Discussiones Mathematicae
Differential Inclusions, Control and Optimization 35 (2015) 89–100
doi:10.7151/dmdico.1168

EXPONENTIAL STABILITY OF NONLINEAR NON-AUTONOMOUS MULTIVARIABLE SYSTEMS

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Abstract

We consider nonlinear non-autonomous multivariable systems governed by differential equations with differentiable linear parts. Explicit conditions for the exponential stability are established. These conditions are formulated in terms of the norms of the derivatives and eigenvalues of the variable matrices, and certain scalar functions characterizing the nonlinearity. Moreover, an estimate for the solutions is derived. It gives us a bound for the region of attraction of the steady state. As a particular case we obtain absolute stability conditions.

Our approach is based on a combined usage of the properties of the "frozen" Lyapunov equation, and recent norm estimates for matrix functions. An illustrative example is given.

Keywords: nonlinear nonautonomous systems, exponential stability, absolute stability.

2010 Mathematics Subject Classification: 93D20,34D20.

1. Introduction and statement of the main result

The problem of stability analysis of nonlinear nonautonomous systems continues to attract the attention of many specialists despite its long history. It is still one of the most burning problems of control theory, because of the absence of its complete solution. The problem of the synthesis of a stable system is closely connected with the problem of stability analysis. Any progress in the problem of analysis implies success in the problem of synthesis of stable systems. The basic method for the stability analysis of nonlinear continuous systems is the direct Lyapunov method, cf. [8, 10]. By that method many very strong results are obtained, but finding Lyapunov's functions is often connected with serious mathematical difficulties. In the interesting papers [1, 11, 12] the authors consider essentially nonlinear nonautonomous ordinary differential equations (i.e., equations without separated linear parts) with locally Lipschitz entries, as well as partially slowly nonlinear time-varying systems. Besides, the classical averaging methods are extended and Lyapunov's theory is developed. About other interesting relevant results see [2, 13] and references therein.

In this note, for a class of nonlinear non-autonomous systems we establish explicit conditions for the exponential stability.

Introduce the notations. Let $\mathbb{C}^n$ be the complex n-dimensional Euclidean space with a scalar product $(\cdot,\cdot)$, the Euclidean norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$ and unit matrix I. For a linear operator A in $\mathbb{C}^n$ (matrix), $\|A\| = \sup_{x\in\mathbb{C}^n} \|Ax\|/\|x\|$ is the spectral (operator) norm, A^* is the adjoint operator, $N_2(A)$ is the Hilbert-Schmidt (Frobenius) norm of A: $N_2(A) = \sqrt{trace\ AA^*}$; $\lambda_k(A)$ $(k = 1, \dots, n)$ are the eigenvalues with their multiplicities, $\alpha(A) = \max_k\ Re\ \lambda_k(A)$. The quantity

$$g(A) = (N_2^2(A) - \sum_{k=1}^{n} |\lambda_k(A)|^2)^{1/2}$$

plays an essential role hereafter. In addition, $\Omega(r) = \{w \in \mathbb{C}^n : \|w\| \le r\}$ for a positive $r \le \infty$.

Everywhere below $A(t)$ is a variable $n \times n$ matrix, defined, uniformly bounded on $[0,\infty)$ and having a derivative measurable and uniformly bounded on $[0,\infty)$. Our main object in this paper is the equation

$$u'(t) = A(t)u(t) + F(u(t), t) \quad (t \ge 0), \tag{1.1}$$

where $F : \Omega(r) \times [0,\infty) \to \mathbb{C}^n$ is continuous and satisfies the inequality

$$\|F(w,t)\| \le \nu(t)\|w\| \quad (w \in \Omega(r); t \ge 0), \tag{1.2}$$

where $\nu(t)$ is a scalar continuous function uniformly bounded on $[0,\infty)$.

The aim of the present paper is to extend the freezing method for linear systems [3, 5, 15, 7] to equation (1.1).

A (global) solution of (1.1) is a continuously differentiable vector valued function satisfying (1.1) for all $t \ge 0$. The existence and uniqueness of solutions is assumed.

The zero solution of system (1.1) *is said to be exponentially stable* in the class of nonlinearities (1.2) if there are constants $M \ge 1, \epsilon > 0$ and $\delta > 0$, such that $\|u(t)\| \le M\ exp(-\epsilon t)\ \|u(0)\|$ $(t \ge 0)$ for any solution $u(t)$ of (1.1), provided $\|u(0)\| < \delta$.

