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CONTROLLABILITY FOR SOME PARTIAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS IN BANACH SPACES

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Abstract

This work concerns the study of the controllability of some partial functional integrodifferential equation with nonlocal initial conditions in Banach spaces. It gives sufficient conditions that ensure the controllability of the system by supposing that its linear homogeneous part admits a resolvent operator in the sense of Grimmer, and by making use of the measure of noncompactness and the Mönch fixed-point theorem. As a result, we obtain a generalization of the work of Y.K. Chang, J.J. Nieto and W.S. Li (J. Optim. Theory Appl. 142, 267–273 (2009)), without assuming the compactness of the resolvent operator. An example of application is given for illustration.

Keywords: controllability, integrodifferential equations, nonlocal initial condition, resolvent operator, Mönch fixed-point theorem.

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1. Introduction

Control theory arises in many modern applications in engineering and environmental sciences [1], is one of the most interdisciplinary research areas [8, 15] and its empirical concept for technology goes back to antiquity with the works of Archimede, Philon, etc... [18]. A control system is a dynamical system on which one can act by the use of suitable parameters (i.e., the controls) in order to achieve a desired behavior or state of the system. Control systems are usually modeled by mathematical formalism involving mainly ordinary differential equations, partial differential equations or functional differential equations. In condensed expression, they often take the form of differential equation:

$$x'(t) = F(t, x(t), u(t)) \qquad \text{for} \quad t \geq 0,$$

where x is the state and u is the control. While studying a control system, one of the most common problems that appear is the controllability problem, which consists in checking the possibility of steering the control system from an initial state (initial condition) to a desired terminal one (boundary condition), by an appropriate choice of the control u. However in many real world contexts such as engineering, environmental sciences, demography, etc..., nonlocal constraints (such as isoperimetric or energy condition, multipoint boundary condition and flux boundary condition) appear and have received considerable attention during the last decades, cf. [5] and [6]. So the concept of nonlocal initial condition not only extends that of Cauchy initial condition, but also turns out to have better effects in applications as it may take into account future measurements over a certain period after the initial time t equals 0.

Several authors have studied the controllability problem of nonlinear systems described by functional integrodifferential equations with nonlocal conditions in infinite dimensional Banach spaces: see for instance [2, 3, 7, 20, 21], and the references therein. For example in [21], the authors proved the controllability of an integrodifferential system with nonlocal conditions basing on the measure of noncompactness and the Sadovskii fixed-point theorem, and in [2], R. Atmania and S. Mazouzi proved the controllability of a semilinear integrodifferential system using Schaefer fixed-point theorem and requiring the compactness of the semigroup.

In [12], R. Grimmer proved the existence and uniqueness of resolvent operators for these integrodifferential equations that gave the variation of parameter formula for the solution. In [9], W. Desch, R. Grimmer and W. Schappacher proved the equivalence of the compactness of the resolvent operator and that of the operator semigroup. In [2], the authors assumed the compactness of the operator semigroup and in [3], the authors assumed the compactness of the resolvent operator whereas in [7, 20, 21], the authors managed to drop this condition which motivates our current work.

The organization of this work is as follows: In Section 2, some definitions and results are given. In Section 3, the controllability result for equation (2.1) is given. In Section 4, an example is given to illustrate the results.

2. Preliminaries

Integrodifferential equations have applications in many problems arising in physical systems, the following one-dimensional model in viscoelasticity is one of the applications of that theory

$$\begin{aligned}
&\alpha\frac{\partial^2\omega}{\partial t^2}(t,\xi)+\beta\frac{\partial\omega}{\partial t}(t,\xi)=\varphi(t,\xi)+h(t,\xi),\\
&\gamma\frac{\partial\omega}{\partial\xi}(t,\xi)+\int_0^t a(t-s)\frac{\partial\omega}{\partial\xi}(s,\xi)ds=\varphi(t,\xi),\ \ (t,\xi)\in\mathbb{R}^+\times[0,1],\\
&\omega(t,0)=\omega(t,1)=0,\quad t\in\mathbb{R}^+,\\
&\omega(0,\xi)=\omega_0(\xi),\quad \xi\in[0,1],
\end{aligned}$$

