0(u)$-path and one vertex from each $L^2(u)$-path, implying that $|D_u| \geq 2k_0 + k_2 = |X_H|$. The set $(D \setminus D_u) \cup \{u_1, u_3\}$ is a semi-TD-set of $T'$, and so $\gamma_{t2}(T') \leq |D| - |D_u| + 2 \leq \gamma_{t2}(T) - |X_H| + 2$. Consequently, $\gamma_{t2}(T) = \gamma_{t2}(T') + |X_H| - 2 = \gamma_{t2}(T') + 2k_0 + k_2 - 2$.

Suppose $v \notin A_{t2}(T')$ and let $S'$ be a $\gamma_{t2}(T')$-set that does not contain the vertex $v$. Analogous to our earlier arguments, we can choose such a set $S'$ so that $S' \cap \{u_1, u_2, u_3, u_4\} = \{u_1, u_3\}$. The set $(S' \setminus \{u_1, u_3\}) \cup X_H$ is a semi-TD-set of cardinality $|S'| + |X_H| - 2 = \gamma_{t2}(T') + |X_H| - 2 = \gamma_{t2}(T)$ and is thus a $\gamma_{t2}(T)$-set not containing $v$, implying that $v \notin A_{t2}(T)$. Therefore, by contraposition, if $v \in A_{t2}(T)$, then $v \in A_{t2}(T')$.

Conversely, suppose $v \in A_{t2}(T')$. Suppose to the contrary that $v \notin A_{t2}(T)$. Let $D$ be a $\gamma_{t2}(T)$-set that does not contain $v$. Analogous to our earlier arguments, we can choose a set $D$ so that $D \cap D(u) = X_H$. Therefore, $(D \setminus X_H) \cup \{u_1, u_3\}$ is a semi-TD-set of cardinality $|D| - |X_H| + 2 = \gamma_{t2}(T) - |X_H| + 2 = \gamma_{t2}(T')$ and is
Thus a $\gamma_2(T')$-set that does not contain $v$, a contradiction. Hence, if $v \in A_2(T')$, then $v \in A_2(T)$. Thus, property $P_A$ holds. Analogous arguments show that property $P_N$ holds.

Claim L follows from Claim L.1 and Claim L.2. This completes the proof of Claim L.

By Claim L, we may assume that $k_2 = 0$, for otherwise the desired result follows. By our earlier assumptions, $k_1 = k_3 = k_4 = 0$. Thus, $L(u) = L^0(u)$. Since $u$ is a branch vertex, $k_0 \geq 2$.

Claim M. Suppose $k_0 \geq 2$. Then, $T'$ is obtained from $T$ by operation $O_7$ and $\gamma_2(T') = \gamma_2(T) - 2k_0 + 2$ and properties $P_A$ and $P_N$ hold.

Proof. Let $\{u_5, v_5\} \subseteq L^0(u)$ and let $u_1u_2u_3u_4u_5$ and $u_1v_2v_3v_4v_5$ be the respective $(u, u_5)$-path and $(u, v_5)$-path in $T$. Thus, $T'$ is obtained from $T$ by operation $O_7$. Renaming vertices if necessary, we may assume that $T' = T - (D(u) \setminus \{u_1, u_2, u_3, u_4, u_5\})$. Let $H = T[D(u) \setminus \{u_1, u_2, u_3, u_4, u_5\}]$ and let $X_H = (S(T) \cup \text{Gr}(u)) \cap V(H)$. We note that $|X_H| = 2k_0 - 2$. Every $\gamma_2(T')$-set can be extended to a semi-TD-set of $T$ by adding to it the set $X_H$, implying that $\gamma_2(T) \leq \gamma_2(T') + |X_H|$. 

Conversely, let $D$ be a $\gamma_2(T)$-set and let $D_u = D \cap D(u)$. The set $D_u$ contains at least two vertices from each $L^0(u)$-path, implying that $|D_u| \geq 2k_0 = |X_H| + 2$. The set $(D \setminus D_u) \cup \{u_2, u_4\}$ is a semi-TD-set of $T'$, and so $\gamma_2(T') \leq |D| - |D_u| + 2 \leq \gamma_2(T) - |X_H|$. Consequently, $\gamma_2(T) = \gamma_2(T') + |X_H| = \gamma_2(T') + 2k_0 - 2$.

