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SOME AVERAGING RESULTS FOR ORDINARY DIFFERENTIAL INCLUSIONS

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Abstract

We consider ordinary differential inclusions and we state and discuss some averaging results for these inclusions. Our results are proved under weaker conditions than the results in the literature.

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1. Introduction and notations

The averaging method was studied for differential inclusions by many authors using different and rather restrictive conditions on the regularity of their right-hand sides (see, for instance, [3, 5, 6, 7, 8, 9] and references therein).

Our aim in this paper is to prove some results on averaging for ordinary differential inclusions of the form

$$\dot{x}(t) \in F\left(\frac{t}{\varepsilon}, x(t)\right) \tag{1.1}$$

where $\varepsilon > 0$ denotes the small perturbation parameter, the time variable $t \in [0, L]$ and F is a multifunction with values that are nonempty compact convex subsets of $\mathbb{R}^d$. Our main contribution is the weakening of the regularity conditions on the multifunction F in (1.1) under which the averaging method is justified in the existing literature. Indeed, usually averaging results require that F is at least Lipschitz with respect to the second variable and sometimes (as in [9]) this condition is relaxed to the uniform continuity in the second variable uniformly with respect to the first one. In our main results these conditions are weakened and in one case it is only assumed that F is continuous in the second variable uniformly with respect to the first one and in an other case it is assumed that the indefinite integral of F satisfies a Lipschitz-type condition. Usually, it is often assumed that F is uniformly bounded. In this paper we assume only that F is bounded by some locally Lebesgue integrable function with the property that its average exists.

The structure of the paper is as follows. In Section 2 we present our main results: Theorems 2.1, 2.3 and 2.4. We state and prove some preliminary results in Section 3 and then we give the proofs of Theorems 2.1, 2.3 and 2.4.

We finish this section with some definitions and notations.
Let $\mathbb{R}^d$ denotes the d-dimensional space with the Euclidean norm $|\cdot|$. $\mathrm{Comp}(\mathbb{R}^\mathrm{d})$ ($\mathrm{Conv}(\mathbb{R}^\mathrm{d})$, respectively) stands for the class of all nonempty compact (nonempty compact and convex, respectively) subsets of $\mathbb{R}^d$. In $\mathrm{Comp}(\mathbb{R}^\mathrm{d})$ the so-called Hausdorff metric is defined by

$$\rho(A, B) := \max\left(\sup_{a\in A}\delta(a, B), \sup_{b\in B}\delta(b, A)\right), \quad \forall A, B \in \mathrm{Comp}(\mathbb{R}^\mathrm{d})$$

where $\delta(x, A) = \inf\{|x - a| : a \in A\}$, for any $x \in \mathbb{R}^d$ and any $A \in \mathrm{Comp}(\mathbb{R}^\mathrm{d})$. The metric space $(\mathrm{Comp}(\mathbb{R}^\mathrm{d}), \rho)$ is complete and $\mathrm{Conv}(\mathbb{R}^\mathrm{d})$ is a closed subset of this space.

The norm of any $A \in \mathrm{Comp}(\mathbb{R}^\mathrm{d})$ is given by: $\|A\| = \rho(A, \{0\}) = \sup\{|a| : a \in A\}$. Let $F : \mathbb{R}_+ \times \mathbb{R}^d \to \mathrm{Comp}(\mathbb{R}^\mathrm{d})$ be a multifunction. By a solution of the ordinary differential inclusion $\dot{x} \in F(t, x)$ we understand an absolutely continuous function x defined on some interval and satisfying $\dot{x}(t) \in F(t, x(t))$ for almost every t.

For the classical theory of differential inclusions we refer to the books of Aubin and Cellina [1], Deimling [2], Hu and Papageorgiou [4] and Smirnov [10].

2. Averaging results

Let U be an open subset of $\mathbb{R}^d$, $x_0 \in U$ and $F : \mathbb{R}_+ \times U \to \mathrm{Conv}(\mathbb{R}^\mathrm{d})$ be a multifunction. Let $\varepsilon > 0$ be a small parameter. We are interested in the limiting behavior of the trajectories of the initial value problem

$$\dot{x}(t) \in F\left(\frac{t}{\varepsilon}, x(t)\right), \qquad x(0) = x_0 \tag{2.1}$$

on finite intervals of time $t \in [0, L]$ as the perturbation parameter ε tends to zero. For this purpose we make use of the averaging method.

First, let us formulate the assumptions on F we will need to prove our averaging results.

Assumption 1. For all $x \in U$, $F(\cdot, x) : \mathbb{R}_+ \to \mathrm{Conv}(\mathbb{R}^{\mathrm{d}})$ is measurable.

Assumption 2. For all $t \in \mathbb{R}_+$, $F(t, \cdot) : U \to \mathrm{Conv}(\mathbb{R}^{\mathrm{d}})$ is continuous.

Assumption 3. The continuity of F in the second variable is uniform with respect to the first one.

Assumption 4. There exist a locally Lebesgue integrable function $b : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $B > 0$ such that

$$\|F(t, x)\| \leq b(t), \quad \forall t \in \mathbb{R}_+, \forall x \in U$$

with

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T b(t)dt = B.$$

Assumption 5. For all $x \in U$ there exists a limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(\tau, x)d\tau =: \overline{F}(x).$$

Assumption 6. There exists a constant $\lambda > 0$ such that for continuous functions $u, v : \mathbb{R}_+ \to U$ and $t_1, t_2 \in \mathbb{R}_+$, $t_1 \leq t_2$,

$$\rho\left(\int_{t_1}^{t_2} F(\tau, u(\tau))d\tau, \int_{t_1}^{t_2} F(\tau, v(\tau))d\tau\right) \leq \lambda \int_{t_1}^{t_2} |u(\tau) - v(\tau)|d\tau. \tag{2.2}$$

In Assumption 5 the integral is understood in Aumann-Hukuhara sense and the convergence in sense of the Hausdorff metric.