Suppose that

$$\sup_{t\geq 0} \alpha(A(t)) < 0, \tag{1.3}$$

denote by $\lambda_R(t)$ the smallest eigenvalue of $(A(t) + A^*(t))/2$ and put

$$\mu(t) := \sum_{j,k=0}^{n-1} \frac{(k+j)!g^{k+j}(A(t))}{2^{k+j}|\alpha(A(t))|^{k+j+1}(k!j!)^{3/2}}.$$

Now we are in a position to formulate our main result.

Theorem 1. *Let the conditions* (1.2), (1.3) *and*

$$\sup_{t\geq 0} \frac{1}{t}\int_0^t (-2 + \mu^2(s)\|A'(s)\| + 2\mu(s)\nu(s))|\lambda_R(s)|ds < 0 \tag{1.4}$$

hold. Then the zero solution of (1.1) *is exponentially stable.*

This theorem is proved in the next two sections. Below in this section we check that it is sharp. In addition, we will show that the proof Theorem 1 gives us the absolute stability conditions.

From (1.3) it follows $\sup_t \lambda_R(s) < 0$. Now Theorem 1 implies

Corollary 2. *Under conditions* (1.2) *and* (1.3), *let*

$$\sup_{t\geq 0} (\mu^2(t)\|A'(t)\| + 2\mu(t)\nu(t)) < 2. \tag{1.5}$$

Then the zero solution of (1.1) *is exponentially stable.*

The following relations are checked in [6, Section 1.5]:

$$g^2(A) \leq N_2^2(A) - |Trace\ A^2|, g(A) \leq \frac{1}{\sqrt{2}}N_2(A - A^*)$$

and $g(e^{ia}A + zI_H) = g(A)$ $(a \in \mathbb{R}, z \in \mathbb{C})$; if A is a normal matrix: $AA^* = A^*A$, then $g(A) = 0$. If A_1 and A_2 are commuting matrices, then $g(A_1+A_2) \leq g(A_1)+g(A_2)$. In addition, by the inequality between the geometric and arithmetic mean values,

$$\left(\frac{1}{n}\sum_{k=1}^n |\lambda_k(A)|^2\right)^n \geq \left(\prod_{k=1}^n |\lambda_k(A)|\right)^2.$$

Hence $g^2(A) \leq N_2^2(A) - n|det\ A|^{2/n}$.

Theorem 1 and Corollary 2 are sharp in the following sense. Let $F(w,t) = \nu_0 w$ ($\nu_0 \equiv const > 0$) and $A(t) = A_0$ be a constant normal matrix. Then $g(A(t)) = 0$, $\mu(t) = \frac{1}{|\alpha(A_0)|}$ and (1.5) takes the form

$$\nu_0 < |\alpha(A_0)|. \tag{1.6}$$

But this inequality is the necessary and sufficient stability condition in the considered case. Moreover, if (1.1) is linear, then $\nu(t) \equiv 0$ and Theorem 1 yields the stability result obtained in [7] in the framework of the freezing method.

2. Preliminaries

Put

$$Q(t) = 2\int_0^\infty e^{A^*(t)s} e^{A(t)s} ds \text{ and } q(t) = 2\int_0^\infty \|e^{A(t)s}\|^2 ds \ (t \ge 0).$$

As it is well known, $Q(t)$ is a unique solution of the equation

$$A^*(t)Q(t) + Q(t)A(t) = -2I, \tag{2.1}$$

cf. [4, Section I.5]. Clearly,

$$\|Q(t)\| \le q(t) \quad (t \ge 0). \tag{2.2}$$

Lemma 3. *Let condition* (1.2) *hold. Then $Q(t)$ is differentiable and $\|Q'(t)\| \le q^2(t)\|A'(t)\|$.*

Proof. Differentiating (2.1), we have

$$A^*(t)Q'(t) + Q'(t)A(t) = -(A^*(t))'Q(t) + Q(t)A'(t)) \quad (t \ge 0).$$

Hence

$$Q'(t) = \int_0^\infty e^{A^*(t)s}((A^*(t))'Q(t) + Q(t)A'(t))e^{A(t)s} ds.$$

Thus,

$$\|Q(t)\| \le \frac{1}{2} q(t)\|(A^*(t))'Q(t) + Q(t)A'(t)\| \le q(t)\|Q(t)\|\|A'(t)\|.$$

