Discussiones Mathematicae
Differential Inclusions, Control and Optimization 35 (2015) 105–122
doi:10.7151/dmdico.1172

UPPER AND LOWER SOLUTIONS METHOD FOR PARTIAL HADAMARD FRACTIONAL INTEGRAL EQUATIONS AND INCLUSIONS

Saïd Abbas [a], Eman Alaidarous [c], Wafaa Albarakati [c]

and

Mouffak Benchohra [b,c,1]

[a] *Laboratory of Mathematics, University of Saïda*
P.O. Box 138, 20000 Saïda, Algeria

[b] *Laboratory of Mathematics, University of Sidi Bel-Abbès*
P.O. Box 89, Sidi Bel-Abbès 22000, Algeria

[c] *Department of Mathematics, Faculty of Science, King Abdulaziz University*
P.O. Box 80203, Jeddah 21589, Saudi Arabia

e-mail: abbasmsaid@yahoo.fr
benchohra@univ-sba.dz

Abstract

In this paper we use the upper and lower solutions method combined with Schauder's fixed point theorem and a fixed point theorem for condensing multivalued maps due to Martelli to investigate the existence of solutions for some classes of partial Hadamard fractional integral equations and inclusions.

Keywords: functional integral equation, integral inclusion, Hadamard partial fractional integral, condensing multivalued map, existence, upper solution, lower solution, fixed point.

2010 Mathematics Subject Classification: 34A08, 34K05.

1. Introduction

The fractional calculus deals with extensions of derivatives and integrals to non-integer orders. There has been a significant development in ordinary and partial

[1] Corresponding author.

fractional differential and integral equations in recent years; see the monographs of Abbas *et al.* [7, 8], Kilbas *et al.* [21], Miller and Ross [26], the papers of Abbas *et al.* [1, 9], Benchohra *et al.* [11], Vityuk and Golushkov [30, 31], and the references therein.

The method of upper and lower solutions has been successfully applied to study the existence of solutions for ordinary and partial differential equations and inclusions. See the monographs by Benchohra *et al.* [12], Ladde *et al.* [23], the papers of Abbas *et al.* [2, 3, 4, 5, 6, 9], Pachpatte [27], and the references therein.

In [13], Butzer *et al.* investigate properties of the Hadamard fractional integral and the derivative. In [14], they obtained the Mellin transforms of the Hadamard fractional integral and differential operators and in [28], Pooseh *et al.* obtained expansion formulas of the Hadamard operators in terms of integer order derivatives. Many other interesting properties of those operators and others are summarized in [29] and the references therein.

In the paper, we use the method of upper and lower solutions for the existence of solutions of the following Hadamard partial fractional integral equation

$$u(x,y) = \mu(x,y)$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y\left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t,u(s,t))}{st}dtds;\ \text{if } (x,y)\in J, \tag{1}$$

where $J := [1,a]\times[1,b]$, $a,b>1$, $r_1,r_2>0$, $\mu : J\to\mathbb{R}$, $f : J\times\mathbb{R}\to\mathbb{R}$ are given continuous functions, and $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\zeta) = \int_0^\infty t^{\zeta-1}e^{-t}dt;\ \zeta>0.$$

Next, we we discuss the existence of solutions of the following Hadamard partial fractional integral inclusion

$$u(x,y)-\mu(x,y)\in({}^HI_\sigma^rF)(x,y,u(x,y));\ (x,y)\in J, \tag{2}$$

where $\sigma=(1,1)$, $F : J\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a compact valued multi-valued map, ${}^HI_\sigma^rF$ is the definite Hadamard integral for the set-valued function F of order $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, $\mu : J\to\mathbb{R}$ is a given continuous function, and $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

Our approach is based on a combination of Schauder's fixed point theorem [18] with the concept of upper and lower solutions for the integral equation (1), and on a combination of a fixed-point theorem for condensing multivalued maps due to Martelli [25] with the concept of upper and lower solutions for the integral inclusion (2).

The paper initiates the application of upper and lower solutions method to these new classes of problems.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts which are used throughout the paper. Denote by $L^1(J,\mathbb{R})$ the Banach space of functions $u : J \longrightarrow \mathbb{R}$ that are Lebesgue integrable with norm

$$\|u\|_{L^1} = \int_1^a \int_1^b |\, u(x,y)|\, dxdy.$$

Let $C := C(J,\mathbb{R})$ be the Banach space of continuous functions $u : J \to \mathbb{R}$ with the norm

$$\|u\|_C = \sup_{(x,y)\in J} |\, u(x,y)|.$$

Let (X,d) be a metric space. We use the following notation:

$$\mathcal{P}(X) = \{Y \subset X : Y \text{ is nonempty}\}$$

$$\mathcal{P}_{cp,cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}.$$

A multivalued map $G : X \to \mathcal{P}(X)$ has *convex (closed) values* if $G(x)$ is convex (closed) for all $x \in X$. We say that G is *bounded* on bounded sets if $G(B)$ is bounded in X for each bounded set B of X, i.e.,

$$\sup_{x\in B} \{\sup\{\ \|u\| :\ u \in G(x)\}\} < \infty.$$

G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. Finally, we say that G has a *fixed point* if there exists $x \in X$ such that $x \in G(x)$.