where, ω is the displacement, φ is the stress, h is the external force, $\alpha,\gamma>0$ and β are constants. In this model, the first equation describes the linear momentum equation while the second equation describes the constitutive relation between stress and strain. Setting $\gamma=1$, $v=\frac{\partial\omega}{\partial t}$, and $u=\frac{\partial\omega}{\partial\xi}$, the above equations can be rewritten as follows

$$\begin{bmatrix} u'(t)\\ v'(t)\end{bmatrix}=\begin{bmatrix}0 & \partial_\xi\\ \frac{\partial_\xi}{\alpha} & 0\end{bmatrix}\left\{\begin{bmatrix}u(t)\\ v(t)\end{bmatrix}+\int_0^t\begin{bmatrix}a(t-s) & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}u(s)\\ v(s)\end{bmatrix}ds\right\}$$
$$+\begin{bmatrix}0 & 0\\ 0 & -\frac{\beta}{\alpha}\end{bmatrix}\begin{bmatrix}u(t)\\ v(t)\end{bmatrix}+\begin{bmatrix}0\\ \frac{h(t)}{\alpha}\end{bmatrix},\ t\geq 0.$$

Setting

$$x(t)=\begin{bmatrix}u(t)\\ v(t)\end{bmatrix},\quad A=\begin{bmatrix}0 & \partial_\xi\\ \frac{\partial_\xi}{\alpha} & 0\end{bmatrix},\quad G(t)=\begin{bmatrix}a(t-s) & 0\\ 0 & 0\end{bmatrix}$$

$$K=\begin{bmatrix}0 & 0\\ 0 & -\frac{\beta}{\alpha}\end{bmatrix},\ \text{ and }\ p(t)=\begin{bmatrix}0\\ \frac{h(t)}{\alpha}\end{bmatrix},$$

we can rewrite the above equation into the following abstract form

$$\begin{cases} x'(t)=A\Big[x(t)+\displaystyle\int_0^t G(t-s)x(s)\Big]ds+Kx(t)+p(t)\ \text{ for }\ \ \mathrm{t}\geq 0\\ x(0)=x_0.\end{cases}$$

The operator A is unbounded here, while K and $G(t)$ are bounded operators for $t \geq 0$ on a Banach space X. When $AG(t) = G(t)A$, we obtain the following equation

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t G(t-s)Ax(s)ds + Kx(t) + p(t) \quad \text{for} \quad \mathrm{t} \geq 0 \\ x(0) = x_0. \end{cases}$$

which has been studied in [10]. We note that in general, the equality $AG(t) = G(t)A$ does not hold.

In this paper, we study the controllability of some systems that arise in the analysis of heat conduction in materials with memory [12], and viscoelasticity, and take the form of the following model of a partial functional integrodifferential equation subject to a nonlocal initial condition in a Banach space $(X, \ \|\cdot\|)$:
(2.1)

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t B(t-s)x(s)ds + f(t, x(t)) + Cu(t) \quad \text{for} \quad \mathrm{t} \in \mathrm{I} = [0, \mathrm{b}] \\ x(0) = x_0 + g(x), \end{cases}$$

where $x_0 \in X, \ g : \mathcal{C}(I, X) \to X$ and $f : I \times X \to X$ are functions satisfying some conditions; $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t \geq 0}$ on X; for $t \geq 0$, $B(t)$ is a closed linear operator with domain $\mathcal{D}(B(t)) \supset \mathcal{D}(A)$. The control u belongs to $L^2(I, U)$ which is a Banach space of admissible controls, where U is a Banach space. The operator C belongs to $\mathcal{B}(U, X)$ which is the Banach space of bounded linear operators from U into X, and $\mathcal{C}(I, X)$ denotes the Banach space of continuous functions $x : I \to X$ with supremum norm $\|x\|_\infty = \sup_{t \in I} \|x(t)\|_X$.

The particular case $B(t) = 0$ has been considered by Y.K. Chang, J.J. Nieto and W.S. Li [7], where the authors used Sadovskii fixed-point theorem and the operator semigroup to prove their result.

Here our goal is to study equation (2.1) where $B(t) \neq 0$ as a generalization of the result by Y.K. Chang, J.J. Nieto and W.S. Li without requiring the compactness of the operator semigroup, in the same way as J. Wang, Z. Fan and Y. Zhou [20] have done for the nonlocal controllability of some semilinear dynamic systems with fractional derivative. Our approach constists in transforming the problem (2.1) into a fixed-point problem of an appropriate operator and to apply the Mönch fixed-point theorem.