Now (2.2) yields the result. ■

For a constant Hurwitz matrix A_0, due to [6, Lemma 1.9.2],

$$2\int_0^\infty \|e^{A_0 s}\|^2 ds \le \hat{\mu}(A_0), \text{ where } \hat{\mu}(A_0) := \sum_{j,k=0}^{n-1} \frac{(k+j)!g^{k+j}(A_0)}{2^{k+j}|\alpha(A_0)|^{k+j+1}(k!j!)^{3/2}}.$$

So $\mu(t) = \hat{\mu}(A(t))$ and therefore,

$$\|Q(t)\| \le q(t) \le \mu(t) \ \ (t \ge 0). \tag{2.3}$$

Now Lemma 3 implies

$$\|Q'(t)\| \le \mu^2(t)\|A'(t)\|. \tag{2.4}$$

Furthermore, put $w(t) = e^{A_0 t}v$ $(v \in \mathbb{C}^n)$. Then $w'(t) = A_0 w(t)$, and

$$\frac{d(w(t), w(t))}{dt} = ((A_0 + A_0^*)w(t), w(t)).$$

Hence

$$\frac{d(w(t), w(t))}{dt} \ge \lambda(A_0 + A_0^*)(w(t), w(t)),$$

where $\lambda(A_0 + A_0^*)$ is the smallest eigenvalue of $A_0 + A_0^*$. Therefore,

$$\|e^{A_0 t}v\|^2 = (w(t), w(t) \ge e^{t\lambda(A_0+A_0^*)}(w(0), w(0)) = e^{t\lambda(A_0+A_0^*)}\|v\|^2.$$

Recall that A_0 is Hurwitzian, so $\lambda(A_0 + A_0^*) < 0$. Put

$$Q_0 = 2\int_0^\infty e^{A_0^* s}e^{A_0 s}ds.$$

Then

$$(Q_0 h, h) = 2\int_0^\infty (e^{A_0 s}h, e^{A_0 s}h)ds \ge 2\int_0^\infty e^{\lambda(A_0+A_0^*)s}ds\ \|h\|^2 = \frac{2\|h\|^2}{|\lambda(A_0 + A_0^*)|}$$

$(h \in \mathbb{C}^n)$. Hence,

$$\|Q^{-1}(t)\| \le |\lambda_R(t)|. \tag{2.5}$$

3. Proof of Theorem 1

Lemma 4. *Let conditions* (1.2) *with* $r = \infty$ *and* (1.3) *hold. Then a solution* $u(t)$ *of* (1.1) *satisfies the inequality*

$$(Q(t)u(t), u(t)) \le y(0) exp\,[\int_0^t (-2 + \|Q'(t_1)\| + 2\nu(t_1)\|Q(t_1)\|)\|Q^{-1}(t_1)\|dt_1$$

$$(y(0) = (Q(0)u(0), u(0))).$$

Proof. Put $b(t) = 1/\|Q(t)\|$ and substitute

$$u(t) = e^{-\int_0^t b(s)ds} x(t) \tag{3.1}$$

into (1.1). Then we obtain

$$x' = (b(t)I + A(t))x + F_1(x, t) \ \ (x = x(t)), \tag{3.2}$$

where

$$F_1(x, t) = e^{\int_0^t b(s)ds} F(xe^{-\int_0^t b(s)ds}, t).$$

Let $Q(t)$ be a solution of (2.1), again. Multiplying equation (3.2) by $Q(t)$ and taking the scalar product, we get

$$(Q(t)x'(t), x(t)) = (Q(t)A(t)x(t), x(t)) + (Q(t)F_1(x(t), t), x(t)).$$

Since

$$\frac{d}{dt}(Q(t)x(t), x(t)) = (Q(t)x'(t), x(t)) + (x(t), Q(t)x'(t)) + (Q'(t)x(t), x(t)),$$

we obtain

$$\frac{d}{dt}(Q(t)x(t), x(t)) = (Q(t)(b(t)I + A(t))x(t), x(t)) + (x(t), Q(t)(b(t)I + A(t))x(t))$$