Definition 2.1 [10]**.** An upper semicontinuous map $G : X \to \mathcal{P}(X)$ is said to be condensing, if for any bounded subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompactness.

Remark 2.2. A completely continuous multivalued map is the easiest example of a condensing map.

Definition 2.3. The selection set of a multivalued map $G : J \to \mathcal{P}(\mathbb{R})$ is defined by

$$S_G = \{u \in L^1(J) : u(x,y) \in G(x,y)\ ,\ a.e.\ (x,y) \in J\}.$$

For each $u \in C$, the set $S_{F\circ u}$ known as *the set of selectors* of $F \circ u$ is defined by

$$S_{F\circ u} = \{v \in L^1(J) : v(x,y) \in F(x,y,u(x,y))\ ,\ a.e.\ (x,y) \in J\}.$$

For more details on multivalued maps we refer to the books of Deimling [15], Djebali *et al.* [16], Hu and Papageorgiou [20], Kisielewicz [22], Górniewicz [17].

Definition 2.4. A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called a Carathéodory map if

(i) $(x,y) \longmapsto F(x,y,u)$ is measurable for each $u \in \mathbb{R}$;

(ii) $u \longmapsto F(x,y,u)$ is upper semicontinuous for almost all $(x,y) \in J$.

F is said to be L^1-Carathéodory if (i), (ii) and the following condition hold;

(iii) for each $c > 0$, there exists $\sigma_c \in L^1(J, \mathbb{R}_+)$ such that

$$\begin{aligned}\|F(x,y,u)\|_{\mathcal{P}} &= \sup\{\|f\| : f \in F(x,y,u)\} \\ &\leq \sigma_c(x,y) \text{ for all } |u| \leq c \text{ and for a.e. } (x,y) \in J.\end{aligned}$$

Lemma 2.5 [20]**.** *Let G be a completely continuous multivalued map with non-empty compact values. G is u.s.c. if and only if G has a closed graph.*

Lemma 2.6 [24]**.** *Let X be a Banach space. Let $F : J \times X \longrightarrow \mathcal{P}_{cp,cv}(X)$ be an L^1-Carathéodory multivalued map and let Λ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$\begin{aligned}\Lambda \circ S_{F\circ u} : C(J,X) &\longrightarrow \mathcal{P}_{cp,cv}(C(J,X)), \\ w &\longmapsto (\Lambda \circ S_{F\circ u})(w) := (\Lambda S_{F\circ u})(w)\end{aligned}$$

is a closed graph operator.

Definition 2.7 [19, 21]**.** The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1([1,a], \mathbb{R})$, is defined as

$$({}^H I_1^r g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\log \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds.$$

Example 2.8. The Hadamard fractional integral of order $q > 0$ for the function $w : [1,e] \to \mathbb{R}$, defined by $w(x) = (\log x)^{\beta-1}$ with $\beta > 0$, is

$$({}^H I_1^q w)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+q)} (\log x)^{\beta+q-1}.$$

Definition 2.9. Let $r_1, r_2 \geq 0$, $\sigma = (1,1)$ and $r = (r_1, r_2)$. For $w \in L^1(J)$, we define the Hadamard partial fractional integral of order r by the expression

$$({}^H I_\sigma^r w)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{w(s,t)}{st} dtds.$$

Definition 2.10. Let $G : J \to \mathcal{P}(\mathbb{R})$ and $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be set-valued functions with nonempty values in $\mathbb{R}$. $({}^H I_\sigma^r G)(x,y)$ is defined as

$$({}^H I_\sigma^r G)(x,y) = \left\{ \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{g(s,t)}{st\Gamma(r_1)\Gamma(r_2)} dtds : g \in S_G \right\}$$

is the definite Hadamard integral for the set-valued functions G of order $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$.

Similarly,

$$\begin{aligned} &\left({}^H I_\sigma^r F\right)(x,y,u(x,y)) \\ &= \left\{ \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s,t)}{st\Gamma(r_1)\Gamma(r_2)} dtds : f \in S_{F\circ u} \right\} \end{aligned}$$

is the definite Hadamard integral for the set-valued functions F of order r which is defined as

Theorem 2.11 (Martelli's fixed point theorem) [25]**.** *Let X be a Banach space and $N : X \to \mathcal{P}_{cl,cv}(X)$ be an u. s. c. and condensing map. If the set $\Omega := \{u \in X : \lambda u \in N(u)$ for some $\lambda > 1\}$ is bounded, then N has a fixed point.*

3. EXISTENCE RESULTS FOR PARTIAL HADAMARD FRACTIONAL INTEGRAL EQUATIONS

Let us start by defining what we mean by a solution of the integral equation (1).

Definition 3.1. A function $u \in C$ is said to be a solution of (1) if u satisfies equation (1) on J.

Definition 3.2. A function $z \in C$ is said to be a lower solution of the integral equation (1) if z satisfies

$$z(x,y) \le \mu(x,y) + \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s,t,z(s,t))}{st\Gamma(r_1)\Gamma(r_2)} dtds; \ (x,y) \in J.$$

The function z is said to be an upper solution of (1) if the reversed inequality holds.

Further, we present our main result for the equation (1).