We now introduce some definitions and lemmas that will be used throughout the paper.

Let $I = [0, b], \ b > 0$ and let X be a Banach space. A measurable function $x : I \to X$ is Bochner integrable if and only if $\|x\|$ is Lebesgue integrable. We denote by $L^1_B(I, X)$ the Banach space of functions $x : I \to X$ which are Bochner integrable and normed by

$$\|x\|_{L^1} = \int_0^b \|x(t)\|dt.$$

Consider the following linear homogeneous equation:

$$\text{(2.2)} \qquad \begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t B(t-s)x(s)ds \quad \text{for} \quad \mathrm{t} \geq 0 \\ x(0) = x_0 \in X. \end{cases}$$

where A and $B(t)$ are closed linear operators on a Banach space X.

In the sequel, we assume A and $\big(B(t)\big)_{t\geq 0}$ satisfy the following conditions:

($\mathbf{H_1}$) A is a densely defined closed linear operator in X. Hence $\mathcal{D}(A)$ is a Banach space equipped with the graph norm defined by $|y| = \|Ay\| + \|y\|$, which will be denoted by $(Y, |\cdot|)$.

($\mathbf{H_2}$) $\big(B(t)\big)_{t\geq 0}$ is a family of linear operators on X such that $B(t)$ is continuous when regarded as a linear map from $(Y, |\cdot|)$ into $(X, \|\cdot\|)$ for almost all $t \geq 0$, the map $t \mapsto B(t)y$ is measurable for all $y \in Y$ and $t \geq 0$ and belongs to $W^{1,1}(\mathbb{R}^+, X)$. Moreover, there is a locally integrable function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|B(t)y\| \leq f(t)|y| \quad \text{and} \quad \left\|\frac{d}{dt}B(t)y\right\| \leq f(t)|y|.$$

Remark 1. Note that ($\mathbf{H_2}$) is satisfied in the modelling of heat conduction in materials with memory and viscosity. More details can be found in [13].

Definition 2 (see e.g., [11])**.** A resolvent operator for equation (2.2) is a family $\big(R(t)\big)_{t\geq 0}$ of bounded linear operators valued function

$$R : [0, +\infty) \longrightarrow \mathcal{B}(X)$$

such that

(i) $R(0) = Id_X$ and $\|R(t)\| \leq Ne^{\beta t}$ for some constants N and β.

(ii) For all $x \in X$, the map $t \mapsto R(t)x$ is continuous for $t \geq 0$.

(iii) Moreover, for $x \in Y$, $R(\cdot)x \in \mathcal{C}^1(I; X) \cap \mathcal{C}(I; Y)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds. \end{aligned}$$

Observe that the map defined on I by $t \mapsto R(t)x_0$ solves equation (2.2) for $x_0 \in \mathcal{D}(A)$.

We have the following example of a resolvent operator for equation (2.2) in $\mathbb{R}$.

Example ([9]). Let $X = \mathbb{R}$, $Ay = 2y$, and $B(t)y = -2y$ in (2.2). Then we have

$$R(t)x_0 = e^t(\cos t + \sin t)x_0 \quad \text{ and } \quad T(t)x_0 = e^{2t}x_0.$$

Remark 3. The above example also shows that, in general, the resolvent operator $\big(R(t)\big)_{t\geq 0}$ for equation (2.2) does not satisfy the semigroup law, namely,

$$R(t+s) \neq R(t)R(s) \quad \text{for some } \ t, s > 0.$$

Now consider the following system:

$$\text{(2.3)} \qquad \begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t \left[B_1(t-s) + B_2(t-s)\right] x(s)ds \quad \text{for t} \geq 0 \\ x(0) = x_0 \in X, \end{cases}$$

where $B_1(t)$ and $B_2(t)$ are closed linear operators in X and satisfy $(\mathbf{H_2})$.

Then we have the following Lemma coming from [9].