$$+ (Q'(t)x(t), x(t)) + (Q(t)F_1(x(t), t), x(t)) + (x(t), Q(t)F_1(x(t), t))$$

$$= 2b(t)(Q(t)x(t), x(t)) + ((Q(t)A(t) + A^*(t)Q(t))x(t), x(t))$$

$$+ (Q'(t)x(t), x(t)) + (Q(t)F_1(x(t), t), x(t)) + (x(t), Q(t)F_1(x(t), t))$$

$$= ((-2I + Q'(t) + \frac{2}{\|Q(t)\|}Q(t))x(t), x(t)) + (Q(t)F_1(x(t), t), x(t))$$

$$+ (x(t), Q(t)F_1(x(t), t)).$$

But

$$\left(\frac{2}{\|Q(t)\|}Q(t)x(t), x(t)\right) \le 2\|x(t)\|^2.$$

Therefore,

$$\frac{d}{dt}(Q(t)x(t), x(t)) \leq \|Q'(t)\|\|x(t)\|^2 + 2Re\ (Q(t)F_1(x(t), t), x(t)). \tag{3.3}$$

Take into account that due to (1.2)

$$\|F_1(x(t), t)\| = e^{\int_0^t b(s)ds}\|F(x(t)e^{-\int_0^t b(s)ds})\|$$

$$\leq e^{\int_0^t b(s)ds}\nu(t)\|x(t)\|e^{-\int_0^t b(s)ds} = \nu(t)\|x(t)\|.$$

Consequently,

$$|(Q(t)F_1(x(t), t), x(t))| \leq \nu(t)\|Q(t)\|\|x(t)\|^2.$$

Now (3.3) yields

$$\frac{d}{dt}(Q(t)x(t), x(t))) \leq (\|Q'(t)\| + 2\nu(t)\|Q(t)\|)\|x(t)\|^2.$$

Hence,

$$\frac{d}{dt}(Q(t)x(t), x(t))) \leq (\|Q'(t)\| + 2\nu(t)\|Q(t)\|)(x(t), x(t))$$

$$\leq (\|Q'(t)\| + 2\nu(t)\|Q(t)\|)\|Q^{-1}(t)\|(Q(t)x(t), x(t)).$$

Integrating this inequality with $y(t) = (Q(t)x(t), x(t))$, we get

$$y(t) \leq y(0) + \int_0^t (\|Q'(t_1)\| + 2\nu(t_1)\|Q(t_1)\|)\|Q^{-1}(t_1)\|y(t_1)dt_1.$$

Now the Gronwall lemma implies

$$y(t) \leq y(0)exp\ [\int_0^t (\|Q'(t_1)\| + 2\nu(t_1)\|Q(t_1)\|)\|Q^{-1}(t_1)\|dt_1].$$

Due to (3.1) this yields the required result. ■

Corollary 5. *Under the hypothesis of Lemma* 4, *let*

$$\theta_0 := \sup_{t\geq 0} \frac{1}{2t}\int_0^t (-2 + \|Q'(t_1)\| + 2\nu(t_1)\|Q(t_1)\|)\|Q^{-1}(t_1)\|dt_1 < \infty.$$

Then a solution $u(t)$ *of* (1.1) *satisfies the inequality*

$$(Q(t)u(t), u(t)) \leq y(0)exp\ (2t\theta_0) \ \ (t \geq 0)$$

and therefore,

(3.4) $$\|u(t)\| \le d_0\|u(0)\| exp\ (t\theta_0) \ \ (t \ge 0),$$

where

$$d_0 = \sup_t \sqrt{\|Q(0)\|\|Q^{-1}(t)\|}.$$

Indeed, this result is due to Lemma 4, since

$$(Q(t)u(t), u(t)) \ge \|u(t)\|^2/\|Q^{-1}(t)\|.$$

Lemma 6. *Let conditions* (1.2) *and* (1.3) *hold with* $r < \infty$. *Then inequality* (3.4) *is valid, provided the conditions* $\theta_0 < 0$ *and* $\|u(0)\| < d_0/r$ *are fulfilled, and consequently the zero solution to* (1.1) *is exponentially stable.*

Proof. For a sufficiently small $t_0 > 0$ we have $\|u(t)\| < r \ \ (t \le t_0)$ and therefore, (3.4) holds for $t \le t_0$. Extending it to all $t \ge 0$ we arrive at the result. ■

***Proof of Theorem* 1.** Due to (2.3)–(2.5) we have

$$d_0 \le \hat{d} := \sup_t \sqrt{\|\mu(0)\|\|\lambda_R(t)\|},$$

and $\theta_0 \le \hat{\theta}$, where according to (1.4),

$$\hat{\theta} := \sup_{t\ge 0} \frac{1}{2t}\int_0^t (-2 + \mu^2(s)\|A'(s)\| + 2\mu(s)\nu(s))|\lambda_R(s)|ds < 0.$$