Theorem 3.3. *Assume that the following hypothesis holds:*

(H) *There exist v and $w \in C$, lower and upper solutions for the equation* (1) *and such that $v \leq w$.*

Then the integral equation (1) *has at least one solution u such that*

$$v(x,y) \leq u(x,y) \leq w(x,y) \quad \textit{for all } (x,y) \in J.$$

Proof. Consider the following modified integral equation:
(3)

$$u(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{g(s,t,u(s,t))}{st} dtds,$$

where

$$g(x,y,u(x,y)) = f(x,y,h(x,y,u(x,y))),$$

$$h(x,y,u(x,y)) = \max\{v(x,y), \min\{u(x,y), w(x,y)\}\},$$

for each $(x,y) \in J$.

A solution of (3) is a fixed point of the operator $N : C \to C$ defined by

$$(Nu)(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{g(s,t,u(s,t))}{st} dtds.$$

Notice that g is a continuous function, and from (H) there exists $M > 0$ such that

$$|g(x,y,u)| \leq M, \text{ for each } (x,y) \in J, \text{ and } u \in \mathbb{R}. \tag{4}$$

Set

$$\eta = \|\mu\|_C + \frac{M(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}$$

and

$$D = \{u \in C : \|u\|_C \leq \eta\}.$$

Clearly D is a closed convex subset of C and N maps D into itself. We shall show that N satisfies the assumptions of Schauder fixed point theorem. The proof will be given in several steps.

Step 1. *N is continuous.*

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in D. Then

$$\begin{aligned}
&|(Nu_n)(x,y) - (Nu)(x,y)| \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1-1} \left|\log \frac{y}{t}\right|^{r_2-1} \\
&\quad \times \frac{|g(s,t,u_n(s,t)) - g(s,t,u(s,t))|}{st} dtds \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1-1} \left|\log \frac{y}{t}\right|^{r_2-1} \\
&\quad \times \frac{\sup_{(s,t)\in J} |g(s,t,u_n(s,t)) - g(s,t,u(s,t))|}{st} dtds \\
&\leq \frac{(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \sup_{(s,t)\in J} |g(s,t,u_n(s,t)) - g(s,t,u(s,t))|.
\end{aligned}$$

For each $(x,y) \in J$, set $(g \circ u)(x,y) := g(x,y,u(x,y))$. Thus, we get

$$\begin{aligned}
&|(Nu_n)(x,y) - (Nu)(x,y)| \\
&\leq \frac{(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \sup_{(s,t)\in J} |(g \circ u_n)(s,t) - (g \circ u)(s,t))| \\
&\leq \frac{(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \|g \circ u_n - g \circ u\|_C.
\end{aligned}$$

From the Lebesgue's dominated convergence theorem and the continuity of the function g, we get

$$|(Nu_n)(x,y) - (Nu)(x,y)| \to 0 \text{ as } n \to \infty.$$

Step 2. *$N(D)$ is bounded.*
This is clear since $N(D) \subset D$ and D is bounded.

Step 3. *$N(D)$ is equicontinuous.*
Let $(x_1,y_1),(x_2,y_2) \in (1,a] \times (1,b]$, $x_1 < x_2$, $y_1 < y_2$, and let $u \in D$. Then

$$\begin{aligned}
&|(Nu)(x_2,y_2) - (Nu)(x_1,y_1)| \leq |\mu(x_1,y_1) - \mu(x_2,y_2)| \\
&+ \int_1^{x_1} \int_1^{y_1} \left| \left(\log \frac{x_2}{s}\right)^{r_1-1} \left(\log \frac{y_2}{t}\right)^{r_2-1} - \left(\log \frac{x_1}{s}\right)^{r_1-1} \left(\log \frac{y_1}{t}\right)^{r_2-1} \right| \\
&\times \frac{|g(s,t,u(s,t))|}{st\Gamma(r_1)\Gamma(r_2)} dtds
\end{aligned}$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{|g(s,t,u(s,t))|}{st}dtds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{1}^{x_1}\int_{y_1}^{y_2}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{|g(s,t,u(s,t))|}{st}dtds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{1}^{y_1}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{|g(s,t,u(s,t))|}{st}dtds.$$

Thus

$$|(Nu)(x_2,y_2)-(Nu)(x_1,y_1)|\le|\mu(x_1,y_1)-\mu(x_2,y_2)|$$
$$+\int_1^{x_1}\int_1^{y_1}\left|\left(\log\frac{x_2}{s}\right)^{r_1-1}\left(\log\frac{y_2}{t}\right)^{r_2-1}-\left(\log\frac{x_1}{s}\right)^{r_1-1}\left(\log\frac{y_1}{t}\right)^{r_2-1}\right|$$
$$\times\frac{M}{st\Gamma(r_1)\Gamma(r_2)}dtds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{M}{st}dtds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{1}^{x_1}\int_{y_1}^{y_2}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{M}{st}dtds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{1}^{y_1}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{M}{st}dtds$$
$$\le|\mu(x_1,y_1)-\mu(x_2,y_2)|+\frac{M}{\Gamma(1+r_1)\Gamma(1+r_2)}$$
$$\times[2(\log y_2)^{r_2}(\log x_2-\log x_1)^{r_1}+2(\log x_2)^{r_1}(\log y_2-\log y_1)^{r_2}$$
$$+(\log x_1)^{r_1}(\log y_1)^{r_2}-(\log x_2)^{r_1}(\log y_2)^{r_2}-2(\log x_2-\log x_1)^{r_1}(\log y_2-\log y_1)^{r_2}].$$

As $x_1\to x_2$ and $y_1\to y_2$, the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that N is completely continuous.