Notice that Assumption 6 is a Lipschitz-type condition on the indefinite integral of F and not on F itself. Naturally, in the case where F is Lipschitz with respect to its second variable, Assumption 6 is satisfied in an obvious way.

If a multifunction $F : \mathbb{R}_+ \times U \to \mathrm{Conv}(\mathbb{R}^{\mathrm{d}})$ satisfies Assumptions 1–5 (resp. Assumptions 1 and 4–6), its average, that is, the multifunction $\overline{F} : U \to \mathrm{Conv}(\mathbb{R}^{\mathrm{d}})$ in Assumption 5, is continuous (resp. Lipschitz) (see Lemma 3.1 below).

Consider now problem (2.1) together with the initial value averaged problem

$$\dot{y}(t) \in \overline{F}(y(t)), \qquad y(0) = x_0. \tag{2.3}$$

The first main result of the paper establishes the approximation of solutions of problem (2.1) by those of the averaged problem (2.3) on finite time intervals, and reads as follows.

Theorem 2.1. *Let $F : \mathbb{R}_+ \times U \to \mathrm{Conv}(\mathbb{R}^{\mathrm{d}})$ be a multifunction and let $x_0 \in U$. Suppose that Assumptions 1–5 are fulfilled. Then, for any $L > 0$ and $\mu > 0$, there exists $\varepsilon_0 = \varepsilon_0(x_0, L, \mu) > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$ and any solution x_ε of (2.1) which is defined on $[0, L]$, there exists a solution y of (2.3) such that y is defined on $[0, L]$ and satisfies $|x_\varepsilon(t) - y(t)| < \mu$ for all $t \in [0, L]$.*

We point out that under Assumptions 2–3 it is only possible to obtain unilateral approximations, that is, the approximation of solutions of problems (2.1) by those of the averaged problem (2.3) (as stated in Theorem 2.1). The converse approximation is, in general, false, even for ordinary differential equations, as showed in the following example.

Example 2.2. Consider the initial value problem

$$\dot{x}(t) = \sqrt{|x(t)|} + \sin\left(\frac{t}{\varepsilon}\right), \qquad x(0) = 0. \tag{2.4}$$

The associated averaged initial value problem is

$$\dot{y}(t) = \sqrt{|y(t)|}, \qquad y(0) = 0. \tag{2.5}$$

There is no solution of problem (2.4) approximating the trivial solution $y(t) \equiv 0$ of the averaged problem (2.5).

However, when problem (2.3) has a unique solution (in this case, the ordinary differential inclusion in (2.3) is, in fact, an ordinary differential equation), this one is approximated by any solution of problem (2.1) as it is stated by the following result.

Theorem 2.3. *Let $F : \mathbb{R}_+ \times U \to \mathrm{Conv}(\mathbb{R}^{\mathrm{d}})$ be a multifunction and let $x_0 \in U$. Suppose that Assumptions 1–5 are fulfilled. Suppose also that the initial value problem (2.3) has a unique solution. Let y be the (unique) solution of (2.3). Then, for any $L > 0$ such that y is defined on $[0, L]$ and any $\mu > 0$, there exists $\varepsilon_0 = \varepsilon_0(x_0, L, \mu) > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, any solution x_ε of (2.1) is defined on $[0, L]$ and satisfies $|x_\varepsilon(t) - y(t)| < \mu$ for all $t \in [0, L]$.*

Now, if in Theorem 2.1 we remove Assumptions 2–3 and together with Assumptions 1 and 4–5 we suppose Assumption 6, the conclusion of Theorem 2.1 remains true, that is, the approximation of solutions of the problem (2.1) by those of the averaged problem (2.3) holds (see assertion (i) in Theorem 2.4 below). Furthermore, the converse approximation is now true as it is proved in the following second main result of the paper.

Theorem 2.4. *Let* $F : \mathbb{R}_+ \times U \to \mathrm{Conv}(\mathbb{R}^{\mathrm{d}})$ *be a multifunction and let* $x_0 \in U$. *Suppose that Assumptions* 1 *and* 4–6 *are fulfilled. Then, for any* $L > 0$ *and* $\mu > 0$, *there exists* $\varepsilon_0 = \varepsilon_0(x_0, L, \mu) > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_0]$ *the following conditions are satisfied:*

(i) *for any solution* x_ε *of* (2.1) *which is defined on* $[0, L]$, *there exists a solution* y *of* (2.3) *such that* y *is defined on* $[0, L]$ *and satisfies* $|x_\varepsilon(t) - y(t)| < \mu$ *for all* $t \in [0, L]$;

(ii) *for any solution* y *of* (2.3) *which is defined on* $[0, L]$, *there exists a solution* x_ε *of* (2.1) *such that* x_ε *is defined on* $[0, L]$ *and satisfies* $|x_\varepsilon(t) - y(t)| < \mu$ *for all* $t \in [0, L]$.

3. PROOFS OF THE RESULTS

The proof of Theorem 2.3 is based on the result of Theorem 2.1 and on a well-known property of continuation of solutions of ordinary differential inclusions. It is postponed to subsection 3.3 below.

To prove Theorems 2.1 and 2.4 we need to establish the following preliminary lemmas. So the proofs of Theorems 2.1 and 2.4 are postponed to subsections 3.2 and 3.4, respectively.