Now the previous lemma implies

(3.5) $$\|u(t)\| \le \hat{d}\|u(0)\| exp\ (t\hat{\theta}) \ \ (t \ge 0), \ \text{provided } \|u(0)\| < \hat{d}/r.$$

The assertion of Theorem 1 directly follows from (3.5). ■

4. Additional stability conditions

4.1. Absolute stability

Assume that $F : \mathbb{C}^n \times [0, \infty) \to \mathbb{C}^n$ is continuous and satisfies the inequality

(4.1) $$\|F(w, t)\| \le \nu(t)\|w\| \ \ (w \in \mathbb{C}^n;\ t \ge 0),$$

where $\nu(t)$ again is continuous and uniformly bounded on $[0, \infty)$.

The zero solution of system (1.1) *is said to be absolutely exponentially stable* in the class of nonlinearities (4.1) if there are constants $M \ge 1, \epsilon > 0$, which do not depend on a concrete form of F (but which depend on $\nu(t)$ and $A(t)$), such that $\|x(t)\| \le M\ exp(-\epsilon t)\ \|x(0)\|\ (t \ge 0)$ for any solution $x(t)$ of (1.1).

Directly from (3.5) it follows

Corollary 7. *Under inequality* (1.3), *let one of the conditions, either* (1.4) *or* (1.5) *hold. Then the zero solution of* (1.1) *is absolutely exponentially stable in the class of nonlinearities* (4.1).

The literature on the absolute stability of continuous systems is very rich, but mainly systems with autonomous linear parts were considered, cf. the survey [9]. One-contour systems with non-stationary linear parts were explored, in particular, in the the well-known paper by Yakubovich [14]. Absolute stability of multivariable systems with non-stationary linear parts to the best of our knowledge was almost not investigated in the available literature.

4.2. Stability tests in terms of the Hurwitzness of auxiliary matrices

We are going to reformulate the condition

$$\mu^2(t)\|A'(t)\| + 2\mu(t)\nu(t) < 2 \tag{4.2}$$

for a fixed t in the terms of the Hurwitzness of some matrices. In this subsection, for the brevity sometimes we put $g(A(t)) = g$, $\nu(t) = \nu$, $\alpha(A(t)) = \alpha$ and $a = \sqrt{\|A'(t)\|}$. In addition, let

$$P(y) = \sum_{j,k=0}^{n-1} \frac{(k+j)! y^{2n-k-j-2}}{(k!j!)^{3/2}} \quad (y > 0).$$

Denote by $r_0(b)$ the unique positive root of the equation

$$y^{2n-1} = bP(y) \;\; (b = const > 0) \tag{4.3}$$

and assume that $\nu \neq 0, g \neq 0$. Then (4.2) can be rewritten as

$$\mu^2(t)a^2 + 2\mu(t)\nu + \nu^2/(a^2) = (\mu(t)a + \nu/a)^2 < 2 + \nu^2/(a^2),$$

and therefore, $cg\mu(t) < 2$, where

$$c = c(t) = \frac{2a^2}{(\sqrt{2a^2+\nu^2} - \nu)g} = \frac{2\|A'(t)\|}{(\sqrt{2\|A'(t)\| + \nu^2(t)} - \nu(t))\; g(A(t))}.$$

We can write

$$\frac{1}{2}cg\mu(t) = cg \sum_{j,k=0}^{n-1} \frac{(k+j)! g^{k+j}}{2^{k+j+1}|\alpha|^{k+j+1}(k!j!)^{3/2}} < 1.$$

Substitute $|\alpha| = yg/2$ into this inequality. Then

$$c \sum_{j,k=0}^{n-1} \frac{(k+j)!}{y^{k+j+1}(k!j!)^{3/2}} < 1.$$

Multiplying this inequality by y^{2n-1}, we get $y^{2n-1} > cP(y)$. So $y > r_0(c)$ and therefore $|\alpha| > gr_0(c)/2$. So (4.2) holds, provided $\alpha + gr_0(c)/2 < 0$, since $A(t)$ is Hurwitzian. Now from Corollary 2 it follows

Corollary 8. *For all $t \geq 0$, let $A'(t) \neq 0$, $g(A(t)) \neq 0$ and*

$$A(t) + \frac{1}{2} g(A(t)) r_0(c(t)) I$$

be a Hurwitz matrix. Then the zero solution to (1.1) *is exponentially stable.*

The cases $A'(t) = 0$ or (and) $g(A(t)) = 0$ are left to the reader. Let us estimate $r_0(b)$.