Therefore, N has a fixed point u which is a solution of the equation (3).

Step 4. *The solution u of* (3) *satisfies*

$$v(x,y)\le u(x,y)\le w(x,y)\ \text{ for all }\ (x,y)\in J.$$

Let u be the solution of (3). We prove that

$$u(x,y)\le w(x,y)\ \text{ for all }\ (x,y)\in J.$$

Assume that $u-w$ attains a positive maximum on J at $(\overline{x},\overline{y}) \in J$, that is,

$$(u-w)(\overline{x},\overline{y}) = \max\{u(x,y)-w(x,y) : (x,y) \in J\} > 0.$$

We distinguish two cases.

Case 1. If $(\overline{x},\overline{y}) \in (1,a)\times[1,b]$ then, there exists $(x^*,y^*) \in (1,a)\times[1,b]$ such that

$$[u(x,y^*)-w(x,y^*)]+[u(x^*,y)-w(x^*,y)]-[u(x^*,y^*)-w(x^*,y^*)] \leq 0 \tag{5}$$

for all $(x,y) \in ([x^*,\overline{x}]\times\{y^*\})\cup(\{x^*\}\times[y^*,b])$ and

$$u(x,y)-w(x,y) > 0; \text{ for all } (x,y) \in (x^*,\overline{x}]\times(y^*,b]. \tag{6}$$

By the definition of h one has

$$u(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{g(s,t,u(s,t))}{st}dtds \tag{7}$$

for all $(x,y) \in [x^*,\overline{x}]\times[y^*,b]$, where

$$g(x,y,u(x,y)) = f(x,y,w(x,y))); \ (x,y) \in [x^*,\overline{x}]\times[y^*,b].$$

Thus equation (7) gives

$$\begin{aligned} & u(x,y)+u(x^*,y^*)-u(x,y^*)-u(x^*,y) \\ & = \int_{x^*}^x\int_{y^*}^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{g(s,t,u(s,t))}{st\Gamma(r_1)\Gamma(r_2)}dtds. \end{aligned} \tag{8}$$

Since w is an upper solution to (1) then using (8) we get

$$u(x,y)+u(x^*,y^*)-u(x,y^*)-u(x^*,y) \leq w(x,y)+w(x^*,y^*)-w(x,y^*)-w(x^*,y),$$

which gives,

$$\begin{aligned} [u(x,y)-w(x,y)] \leq & \ [u(x,y^*)-w(x,y^*)]+[u(x^*,y)-w(x^*,y)] \\ & - [u(x^*,y^*)-w(x^*,y^*)]. \end{aligned} \tag{9}$$

Thus from (5), (6) and (9) we obtain the contradiction

$$0 < [u(x,y) - w(x,y)] \leq [u(x,y^*) - w(x,y^*)]$$

$$+[u(x^*,y) - w(x^*,y)] - [u(x^*,y^*) - w(x^*,y^*)] \leq 0; \text{ for all } (x,y) \in [x^*,\overline{x}] \times [y^*,b].$$

Case 2. If $\overline{x} = 1$, then

$$w(1,\overline{y}) < u(1,\overline{y}) \leq w(1,\overline{y})$$

which is a contradiction. Thus

$$u(x,y) \leq w(x,y) \text{ for all } (x,y) \in J.$$

Analogously, we can prove that

$$u(x,y) \geq v(x,y), \text{ for all } (x,y) \in J.$$

This shows that the integral equation (3) has a solution u satisfying $v \leq u \leq w$ which is also a solution of (1).

4. EXISTENCE RESULTS FOR PARTIAL HADAMARD FRACTIONAL INTEGRAL INCLUSIONS

Definition 4.1. A function $v \in C$ is said to be a lower solution of (2) if there exists a function $f_1 \in S_{F\circ v}$ such that v satisfies

$$v(x,y) \leq \mu(x,y) + \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f_1(s,t)}{st\Gamma(r_1)\Gamma(r_2)} dtds; \ (x,y) \in J.$$

A function w is said to be an upper solution of (2) if there exists a function $f_2 \in S_{F\circ w}$ such that w satisfies

$$w(x,y) \geq \mu(x,y) + \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f_2(s,t)}{st\Gamma(r_1)\Gamma(r_2)} dtds; \ (x,y) \in J.$$

Theorem 4.2. *Assume that the following hypotheses*

(H_1) *The multifunction $F : J \times \mathbb{R} \longrightarrow \mathcal{P}_{cp,cv}(\mathbb{R})$ is L^1-Carathéodory,*