3.1. Technical lemmas

Lemma 3.1. *Let* $F : \mathbb{R}_+ \times U \to \mathrm{Conv}(\mathbb{R}^{\mathrm{d}})$ *be a multifunction.*

(i) *If* F *satisfies Assumptions* 1–5, *then its average* $\overline{F} : U \to \mathrm{Conv}(\mathbb{R}^{\mathrm{d}})$ *in Assumption* 5 *is uniformly bounded by the constant* B *in Assumption* 4 *and is continuous.*

(ii) *If* F *satisfies Assumptions* 1 *and* 4–6, *then its average* $\overline{F} : U \to \mathrm{Conv}(\mathbb{R}^{\mathrm{d}})$ *in Assumption* 5 *is uniformly bounded by the constant* B *in Assumption* 4 *and satisfies the Lipschitz condition with constant* λ *as in Assumption* 6.

Proof. (i) Suppose that F satisfies Assumptions 1–5. We first prove the uniform boundedness of $\overline{F}$ by the constant B. Let $x \in U$. Letting $T \to \infty$ in the following inequality

$$\begin{aligned}\|\overline{F}(x)\| &\leq \rho\left(\overline{F}(x), \frac{1}{T}\int_0^T F(\tau,x)d\tau\right) + \left\|\frac{1}{T}\int_0^T F(\tau,x)d\tau\right\| \\ &\leq \rho\left(\overline{F}(x), \frac{1}{T}\int_0^T F(\tau,x)d\tau\right) + \frac{1}{T}\int_0^T b(\tau)d\tau\end{aligned}$$

and taking into account Assumptions 4–5, we obtain $\|\overline{F}(x)\| \leq B$, which finishes the proof of the uniform boundedness of $\overline{F}$ by the constant B.

We now prove continuity of $\overline{F}$. Let $x_0 \in U$. By Assumptions 2–3, for any $\xi > 0$ there exists $\eta > 0$ such that, for all $x \in U$, $|x - x_0| \leq \eta$ implies that

$$\rho(F(\tau,x), F(\tau,x_0)) \leq \xi, \ \forall \tau \in \mathbb{R}_+. \tag{3.1}$$

Now, let $T \to \infty$ in the following inequality

$$\begin{aligned}&\rho(\overline{F}(x), \overline{F}(x_0)) \\ &\leq \rho\left(\overline{F}(x), \frac{1}{T}\int_0^T F(\tau,x)d\tau\right) \\ &\quad + \rho\left(\frac{1}{T}\int_0^T F(\tau,x)d\tau, \frac{1}{T}\int_0^T F(\tau,x_0)d\tau\right) \\ &\quad + \rho\left(\overline{F}(x_0), \frac{1}{T}\int_0^T F(\tau,x_0)d\tau\right) \\ &\leq \rho\left(\overline{F}(x), \frac{1}{T}\int_0^T F(\tau,x)d\tau\right) + \rho\left(\overline{F}(x_0), \frac{1}{T}\int_0^T F(\tau,x_0)d\tau\right) \\ &\quad + \frac{1}{T}\int_0^T \rho\left(F(\tau,x), F(\tau,x_0)\right)d\tau.\end{aligned}$$

From (3.1) and Assumption 5, we deduce that $\rho(\overline{F}(x), \overline{F}(x_0)) \leq \xi$. This finishes the proof of the continuity of $\overline{F}$ at the point x_0.

(ii) Suppose that F satisfies Assumptions 1 and 4–6. The proof of the uniform boundedness of $\overline{F}$ by the constant B is the same as the corresponding proof in the case (i) above. It remains to prove that $\overline{F}$ is λ-Lipschitz. Let $x, x' \in U$. Letting $T \to \infty$ in the following inequality

$$\begin{aligned}&\rho(\overline{F}(x), \overline{F}(x')) \\ &\leq \rho\left(\overline{F}(x), \frac{1}{T}\int_0^T F(\tau,x)d\tau\right)\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{T}\rho\left(\int_0^T F(\tau,x)d\tau, \int_0^T F(\tau,x')d\tau\right)\\
&+ \rho\left(\overline{F}(x'), \frac{1}{T}\int_0^T F(\tau,x')d\tau\right)\\
\le{}& \rho\left(\overline{F}(x), \frac{1}{T}\int_0^T F(\tau,x)d\tau\right) + \rho\left(\overline{F}(x'), \frac{1}{T}\int_0^T F(\tau,x')d\tau\right)\\
&+ \frac{1}{T}\lambda\int_0^T |x-x'|d\tau
\end{aligned}
$$

we deduce that $\rho(\overline{F}(x), \overline{F}(x')) \le \lambda|x-x'|$. This finishes the proof. ■

Lemma 3.2. *Let* $F : \mathbb{R}_+ \times U \to \mathrm{Conv}(\mathbb{R}^{\mathrm{d}})$ *be a multifunction. Suppose that* F *satisfies Assumptions* 1 *and* 4–5*. Then, for all* $x \in U$*,* $t \ge 0$ *and* $\alpha > 0$*, we have*

$$\lim_{\varepsilon\to 0}\rho\left(\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} F(\tau,x)d\tau, \overline{F}(x)\right) = 0.$$

Proof. Let $x \in U$, $t \ge 0$ and $\alpha > 0$.