Lemma 9. *One has $r_0(b) \leq \delta_0(b)$, where*

$$\delta_0(b) = \begin{cases} \sqrt[2n-1]{bP(1)} & \text{if } bP(1) \leq 1, \\ bP(1) & \text{if } bP(1) \geq 1. \end{cases}$$

Proof. If $bP(1) \geq 1$, then from (4.3) it follows that $r_0(b) = r_0 \geq 1$,

$$r_0^{2n-1} = b \sum_{j,k=0}^{n-1} r_0^{2n-k-j-2} \frac{(k+j)!}{(k!j!)^{3/2}} \leq b r_0^{2n-1} P(1)$$

and thus $r_0 \leq bP(1)$. If $bP(1) \leq 1$, then $r_0 \leq 1$ and $r_0^{2n-1} \leq bP(1)$, as claimed. ∎

Due to the previous lemma and Corollary 8, we arrive at the following result.

Corollary 10. *For all $t \geq 0$, let $A'(t) \neq 0$, $g(A(t)) \neq 0$ and*

$$A(t) + \frac{1}{2} g(A(t)) \delta_0(c(t)) I$$

be a Hurwitz matrix. Then the zero solution to (1.1) *is exponentially stable.*

4.3. Example

Consider the system

$$\dot{x}_1(t) = -x_1(t) + a\ sin(\omega t) x_2(t) + f_1(x_1(t), x_2(t), t),$$

$$\dot{x}_2(t) = a\ sin(\omega t) x_1(t) - x_2(t) + f_2(x_1(t), x_2(t), t)$$

with positive constants a and ω. In addition, f_k $(k = 1, 2)$ are real continuous functions defined on $\mathbb{R}^2 \times [0, \infty)$ and satisfying

$$|f_k(y_1, y_2, t)|^2 \le \eta_k^2(t)(y_1^2 + y_2^2)^{p_k} \quad (y_1, y_2 \in \mathbb{R}, t \ge 0)$$

for a $p_k > 1$. Here $\eta_k(t)$ are bounded continuous functions. So for an $r > 0$ we have

$$\|F(y,t)\|^2 = |f_1(y_1, y_2, t)|^2 + |f_2(y_1, y_2, t)|^2$$

$$\le \eta_1^2(t)(y_1^2 + y_2^2)^{p_1} + \eta_2^2(t)(y_1^2 + y_2^2)^{p_2}$$

$$\le (\eta_1^2(t) r^{2(p_1-1)} + \eta_2^2(t) r^{2(p_2-1)}(y_1^2 + y_2^2) \quad (y = (y_1, y_2); y_1^2 + y_2^2 \le r^2).$$

Thus condition (1.2) holds with

$$\nu(t) = (\eta_1^2(t) r^{2(p_1-1)} + \eta_2^2(t) r^{2(p_2-1)})^{1/2}.$$

In addition, under consideration,

$$A(t) = \begin{pmatrix} -1 & a\ sin(\omega t) \\ -a\ sin(\omega t) & -1 \end{pmatrix}.$$

We have $\lambda_{1,2}(A(t)) = -1 \pm ia\ sin(\omega t)$ and $\|A'(t)\| = a\omega |cos(\omega t)|$. So $\alpha(A(t)) \equiv -1$. In addition, $g(A(t)) \equiv 0$, since $A(t)$ under consideration is normal. Thus $\mu(t) \equiv 1$. Due to Corollary 2 the zero solution to the considered equation is exponentially stable, provided

$$\sup_{t \ge 0} (a\omega |cos(\omega t)| + 2\nu(t)) < 2.$$

Concluding remarks:
In this paper we have established a sufficient explicit exponential stability test for a class of nonlinear nonautonomous systems. The test is sharp. It becomes also the necessary condition provided $A(t)$ is a constant normal matrix. As the example shows, in appropriate situations we can avoid the constructing of the Lyapunov functions. Moreover, the solution estimate (3.5) gives us a bound for the region of attraction of the steady state and the absolute stability conditions.

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Received 3 June 2015