(H_2) *There exist v and $w \in C$, lower and upper solutions for the integral inclusion* (2) *such that $v \leq w$,*

hold. Then the Hadamard integral inclusion (2) *has at least one solution u such that*

$$v(x,y) \leq u(x,y) \leq w(x,y) \quad \textit{for all } (x,y) \in J.$$

Proof. Solutions of the inclusion (2) are solutions of the Hadamard integral inclusion

$$u(x,y) \in \left\{\mu(x,y) + ({}^H I_\sigma^r f)(x,y) : \; f \in S_{F\circ u}\right\}; \; (x,y) \in J.$$

Consider the following modified integral inclusion:

$$u(x,y) - \mu(x,y) \in ({}^H I_\sigma^r F)(x,y,(gu)(x,y)); \; (x,y) \in J, \tag{10}$$

where $g : C \longrightarrow C$ be the truncation operator defined by

$$(gu)(x,y) = \begin{cases} v(x,y); & u(x,y) < v(x,y), \\ u(x,y); & v(x,y) \le u(x,y) \le w(x,y), \\ w(x,y); & w(x,y) < u(x,y). \end{cases}$$

A solution to (10) is a fixed point of the operator $N : C \to \mathcal{P}(C)$ defined by

$$(Nu)(x,y) = \left\{ h \in C : \begin{cases} h(x,y) = \mu(x,y) \\ + \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1 - 1} \left(\log \frac{y}{t}\right)^{r_2 - 1} \frac{f(s,t)}{st\Gamma(r_1)\Gamma(r_2)} dtds; \; (x,y) \in J \end{cases} \right\},$$

where

$$\begin{aligned} & f \in \tilde{S}_{F\circ g(u)} \\ & = \{f \in S_{F\circ g(u)} : f(x,y) \ge f_1(x,y) \text{ on } A_1 \text{ and } f(x,y) \le f_2(x,y) \text{ on } A_2\}, \end{aligned}$$

$$A_1 = \{(x,y) \in J : u(x,y) < v(x,y) \le w(x,y)\},$$

$$A_2 = \{(x,y) \in J : v(x,y) \le w(x,y) < u(x,y)\},$$

and

$$S^1_{F\circ g(u)} = \{f \in L^1(J) : f(x,y) \in F(t,x,(gu)(x,y)); \text{ for } (x,y) \in J\}.$$

Remark 4.3. (A) For each $u \in C$, the set $\tilde{S}_{F\circ g(u)}$ is nonempty. In fact, (H_1) implies that there exists $f_3 \in S_{F\circ g(u)}$, so we set

$$f = f_1\chi_{A_1} + f_2\chi_{A_2} + f_3\chi_{A_3},$$

where χ_{A_i} is the characteristic function of A_i; $i = 1,2,3$ and

$$A_3 = \{(x,y) \in J : v(x,y) \le u(x,y) \le w(x,y)\}.$$

Then, by decomposability, $f \in \tilde{S}_{F\circ g(u)}$.

(B) By the definition of g it is clear that $F(\cdot,\cdot,(gu)(\cdot,\cdot))$ is an L^1-Carathéodory multivalued map with compact convex values and there exists $\phi \in C(J,\mathbb{R}_+)$ such that

$$\|F(t,x,(gu)(x,y))\|_{\mathcal{P}} \leq \phi(x,y); \text{ for each } u \in \mathbb{R} \text{ and } (x,y) \in J.$$

Set

$$\phi^* := \sup_{(x,y)\in J} \phi(x,y).$$

From the fact that $g(u) = u$ for all $v \leq u \leq w$, the problem of finding the solutions of the integral inclusion (2) is reduced to finding the solutions of the operator inclusion $u \in N(u)$. We shall show that N is a completely continuous multivalued map, u.s.c. with convex and closed values. The proof will be given in several steps.

Step 1. *$N(u)$ is convex for each $u \in C$.*
Indeed, if $h_1,\ h_2$ belong to $N(u)$, then there exist $f_1^\star,\ f_2^\star \in \tilde{S}_{F\circ g(u)}$ such that for each $(x,y) \in J$ we have

$$h_i(x,y) = \mu(x,y) + \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1} \frac{f_i^\star(s,t)}{st\Gamma(r_1)\Gamma(r_2)}dtds;\ i=1,2.$$

Let $0 \leq \xi \leq 1$. Then, for each $(x,y) \in J$, we have

$$(\xi h_1 + (1-\xi)h_2)(x,y) = \mu(x,y)$$
$$+ \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1} \frac{((\xi f_1^\star + (1-\xi)f_2^\star))(s,t)}{st\Gamma(r_1)\Gamma(r_2)}dtds.$$

Since $\tilde{S}_{F\circ g(u)}$ is convex (because F has convex values), we have

$$\xi h_1 + (1-\xi)h_2 \in N(u).$$

Step 2. *N sends bounded sets of C into bounded sets.*
Indeed, we can prove that $N(C)$ is bounded. It is enough to show that there exists a positive constant ℓ such that for each $h \in N(u)$, $u \in C$ one has $\|h\|_C \leq \ell$. If $h \in N(u)$, then there exists $f \in \tilde{S}_{F\circ g(u)}$ such that for each $(x,y) \in J$ we have

$$h(x,y) = \mu(x,y) + \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1} \frac{f(s,t)}{st\Gamma(r_1)\Gamma(r_2)}dtds.$$