Case 1. $t = 0$. From Assumption 5, it follows immediately that

$$\lim_{\varepsilon\to 0}\rho\left(\frac{\varepsilon}{\alpha}\int_0^{\alpha/\varepsilon} F(\tau,x)d\tau, \overline{F}(x)\right) = 0.$$

Case 2. $t > 0$. We have

$$\frac{\varepsilon}{\alpha}\int_0^{t/\varepsilon+\alpha/\varepsilon} F(\tau,x)d\tau = \frac{\varepsilon}{\alpha}\int_0^{t/\varepsilon} F(\tau,x)d\tau + \frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} F(\tau,x)d\tau$$

or equivalently

$$
\begin{aligned}
&\left(\frac{t}{\alpha}+1\right)\frac{1}{t/\varepsilon+\alpha/\varepsilon}\int_0^{t/\varepsilon+\alpha/\varepsilon} F(\tau,x)d\tau\\
&= \frac{t}{\alpha}\cdot\frac{1}{t/\varepsilon}\int_0^{t/\varepsilon} F(\tau,x)d\tau + \frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} F(\tau,x)d\tau.
\end{aligned}
$$

This implies

$$\begin{aligned}
&\left(\frac{t}{\alpha}+1\right)\rho\left(\frac{1}{t/\varepsilon+\alpha/\varepsilon}\int_0^{t/\varepsilon+\alpha/\varepsilon}F(\tau,x)d\tau,\overline{F}(x)\right)\\
&=\rho\left(\frac{t}{\alpha}\cdot\frac{1}{t/\varepsilon}\int_0^{t/\varepsilon}F(\tau,x)d\tau+\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon}F(\tau,x)d\tau,\frac{t}{\alpha}\overline{F}(x)+\overline{F}(x)\right)\\
&\geq -\frac{t}{\alpha}\rho\left(\frac{1}{t/\varepsilon}\int_0^{t/\varepsilon}F(\tau,x)d\tau,\overline{F}(x)\right)+\rho\left(\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon}F(\tau,x)d\tau,\overline{F}(x)\right).
\end{aligned}$$

So we have

$$\begin{aligned}
&\rho\left(\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon}F(\tau,x)d\tau,\overline{F}(x)\right)\\
&\leq\left(\frac{t}{\alpha}+1\right)\rho\left(\frac{1}{t/\varepsilon+\alpha/\varepsilon}\int_0^{t/\varepsilon+\alpha/\varepsilon}F(\tau,x)d\tau,\overline{F}(x)\right)\\
&\quad+\frac{t}{\alpha}\rho\left(\frac{1}{t/\varepsilon}\int_0^{t/\varepsilon}F(\tau,x)d\tau,\overline{F}(x)\right).
\end{aligned}\tag{3.2}$$

Now, from Assumption 5 we can easily deduce that

$$\lim_{\varepsilon\to0}\rho\left(\frac{1}{t/\varepsilon}\int_0^{t/\varepsilon}F(\tau,x)d\tau,\overline{F}(x)\right)=0$$

and

$$\lim_{\varepsilon\to0}\rho\left(\frac{1}{t/\varepsilon+\alpha/\varepsilon}\int_0^{t/\varepsilon+\alpha/\varepsilon}F(\tau,x)d\tau,\overline{F}(x)\right)=0.$$

Therefore, the right-hand side of (3.2) tends to zero as $\varepsilon\to0$ and the result is proved. ■

Lemma 3.3. *Let* $F:\mathbb{R}_+\times U\to\mathrm{Conv}(\mathbb{R}^{\mathrm{d}})$ *be a multifunction. Suppose that* F *satisfies Assumptions* 1–5 *or Assumptions* 1 *and* 4–6. *Let* $x_0\in U$ *and* $L>0$.

(i) *If a solution* y *of problem* (2.3) *is defined on* $[0,L]$, *then for all* $t\in[0,L]$ *and* $\alpha>0$ *we have*

$$\lim_{\varepsilon\to0}\rho\left(\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon}F(\tau,y(t))d\tau,\overline{F}(y(t))\right)=0.$$

(ii) *If the family* $\{x_\varepsilon\}$ *of solutions of problem* (2.1) *is defined and converges uniformly on* $[0,L]$ *to a continuous function* $\tilde{x}_0$ *when* $\varepsilon \to 0$*, then for all* $t \in [0,L]$ *and* $\alpha > 0$ *we have*

$$\lim_{\varepsilon\to 0} \rho\left(\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} F(\tau, x_\varepsilon(t))d\tau, \overline{F}(\tilde{x}_0(t))\right) = 0.$$

Proof. (i) There is nothing to prove. The result follows directly from Lemma 3.2.

(ii) For $t \in [0,L]$ and $\alpha > 0$ we have

$$\begin{aligned} &\rho\left(\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} F(\tau, x_\varepsilon(t))d\tau, \overline{F}(\tilde{x}_0(t))\right) \\ &\le \frac{\varepsilon}{\alpha}\rho\left(\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} F(\tau, x_\varepsilon(t))d\tau, \int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} F(\tau, \tilde{x}_0(t))d\tau\right) \\ &\quad + \rho\left(\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} F(\tau, \tilde{x}_0(t))d\tau, \overline{F}(\tilde{x}_0(t))\right). \end{aligned} \tag{3.3}$$

By Lemma 3.2 the second term of the right-hand side of (3.3) tends to zero as $\varepsilon \to 0$. For the first term in the right-hand side of (3.3) we write

$$\begin{aligned} &\frac{\varepsilon}{\alpha}\rho\left(\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} F(\tau, x_\varepsilon(t))d\tau, \int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} F(\tau, \tilde{x}_0(t))d\tau\right) \\ &= \frac{1}{\alpha}\rho\left(\int_t^{t+\alpha} F\left(\frac{\tau}{\varepsilon}, x_\varepsilon(t)\right)d\tau, \int_t^{t+\alpha} F\left(\frac{\tau}{\varepsilon}, \tilde{x}_0(t)\right)d\tau\right) := \frac{1}{\alpha}\xi. \end{aligned}$$

By Assumptions 2–3 or by Assumption 6, it follows that $\lim_{\varepsilon\to 0}\xi = 0$ since the sequence $\{x_\varepsilon(t)\}$ converges to $\tilde{x}_0(t)$. So, one can conclude that all terms of the right-hand side of inequality (3.3) tend to zero as $\varepsilon \to 0$, which finishes the proof of (ii).