Suppose $v \notin A_2(T')$ and let $S'$ be a $\gamma_2(T')$-set that does not contain the vertex $v$. Then, the set $S' \cup X_H$ is a $\gamma_2(T)$-set not containing $v$, implying that $v \notin A_2(T)$. Therefore, by contraposition, if $v \in A_2(T)$, then $v \in A_2(T')$.

Conversely, suppose $v \in A_2(T')$. Suppose to the contrary that $v \notin A_2(T)$. Let $D$ be a $\gamma_2(T)$-set that does not contain $v$ and chosen so that $|D \cap D(u)|$ is a minimum. Let $D_u = D \cap D(u)$. If $|D_u| \geq |X_H| + 3$, then the set $(D \setminus D_u) \cup (X_H \cup \{u, u_2, u_4\})$ is a semi-TD-set of $T$ of cardinality $|D| - |D_u| + |X_H| + 3 \leq |D| = \gamma_2(T)$ and is therefore a $\gamma_2(T)$-set containing fewer vertices of $D(u)$ than does $D$, a contradiction. Hence, $|D_u| \leq |X_H| + 2$. Analogous to our earlier arguments, $|D_u| \geq |X_H| + 2$. Consequently, $|D_u| = |X_H| + 2$ and $(D \setminus D_u) \cup \{u_2, u_4\}$ is a semi-TD-set of $T'$ of cardinality $|D| - |D_u| + 2 = \gamma_2(T) - |X_H| = \gamma_2(T')$. Thus, $(D \setminus D_u) \cup \{u_2, u_4\}$ is a $\gamma_2(T')$-set that does not contain $v$, a contradiction. Hence, if $v \in A_2(T')$, then $v \in A_2(T)$. Thus, property $P_A$ holds. Analogous arguments show that property $P_N$ holds.

Theorem 8 follows from Claims I, J, K, L and M.
References


Appendix

We now present an example to illustrate Theorem 2. Applying our pruning process discussed in Section 2 to the rooted tree $T$ with root $v$ illustrated in Figure 1(a), we proceed as follows.

- The branch vertices $b_3$ and $b_4$ are both at maximum distance 3 from $v$ in $T$. We select $b_3$, where $|L^3(b_3)| = 1$. Thus, $b_3$ is a type-(T.1) branch vertex and we delete $D(b_3)$ and attach a path of length 3 to $b_3$.

- The branch vertex at maximum distance from $v$ in the resulting tree (illustrated in Figure 1(b)) is the vertex $b_4$. Since $|L^1(b_4)| > 2$ and every leaf-descendant of $b_4$ belongs to $L^1(b_4)$, the vertex $b_4$ is therefore a type-(T.3) branch vertex and we delete $D(b_4)$ and attach a path of length 1 to $b_4$. 
The branch vertex at maximum distance from $v$ in the resulting tree (illustrated in Figure 1(c)) is the vertex $b_2$. Since $|L^1(b_2)| = 1$ and $L^1(b_2) = L^3(b_2) = \emptyset$, the vertex $b_2$ is a type-(T.4) branch vertex and we delete $D(b_2)$ and attach a path of length 4 to $b_2$.

The branch vertex at maximum distance from $v$ in the resulting tree (illustrated in Figure 1(d)) is the vertex $b_1$. Since $|L^3(b_1)| = 1$, the vertex $b_1$ is a type-(T.1) branch vertex and we delete $D(b_1)$ and attach a path of length 3 to $b_1$. The resulting pruned tree $T_v$ is illustrated in Figure 1(e).

Since $|\overline{L}^1(v)| = 1$ and $|\overline{L}^3(v)| = 1$, by Theorem 2, we deduce that $v \notin A_{t2}(T) \cup N_{t2}(T)$.

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