Then we get

$$|h(x,y)| \leq |\mu(x,y)| + \int_1^x \int_1^y \left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1} \frac{\phi(s,t)}{st\Gamma(r_1)\Gamma(r_2)}dtds.$$

Thus, we obtain

$$
\begin{aligned}
|h(x,y)| &\leq \|\mu\|_C + \int_{\phi^*}^{x}\int_{1}^{y}\left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1}\frac{\phi^*}{st\Gamma(r_1)\Gamma(r_2)}dtds \\
&\leq \|\mu\|_C + \frac{(\log a)^{r_1}(\log b)^{r_2}\phi^*}{\Gamma(1+r_1)\Gamma(1+r_2)} := \ell.
\end{aligned}
$$

Hence

$$\|h\|_C \leq \ell.$$

Step 3. *N sends bounded sets of C into equi-continuous sets.*
Let $(x_1,y_1),\ (x_2,y_2) \in J,\ x_1 < x_2,\ y_1 < y_2$ and $B_\rho = \{u \in C : \|u\|_C \leq \rho\}$ be a bounded set of C. For each $u \in B_\rho$ and $h \in N(u)$, there exists $f \in \tilde{S}_{F\circ g(u)}$ such that for each $(x,y) \in J$, we get

$$
\begin{aligned}
&|h(x_2,y_2) - h(x_1,y_1)| \leq |\mu(x_1,y_1) - \mu(x_2,y_2)| \\
&+ \int_{1}^{x_1}\int_{1}^{y_1}\left|\left(\log\frac{x_2}{s}\right)^{r_1-1}\left(\log\frac{y_2}{t}\right)^{r_2-1} - \left(\log\frac{x_1}{s}\right)^{r_1-1}\left(\log\frac{y_1}{t}\right)^{r_2-1}\right| \\
&\quad\times \frac{|f(s,t)|}{st\Gamma(r_1)\Gamma(r_2)}dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{|f(s,t)|}{st}dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{1}^{x_1}\int_{y_1}^{y_2}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{|f(s,t)|}{st}dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{1}^{y_1}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{|f(s,t)|}{st}dtds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&|h(x_2,y_2) - h(x_1,y_1)| \leq |\mu(x_1,y_1) - \mu(x_2,y_2)| + \frac{\phi^*}{\Gamma(1+r_1)\Gamma(1+r_2)} \\
&\times [2(\log y_2)^{r_2}(\log x_2 - \log x_1)^{r_1} + 2(\log x_2)^{r_1}(\log y_2 - \log y_1)^{r_2} \\
&+ (\log x_1)^{r_1}(\log y_1)^{r_2} - (\log x_2)^{r_1}(\log y_2)^{r_2} \\
&- 2(\log x_2 - \log x_1)^{r_1}(\log y_2 - \log y_1)^{r_2}].
\end{aligned}
$$

As $x_1 \to x_2$ and $y_1 \to y_2$, the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we conclude that N is completely continuous and therefore a condensing multivalued map.

Step 4. *N has a closed graph.*
Let $u_n \to u_*$, $h_n \in N(u_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(u_*)$.
$h_n \in N(u_n)$ means that there exists $f_n \in \tilde{S}_{F\circ g(u_n)}$ such that, for each $(x,y) \in J$, we have

$$h_n(x,y) = \mu(x,y) + \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \left(\log\frac{y}{t}\right)^{r_2-1} \frac{f_n(s,t)}{st\Gamma(r_1)\Gamma(r_2)} dtds.$$

We have to show that there exists $f_* \in \tilde{S}_{F\circ g(u_*)}$ such that, for each $(x,y) \in J$,

$$h_*(x,y) = \mu(x,y) + \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \left(\log\frac{y}{t}\right)^{r_2-1} \frac{f_*(s,t)}{st\Gamma(r_1)\Gamma(r_2)} dtds.$$

Consider the linear continuous operator

$$\begin{aligned} \Lambda : L^1(J) &\longrightarrow C(J), \\ f &\longmapsto \Lambda f \end{aligned}$$

defined by

$$(\Lambda f)(x,y) = \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \left(\log\frac{y}{t}\right)^{r_2-1} \frac{f(s,t)}{st\Gamma(r_1)\Gamma(r_2)} dtds.$$

Remark 4.3 (B) implies that the operator Λ is well defined. From Lemma 2.6, it follows that $\Lambda \circ \tilde{S}_{F\circ u}$ is a closed graph operator. Clearly we have

$$\begin{aligned} |h_n(x,y) - h_*(x,y)| &= \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \left(\log\frac{y}{t}\right)^{r_2-1} \frac{|f_n(s,t) - f_*(s,t)|}{st\Gamma(r_1)\Gamma(r_2)} dtds \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Moreover, from the definition of Λ, we deduce

$$h_n - \mu \in \Lambda(\tilde{S}_{F\circ g(u_n)}).$$

Since $u_n \to u_*$, it follows from Lemma 2.6 that, for some $f_* \in \Lambda(\tilde{S}_{F\circ g(u_*)})$, we have

$$h_*(x,y) = \mu(x,y) + \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \left(\log\frac{y}{t}\right)^{r_2-1} \frac{f_*(s,t)}{st\Gamma(r_1)\Gamma(r_2)} dtds.$$

From Lemma 2.5, we conclude that N is u.s.c.