The proof of the lemma is complete. ∎

Lemma 3.4. *Let* $F : \mathbb{R}_+ \times U \to \mathrm{Conv}(\mathbb{R}^d)$ *be a multifunction. Suppose that* F *satisfies Assumptions* 1–5 *or Assumptions* 1 *and* 4–6*. Let* $x_0 \in U$ *and* $L > 0$*.*

(i) *If a solution* y *of problem* (2.3) *is defined on* $[0,L]$*, then we have*

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,L]} \rho\left(\int_0^t F\left(\frac{\tau}{\varepsilon}, y(\tau)\right)d\tau, \int_0^t \overline{F}(y(\tau))d\tau\right) = 0$$

(ii) *If the family* $\{x_\varepsilon\}$ *of solutions of problem* (2.1) *is defined and converges uniformly on* $[0, L]$ *to a continuous function* $\tilde{x}_0$ *when* $\varepsilon \to 0$, *then we have*

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,L]} \rho\left(\int_0^t F\left(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\right) d\tau, \int_0^t \overline{F}(\tilde{x}_0(\tau))d\tau\right) = 0.$$

Proof. *Case* 1. F satisfies Assumptions 1–5. Let $x_0 \in U$ and $L > 0$. We will prove simultaneously (i) and (ii). For this, let $u = \tilde{u} = y$ in the case (i) and, $u = x_\varepsilon$ and $\tilde{u} = \tilde{x}_0$ in the case (ii). Let $t_0 = 0 < t_1 < \cdots < t_m < \cdots < t_p = L$, $p \in \mathbb{N}$, any partition of $[0, L]$ with $\alpha = \alpha(\varepsilon) := t_{m+1} - t_m$, $m = 0, \dots, p-1$ and be given $\lim_{\varepsilon\to 0} \alpha = 0$. Let $t \in [t_m, t_{m+1}]$ for any $m \in \{0, \dots, p-1\}$. Then

$$\begin{aligned}
&\rho\left(\int_0^t F\left(\frac{\tau}{\varepsilon}, u(\tau)\right) d\tau, \int_0^t \overline{F}(\tilde{u}(\tau))d\tau\right)\\
&\le \sum_{n=0}^{m-1}\rho\left(\int_{t_n}^{t_{n+1}} F\left(\frac{\tau}{\varepsilon}, u(\tau)\right) d\tau, \int_{t_n}^{t_{n+1}} \overline{F}(\tilde{u}(\tau))d\tau\right)\\
&\quad + \rho\left(\int_{t_m}^{t} F\left(\frac{\tau}{\varepsilon}, u(\tau)\right) d\tau, \int_{t_m}^{t} \overline{F}(\tilde{u}(\tau))d\tau\right).
\end{aligned}\tag{3.4}$$

By Assumption 4 and Lemma 3.1 we have

$$\begin{aligned}
&\rho\left(\int_{t_m}^{t} F\left(\frac{\tau}{\varepsilon}, u(\tau)\right) d\tau, \int_{t_m}^{t} \overline{F}(\tilde{u}(\tau))d\tau\right)\\
&\le \left\|\int_{t_m}^{t} F\left(\frac{\tau}{\varepsilon}, u(\tau)\right) d\tau\right\| + \left\|\int_{t_m}^{t} \overline{F}(\tilde{u}(\tau))d\tau\right\|\\
&\le \int_{t_m}^{t} b\left(\frac{\tau}{\varepsilon}\right) d\tau + \alpha B \le \int_{t_m}^{t_{m+1}} b\left(\frac{\tau}{\varepsilon}\right) d\tau + \alpha B\\
&= \alpha\left(\frac{\varepsilon}{\alpha}\int_{t_m/\varepsilon}^{t_m/\varepsilon+\alpha/\varepsilon} b(\tau)d\tau\right) + \alpha B.
\end{aligned}\tag{3.5}$$

However, we have

$$\begin{aligned}
\frac{\varepsilon}{\alpha}\int_{t_m/\varepsilon}^{t_m/\varepsilon+\alpha/\varepsilon} b(\tau)d\tau - B &= \frac{\varepsilon}{\alpha}\int_0^{t_m/\varepsilon+\alpha/\varepsilon} b(\tau)d\tau - \frac{\varepsilon}{\alpha}\int_0^{t_m/\varepsilon} b(\tau)d\tau - B\\
&= \left(\frac{t_m}{\alpha}+1\right)\left[\frac{1}{t_m/\varepsilon+\alpha/\varepsilon}\int_0^{t_m/\varepsilon+\alpha/\varepsilon} b(\tau)d\tau - B\right]\\
&\quad - \left(\frac{t_m}{\alpha}\right)\left[\frac{1}{t_m/\varepsilon}\int_0^{t_m/\varepsilon} b(\tau)d\tau - B\right],
\end{aligned}$$

from which we deduce

$$\left|\frac{\varepsilon}{\alpha}\int_{t_m/\varepsilon}^{t_m/\varepsilon+\alpha/\varepsilon} b(\tau)d\tau - B\right| \le \left(\frac{L}{\alpha}+1\right)\left|\frac{1}{t_m/\varepsilon+\alpha/\varepsilon}\int_0^{t_m/\varepsilon+\alpha/\varepsilon} b(\tau)d\tau - B\right| + \frac{L}{\alpha}\left|\frac{1}{t_m/\varepsilon}\int_0^{t_m/\varepsilon} b(\tau)d\tau - B\right| \tag{3.6}$$

where $|\cdot|$ is here the absolute value in $\mathbb{R}$.