Step 5. *The set $\Omega = \{u \in C : \lambda u \in N(u)$ for some $\lambda > 1\}$ is bounded.*
Let $u \in \Omega$. Then, there exists $f \in \Lambda(\tilde{S}_{F\circ g(u)})$, such that

$$\lambda u(x,y) = \mu(x,y) + \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \left(\log\frac{y}{t}\right)^{r_2-1} \frac{f(s,t)}{st\Gamma(r_1)\Gamma(r_2)} dtds.$$

As in Step 2, this implies that for each $(x,y)\in J$, we have

$$\|u\|_C \le \frac{\ell}{\lambda} < \ell.$$

This shows that Ω is bounded. As a consequence of Theorem 2.11, we deduce that N has a fixed point which is a solution of (10) on J.

Step 6. *The solution u of* (10) *satisfies*

$$v(x,y) \le u(x,y) \le w(x,y);\ \text{for all}\ \ (x,y)\in J.$$

First, we prove that

$$u(x,y) \le w(x,y);\ \ \text{for all}\ \ (x,y)\in J.$$

Assume that $u-w$ attains a positive maximum on J at $(\overline{x},\overline{y})\in J$, that is,

$$(u-w)(\overline{x},\overline{y}) = \max\{u(x,y)-w(x,y);\ (x,y)\in J\} > 0.$$

We distinguish the following cases:

Case 1. If $(\overline{x},\overline{y})\in(1,a)\times[1,b]$ then, there exists $(x^*,y^*)\in(1,a)\times[1,b]$ such that

$$[u(x,y^*)-w(x,y^*)]+[u(x^*,y)-w(x^*,y)]-[u(x^*,y^*)-w(x^*,y^*)] \le 0 \tag{11}$$

for all $(x,y)\in([x^*,\overline{x}]\times\{y^*\})\cup(\{x^*\}\times[y^*,b])$ and

$$u(x,y)-w(x,y) > 0;\ \text{for all}\ (x,y)\in(x^*,\overline{x}]\times(y^*,b]. \tag{12}$$

For all $(x,y)\in[x^*,\overline{x}]\times[y^*,b]$, we have

$$u(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t)}{st}dtds, \tag{13}$$

where $f\in S_{F\circ u}$. Thus equation (13) gives

$$\begin{aligned} &u(x,y)+u(x^*,y^*)-u(x,y^*)-u(x^*,y) \\ &= \int_{x^*}^x\int_{y^*}^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t)}{st\Gamma(r_1)\Gamma(r_2)}dtds. \end{aligned} \tag{14}$$

From (14) and the fact that w is an upper solution to (2) we get

$$u(x,y)+u(x^*,y^*)-u(x,y^*)-u(x^*,y) \le w(x,y)+w(x^*,y^*)-w(x,y^*)-w(x^*,y),$$

which gives,

$$\begin{aligned}[u(x,y)-w(x,y)] &\le [u(x,y^*)-w(x,y^*)]+[u(x^*,y)-w(x^*,y)] \\ &\quad -[u(x^*,y^*)-w(x^*,y^*)].\end{aligned} \tag{15}$$

Thus from (11), (12) and (15) we obtain the contradiction

$$\begin{aligned}0 < [u(x,y)-w(x,y)] &\le [u(x,y^*)-w(x,y^*)] \\ &+ [u(x^*,y)-w(x^*,y)]-[u(x^*,y^*)-w(x^*,y^*)] \le 0;\end{aligned}$$

for all $(x,y) \in [x^*,\overline{x}] \times [y^*,b]$.

Case 2. If $\overline{x}=1$, then

$$w(1,\overline{y}) < u(1,\overline{y}) \le w(1,\overline{y})$$

which is a contradiction. Thus

$$u(x,y) \le w(x,y); \text{ for all } (x,y) \in J.$$

Analogously, we can prove that

$$u(x,y) \ge v(x,y); \text{ for all } (x,y) \in J.$$

This shows that the problem (10) has a solution u satisfying $v \le u \le w$ which is solution of the integral inclusion (2).

Acknowledgement

The authors are grateful to the referee for the careful reading of the paper and for his(her) helpful remarks.

References

[1] S. Abbas and M. Benchohra, *Fractional order integral equations of two independent variables*, Appl. Math. Comput. **227** (2014), 755–761.doi:10.1016/j.amc.2013.10.086

[2] S. Abbas and M. Benchohra, *Upper and lower solutions method for the Darboux problem for fractional order partial differential inclusions*, Int. J. Modern Math. **5** (3) (2010), 327–338.

[3] S. Abbas and M. Benchohra, *Upper and lower solutions method for impulsive partial hyperbolic differential equations with fractional order*, Nonlinear Anal. Hybrid Syst. **4** (2010), 406–413. doi:10.1016/j.nahs.2009.10.004

[4] S. Abbas and M. Benchohra, *Upper and lower solutions method for Darboux problem for fractional order implicit impulsive partial hyperbolic differential equations*, Acta Univ. Palacki. Olomuc. **51** (2) (2012), 5–18.