From Assumption 4 it is easy to see that

$$\lim_{\varepsilon\to 0}\left|\frac{1}{t_m/\varepsilon}\int_0^{t_m/\varepsilon} b(\tau)d\tau - B\right| = 0$$

and

$$\lim_{\varepsilon\to 0}\left|\frac{1}{t_m/\varepsilon+\alpha/\varepsilon}\int_0^{t_m/\varepsilon+\alpha/\varepsilon} b(\tau)d\tau - B\right| = 0.$$

Therefore, the right-hand side of (3.6) tends to zero as $\varepsilon \to 0$, so the inequality (3.5) gives

$$\begin{aligned}&\rho\left(\int_{t_m}^t F\left(\frac{\tau}{\varepsilon}, u(\tau)\right)d\tau, \int_{t_m}^t \overline{F}(\tilde{u}(\tau))d\tau\right)\\ &\le \alpha(B+\xi_m)+\alpha B \le \alpha(B+\xi)+\alpha B = \alpha(2B+\xi),\end{aligned} \tag{3.7}$$

where $\xi = \xi(\varepsilon) = \max\{\xi_m = \xi_m(\varepsilon) : m = 0, \dots, p-1\}$ with $\lim_{\varepsilon\to 0}\xi_m = 0$.

Now let $n \in \{0, \dots, m-1\}$, $\tau \in [t_n, t_{n+1}]$ and consider each case of the function u separately.

1) $u = y$. We have

$$|y(\tau) - y(t_n)| \le \alpha B.$$

2) $u = x_\varepsilon$. As in the case of inequality (3.7), we can easily verify that

$$\begin{aligned}&|x_\varepsilon(\tau) - x_\varepsilon(t_n)| \le \int_{t_n}^{\tau} b\left(\frac{s}{\varepsilon}\right)ds \le \int_{t_n}^{t_{n+1}} b\left(\frac{s}{\varepsilon}\right)ds\\ &= \alpha\left(\frac{\varepsilon}{\alpha}\int_{t_n/\varepsilon}^{t_n/\varepsilon+\alpha/\varepsilon} b(s)ds\right) \le \alpha(B+\xi_n) \le \alpha(B+\xi_m),\end{aligned} \tag{3.8}$$

where $\xi_m = \xi_m(\varepsilon) = \max\{\xi_n = \xi_n(\varepsilon) : n = 0, \dots, m-1\}$ with $\lim_{\varepsilon\to 0}\xi_n = 0$.

In both cases above, by Assumptions 2–3 and the continuity of $\overline{F}$ (Lemma 3.1, (i)) it follows, respectively, that

$$\begin{aligned}&\rho\left(\int_{t_n}^{t_{n+1}} F\left(\frac{\tau}{\varepsilon}, u(\tau)\right) d\tau, \int_{t_n}^{t_{n+1}} F\left(\frac{\tau}{\varepsilon}, u(t_n)\right) d\tau\right)\\ &\le \int_{t_n}^{t_{n+1}} \rho\left(F\left(\frac{\tau}{\varepsilon}, u(\tau)\right), F\left(\frac{\tau}{\varepsilon}, u(t_n)\right)\right) d\tau \le \gamma_n\alpha \le \gamma_m\alpha\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&\rho\left(\int_{t_n}^{t_{n+1}} \overline{F}(\tilde{u}(\tau))d\tau, \int_{t_n}^{t_{n+1}} \overline{F}(\tilde{u}(t_n))d\tau\right)\\ &\le \int_{t_n}^{t_{n+1}} \rho(\overline{F}(\tilde{u}(\tau)), \overline{F}(\tilde{u}(t_n)))d\tau \le \delta_n\alpha \le \delta_m\alpha\end{aligned} \tag{3.10}$$

where $\gamma_m = \gamma_m(\varepsilon) = \max\{\gamma_n = \gamma_n(\varepsilon) : n = 0, \dots, m-1\}$ and $\delta_m = \delta_m(\varepsilon) = \max\{\delta_n = \delta_n(\varepsilon) : n = 0, \dots, m-1\}$ with $\lim_{\varepsilon\to 0}\gamma_n = \lim_{\varepsilon\to 0}\delta_n = 0$.

Hence, from (3.4) it follows that

$$\begin{aligned}&\rho\left(\int_0^t F\left(\frac{\tau}{\varepsilon}, u(\tau)\right) d\tau, \int_0^t \overline{F}(\tilde{u}(\tau))d\tau\right)\\ &\le \sum_{n=0}^{m-1} \rho\left(\int_{t_n}^{t_{n+1}} F\left(\frac{\tau}{\varepsilon}, u(t_n)\right) d\tau, \int_{t_n}^{t_{n+1}} \overline{F}(\tilde{u}(t_n))d\tau\right)\\ &+ (\gamma_m + \delta_m)t + \alpha(2B + \xi).\end{aligned} \tag{3.11}$$

Now we consider the first term in the right-hand side of inequality (3.11). For each $n = 0, \dots, m-1$, we have

$$\begin{aligned}\beta_n &:= \rho\left(\int_{t_n}^{t_{n+1}} F\left(\frac{\tau}{\varepsilon}, u(t_n)\right) d\tau, \int_{t_n}^{t_{n+1}} \overline{F}(\tilde{u}(t_n))d\tau\right)\\ &= \alpha\cdot\rho\left(\frac{\varepsilon}{\alpha}\int_{t_n/\varepsilon}^{t_n/\varepsilon+\alpha/\varepsilon} F(\tau, u(t_n))\, d\tau, \overline{F}(\tilde{u}(t_n))\right) := \alpha\cdot\varrho_n \le \alpha\cdot\varrho_m,\end{aligned}$$

where $\varrho_m = \max\{\varrho_n = \varrho_n(\varepsilon) : n = 0, \dots, m-1\}$ and, by Lemma 3.3, $\lim\limits_{\varepsilon\to 0}\varrho_n = 0$. Then

$$\begin{aligned}&\sum_{n=0}^{m-1} \rho\left(\int_{t_n}^{t_{n+1}} F\left(\frac{\tau}{\varepsilon}, u(t_n)\right) d\tau, \int_{t_n}^{t_{n+1}} \overline{F}(\tilde{u}(t_n))d\tau\right)\\ &= \sum_{n=0}^{m-1}\beta_n \le \varrho_m\sum_{n=0}^{m-1}\alpha = \varrho_m\sum_{n=0}^{m-1}(t_{n+1} - t_n) \le \varrho_m t.\end{aligned}$$

Finally, from (3.11) we obtain

$$\begin{aligned}&\rho\left(\int_0^t F\left(\frac{\tau}{\varepsilon},u(\tau)\right)d\tau,\int_0^t \overline{F}(\tilde{u}(\tau))d\tau\right)\\&\leq(\varrho_m+\gamma_m+\delta_m)t+\alpha(2B+\xi)\leq(\varrho+\gamma+\delta)L+\alpha(2B+\xi)\end{aligned}\tag{3.12}$$

where $\varrho+\gamma+\delta=\max\{\varrho_m+\gamma_m+\delta_m : m=0,\dots,p-1\}$. As the right-hand side of (3.12) tends to zero as $\varepsilon\to 0$, and this limit is uniform with respect to t, this finishes the proof of the lemma.

Case 2. F satisfies Assumptions 1 and 4–6. The proof is similar to the one in the case 1 above. The only difference is that inequalities (3.9) and (3.10) will be obtained by Assumption 6 and the Lipschitz condition of $\overline{F}$ (Lemma 3.1, (ii)) instead of Assumptions 2–3 and the continuity of $\overline{F}$ (Lemma 3.1, (i)). ■

3.2. Proof of Theorem 2.1

We assume that the Assumptions 1–5 are fulfilled.

Let $L>0$. For ε sufficiently small, suppose that the family $\{x_\varepsilon\}$ of solutions of problem (2.1) is defined on $[0,L]$. We have x_ε are equicontinuous and uniformly bounded on $[0,L]$. Indeed, let $t,\tau\in[0,L]$ with $t>\tau$ and $t\to\tau$. As in the inequality (3.8) in the proof of Lemma 3.4 we can verify that

$$\begin{aligned}|x_\varepsilon(t)-x_\varepsilon(\tau)|&\leq\int_\tau^t b\left(\frac{s}{\varepsilon}\right)ds\\&=(t-\tau)\left(\frac{\varepsilon}{t-\tau}\int_{\tau/\varepsilon}^{t/\varepsilon}b(s)ds\right)\leq(t-\tau)(B+1).\end{aligned}\tag{3.13}$$

On the other hand, from (3.13) we deduce that

$$|x_\varepsilon(t)|\leq|x_0|+(B+1)L,\qquad\forall t\in[0,L].$$

So, by the Arzela-Ascoli theorem there exists a continuous function $y:[0,L]\to U$ such that

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,L]}|x_\varepsilon(t)-y(t)|=0.\tag{3.14}$$

Let us now verify that the function y is a solution of problem (2.3). If $t\in[0,L]$

the function y is such that

$$\begin{aligned}
&\delta\left(y(t), x_0 + \int_0^t \overline{F}(y(\tau))d\tau\right) \\
(3.15)\qquad &\leq |y(t) - x_\varepsilon(t)| + \delta\left(x_\varepsilon(t), x_0 + \int_0^t \overline{F}(y(\tau))d\tau\right) \\
&\leq \sup_{t\in[0,L]} |y(t) - x_\varepsilon(t)| + \sup_{t\in[0,L]} \rho\left(\int_0^t F\left(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\right) d\tau, \int_0^t \overline{F}(y(\tau))d\tau\right).
\end{aligned}$$

By (3.14) and Lemma 3.4 (with $u = x_\varepsilon$ and $\tilde{u} = \tilde{x}_0 = y$), the right-hand side of (3.15) tends to zero as $\varepsilon \to 0$, so that one can conclude that the function y is a solution of problem (2.3).

The proof of Theorem 2.1 is complete. ■

3.3. Proof of Theorem 2.3

We assume that the Assumptions 1–5 are fulfilled.

Let $L \in J$. Fix $\zeta > 0$ to be such that the compact neighborhood around $\Gamma = \{y(t) : t \in [0, L]\}$ given by $W = \{z \in \mathbb{R}^d \;/\; \exists t \in [0, L] : |z - y(t)| \leq \zeta\}$ is included in U. Let x_ε be any solution of problem (2.1) and A the subset of $[0, L]$ defined by

$$A = \{L_1 \in [0, L] \;/\; x_\varepsilon \text{ is defined on } [0, L_1] \text{ and } \{x_\varepsilon(t) : t \in [0, L_1]\} \subset W\}.$$

The set A is nonempty ($0 \in A$) and bounded above by L. Let L_0 be the upper bound of A and let $L_1 \in A$ be such that $L_0 - \varepsilon < L_1 \leq L_0$. Since $x_\varepsilon(L_1) \in W$, the solution x_ε can be continued on some interval $[L_1, L_1 + \Delta]$ where $\Delta > 0$ is independent of ε.

Let $L_1' := L_1 + \varepsilon$. For ε sufficiently small we have $[0, L_1'] \subset [0, L_1 + \Delta]$. Now, taking into account the assumption on the uniqueness of the solution of (2.3), by Theorem 2.1, we deduce that y is defined on $[0, L_1']$ and

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,L_1']} |x_\varepsilon(t) - y(t)| = 0. \tag{3.16}$$

It remains to verify that $L \leq L_1'$. If this is not true, then x_ε is defined on $[0, L_1']$ and $\{x_\varepsilon(t) : t \in [0, L_1']\} \subset W$ imply $L_1' \in A$. This contradicts the fact that $L_1' > L_0$. So the proof is complete. ■

3.4. Proof of Theorem 2.4

We assume that the Assumptions 1 and 4–6 are fulfilled.