[5] S. Abbas and M. Benchohra, *The method of upper and lower solutions for partial hyperbolic fractional order differential inclusions with impulses*, Discuss. Math. Differ. Incl. Control Optim. **30** (1) (2010), 141–161. doi:10.7151/dmdico.1116

[6] S. Abbas, M. Benchohra and A. Hammoudi, *Upper, lower solutions method and extremal solutions for impulsive discontinuous partial fractional differential inclusions*, Panamerican Math. J. **24** (1) (2014), 31–52.

[7] S. Abbas, M. Benchohra and G.M. N'Guérékata, Topics in Fractional Differential Equations (Springer, New York, 2012). doi:10.1007/978-1-4614-4036-9

[8] S. Abbas, M. Benchohra and G.M. N'Guérékata, Advanced Fractional Differential and Integral Equations (Nova Science Publishers, New York, 2015).

[9] S. Abbas, M. Benchohra and J.J. Trujillo, *Upper and lower solutions method for partial fractional differential inclusions with not instantaneous impulses*, Prog. Frac. Diff. Appl. **1** (1) (2015), 11–22.

[10] J. Banaś and K. Goebel, Measures of Noncompactness in Banach Spaces (Marcel-Dekker, New York, 1980).

[11] M. Benchohra, J. Henderson, S.K. Ntouyas and A. Ouahab, *Existence results for functional differential equations of fractional order*, J. Math. Anal. Appl. **338** (2008), 1340–1350. doi:10.1016/j.jmaa.2007.06.021

[12] M. Benchohra, J. Henderson and S.K. Ntouyas, Impulsive Differential Equations and Inclusions (Hindawi Publishing Corporation, Vol. 2, New York, 2006). doi:10.1155/9789775945501

[13] P.L. Butzer, A.A. Kilbas and J.J. Trujillo, *Fractional calculus in the mellin setting and Hadamard-type fractional integrals*, J. Math. Anal. Appl. **269** (2002), 1–27. doi:10.1016/S0022-247X(02)00001-X

[14] P.L. Butzer, A.A. Kilbas and J.J. Trujillo, *Mellin transform analysis and integration by parts for Hadamard-type fractional integrals*, J. Math. Anal. Appl. **270** (2002), 1–15. doi:10.1016/S0022-247X(02)00066-5

[15] K. Deimling, Multivalued Differential Equations (Walter de Gruyter, Berlin-New York, 1992). doi:10.1515/9783110874228

[16] S. Djebali, L. Górniewicz and A. Ouahab, Solution Sets for Differential Equations and Inclusions, de Gruyter Series in Nonlinear Analysis and Applications (Walter de Gruyter & Co., Berlin, 2013). doi:10.1515/9783110293562

[17] L. Górniewicz, Topological Fixed Point Theory of Multivalued Mappings, Mathematics and its Applications, 495 (Kluwer Academic Publishers, Dordrecht, 1999). doi:10.1007/978-94-015-9195-9

[18] A. Granas and J. Dugundji, Fixed Point Theory (Springer-Verlag, New York, 2003). doi:10.1007/978-0-387-21593-8

[19] J. Hadamard, *Essai sur l'étude des fonctions données par leur développment de Taylor*, J. Pure Appl. Math. **4** (8) (1892) 101–186.

[20] Sh. Hu and N. Papageorgiou, Handbook of Multivalued Analysis, Volume I: Theory (Kluwer, Dordrecht, Boston, London, 1997). doi:10.1007/978-1-4615-6359-4

[21] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations (Elsevier Science B.V., Amsterdam, 2006).

[22] M. Kisielewicz, Differential Inclusions and Optimal Control (Kluwer Academic Publishers, Dordrecht, Netherlands, 1991).

[23] G.S. Ladde, V. Lakshmikanthan and A.S. Vatsala, Monotone Iterative Techniques for Nonliner Differential Equations (Pitman Advanced Publishing Program, London, 1985).

[24] A. Lasota and Z. Opial, *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. **13** (1965), 781–786.

[25] M. Martelli, *A Rothe's type theorem for noncompact acyclic-valued map*, Boll. Un. Math. Ital. **11** (1975), 70–76.

[26] K.S. Miller and B. Ross, An Introduction to the Fractional Calculus and Differential Equations (John Wiley, New York, 1993).

[27] B.G. Pachpatte, *Monotone methods for systems of nonlinear hyperbolic problems in two independent variables*, Nonlinear Anal. **30** (1997), 235–272.

[28] S. Pooseh, R. Almeida and D. Torres, *Expansion formulas in terms of integer-order derivatives for the hadamard fractional integral and derivative*, Numer. Funct. Anal. Optim. **33** (3) (2012), 301–319. doi:10.1080/01630563.2011.647197

[29] S.G. Samko, A.A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives. Theory and Applications (Gordon and Breach, Yverdon, 1993).

[30] A.N. Vityuk, *On solutions of hyperbolic differential inclusions with a nonconvex right-hand side*, (Russian) Ukran. Mat. Zh. **47** (4) (1995), 531–534; translation in Ukrainian Math. J. **47** (4) (1995), 617–621 (1996).

[31] A.N. Vityuk and A.V. Golushkov, *Existence of solutions of systems of partial differential equations of fractional order*, Nonlinear Oscil. **7** (2004), 318–325. doi:10.1007/s11072-005-0015-9

Received 27 July 2015