***Proof of* (i).** The proof uses Lemma 3.1, (ii) and Lemma 3.4. As this proof is similar to the proof of Theorem 2.1, it is omitted.

***Proof of* (ii).** We will construct a Cauchy sequence of successive approximations $\{x_{\varepsilon,n}\}_n$ which converges to a solution of problem (2.1) when $n \to \infty$.

Let $L > 0$. Let y be a solution of (2.3) and suppose that it is defined on $[0, L]$. For any $t \in [0, L]$ we have

$$\begin{aligned} &\delta\left(y(t), x_0 + \int_0^t F\left(\frac{\tau}{\varepsilon}, y(\tau)\right) d\tau\right) \\ &\leq \sup_{t\in[0,L]} \rho\left(\int_0^t \overline{F}(y(\tau))d\tau, \int_0^t F\left(\frac{\tau}{\varepsilon}, y(\tau)\right) d\tau\right) := \xi \quad (\xi = \xi_\varepsilon). \end{aligned} \tag{3.17}$$

By Lemma 3.4 (with $u = \tilde{u} = y$), $\lim_{\varepsilon\to 0} \xi = 0$.

Let $x_{\varepsilon,1} : [0, L] \to U$ be a continuous function satisfying, for all $t \in [0, L]$:

(1) $x_{\varepsilon,1}(t) \in x_0 + \displaystyle\int_0^t F\left(\frac{\tau}{\varepsilon}, y(\tau)\right) d\tau.$

(2) $|y(t) - x_{\varepsilon,1}(t)| = \delta\left(y(t), x_0 + \displaystyle\int_0^t F\left(\frac{\tau}{\varepsilon}, y(\tau)\right) d\tau\right).$

For the existence of such continuous selection, compare, e.g., [2], Proposition 3.4b with adapted assumptions. Property (2) and inequality (3.17) imply that, for any $t \in [0, L]$

$$|x_{\varepsilon,1}(t) - y(t)| \leq \xi.$$

We claim that we can define a sequence of continuous functions $\{x_{\varepsilon,n}\}_n$, with $x_{\varepsilon,0} = y$, which satisfies the following properties for all $t \in [0, L]$:

(3) $x_{\varepsilon,n}(t) \in x_0 + \displaystyle\int_0^t F\left(\frac{\tau}{\varepsilon}, x_{\varepsilon,n-1}(\tau)\right) d\tau.$

(4) $|x_{\varepsilon,n}(t) - x_{\varepsilon,n-1}(t)| \leq \xi\dfrac{(\lambda t)^{n-1}}{(n-1)!}$, with a constant λ as in Assumption 6.

Assume we have defined on $[0, L]$ functions $x_{\varepsilon,n}$ up to $n = p$, satisfying properties (3) and (4). As above, one can affirm that there exists a continuous function $x_{\varepsilon,p+1} : [0, L] \to U$ satisfying, for all $t \in [0, L]$:

(5) $x_{\varepsilon,p+1}(t) \in x_0 + \displaystyle\int_0^t F\left(\frac{\tau}{\varepsilon}, x_{\varepsilon,p}(\tau)\right) d\tau.$

(6) $|x_{\varepsilon,p}(t) - x_{\varepsilon,p+1}(t)| = \delta\left(x_{\varepsilon,p}(t), x_0 + \displaystyle\int_0^t F\left(\frac{\tau}{\varepsilon}, x_{\varepsilon,p}(\tau)\right) d\tau\right).$

We have for any $t \in [0, L]$

$$\begin{aligned}&|x_{\varepsilon,p+1}(t) - x_{\varepsilon,p}(t)|\\ &\leq \rho\left(\int_0^t F\left(\frac{\tau}{\varepsilon}, x_{\varepsilon,p}(\tau)\right) d\tau, \int_0^t F\left(\frac{\tau}{\varepsilon}, x_{\varepsilon,p-1}(\tau)\right) d\tau\right)\\ &\leq \lambda \int_0^t |x_{\varepsilon,p}(\tau) - x_{\varepsilon,p-1}(\tau)| d\tau \leq \lambda \int_0^t \xi \frac{(\lambda\tau)^{p-1}}{(p-1)!} d\tau = \xi \frac{(\lambda t)^p}{p!},\end{aligned}$$

which finishes the proof by induction.

Now, Property (4) implies that $\{x_{\varepsilon,n}\}_n$ is a Cauchy sequence of continuous functions, converging uniformly to a continuous function x_ε. Moreover, taking into account Assumption 6 we obtain for any $t \in [0, L]$

$$\begin{aligned}0 &= \lim_{n\to\infty} \delta\left(x_{\varepsilon,n}(t), x_0 + \int_0^t F\left(\frac{\tau}{\varepsilon}, x_{\varepsilon,n-1}(\tau)\right) d\tau\right)\\ &= \delta\left(x_\varepsilon(t), x_0 + \int_0^t F\left(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\right) d\tau\right).\end{aligned}$$

Hence, x_ε is a solution of problem (2.1).

Finally, by Property (4) we have

$$\begin{aligned}&\sup_{t\in[0,L]} |x_{\varepsilon,n}(t) - y(t)|\\ &\leq \sup_{t\in[0,L]} (|x_{\varepsilon,n}(t) - x_{\varepsilon,n-1}(t)| + \cdots + |x_{\varepsilon,1}(t) - y(t)|)\\ &\leq \xi \sup_{t\in[0,L]} \left(\frac{(\lambda t)^{n-1}}{(n-1)!} + \cdots + 1\right) \leq \xi \sup_{t\in[0,L]} e^{\lambda t} = \xi e^{\lambda L}.\end{aligned} \tag{3.18}$$

As the right hand side of (3.18) tends to zero as $\varepsilon \to 0$, this finishes the proof of (ii).

The proof of the theorem is complete. ■

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