Lemma 4 (Perturbation result) ([9]). *Suppose A satisfies $(\mathbf{H_1})$ and $\big(B_1(t)\big)_{t\geq 0}$ and $\big(B_2(t)\big)_{t\geq 0}$ satisfy $(\mathbf{H_2})$. Let $\big(R_{B_1}(t)\big)_{t\geq 0}$ be a resolvent operator of equation* (2.2) *and $\big(R_{B_1+B_2}(t)\big)_{t\geq 0}$ be a resolvent operator of equation* (2.3). *Then*

$$R_{B_1+B_2}(t)x - R_{B_1}(t)x = \int_0^t R_{B_1}(t-s)Q(s)x\, ds$$

where the operator Q is defined by

$$Q(t)x = \int_0^t B_2'(t-s)\int_0^s R_{B_1+B_2}(\tau)x\, d\tau\, ds + B_2(0)\int_0^t R_{B_1+B_2}(s)x\, ds,$$

Q is uniformly bounded on bounded intervals, and for each $x \in X$, $Q(\cdot)x$ belongs to $\mathcal{C}([0,\infty), X)$.

Based on this and the following corollary from ([9], p. 224), we prove the operator-norm continuity of the resolvent operator $\big(R(t)\big)_{t\geq 0}$ for $t > 0$.

Corollary 5 ([9]). *Let A be a closed, densely defined linear operator in X, $B(t) = 0$ for all $t \geq 0$, and $\big(R(t)\big)_{t\geq 0}$ be a resolvent operator for equation* (2.2). *Then $\big(R(t)\big)_{t\geq 0}$ is a C_0-semigroup with infinitesimal generator A.*

Theorem 6. *Let A be the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ and let $\big(B(t)\big)_{t\geq 0}$ satisfy $(\mathbf{H_2})$. Then the resolvent operator $\big(R(t)\big)_{t\geq 0}$ for equation* (2.2) *is operator-norm continuous (or continuous in the uniform operator topology) for $t > 0$ if and only if $\big(T(t)\big)_{t\geq 0}$ is operator-norm continuous for $t > 0$.*

Proof. Since $\big(T(t)\big)_{t\geq 0}$ is a C_0-semigroup, there exist M and ω such that

$$||T(t)|| \leq Me^{\omega t} \qquad \text{for all} \quad t \geq 0.$$

Let

$$Q(t)x = B(0)\int_0^t R(s)x\,ds + \int_0^t B'(t-s)\int_0^s R(\tau)x\,d\tau\,ds.$$

By Lemma 4 (applied to equation (2.2)) we deduce that the operators $\big(Q(t)\big)_{t\geq 0}$ are uniformly bounded for t in a bounded interval and that

$$R(t)x = T(t)x + \int_0^t T(t-s)Q(s)x\,ds.$$

Now set $\alpha = \sup_{0\leq t\leq b}\|Q(t)\|$ and let $x \in X$ be arbitrary and such that $\|x\| \leq 1$.

Suppose first that $\big(T(t)\big)_{t\geq 0}$ is operator-norm continuous for $t > 0$. Then on the one hand, we have for every $t \in (0, b)$ and $h \in (0, b-t)$:

$$\begin{aligned}\Big\|R(t+h)x - R(t)x\Big\| \;\leq\; & \Big\|\Big[T(t+h) - T(t)\Big]x\Big\| \\ & + \;\alpha\int_0^t \Big\|\Big[T((t-s)+h) - T(t-s)\Big]x\Big\|\,ds \\ & + \;\alpha\int_t^{t+h} Me^{\omega(t-s+h)}\,ds.\end{aligned}$$

It follows that

$$\begin{aligned}\Big\|R(t+h) - R(t)\Big\| \;\leq\; & \Big\|T(t+h) - T(t)\Big\| + \alpha\int_0^t \Big\|T((t-s)+h) - T(t-s)\Big\|\,ds \\ & + \;\alpha\int_t^{t+h} Me^{\omega(t-s+h)}\,ds.\end{aligned}$$

Therefore, by using the operator-norm continuity of $\big(T(t)\big)_{t\geq 0}$ on $(0, +\infty)$ and the Lebesgue dominated convergence theorem, we deduce that

$$\Big\|R(t+h) - R(t)\Big\| \to 0 \quad \text{as} \quad h \to 0^+.$$

On the other hand, for every $t \in (0, b]$ and $h \in (-t, 0)$, we have

$$\begin{aligned}\Big\|R(t+h) - R(t)\Big\| \;\leq\; & \Big\|T(t+h) - T(t)\Big\| + \alpha\int_0^t \Big\|T((t-s)+h) - T(t-s)\Big\|\,ds \\ & + \;\alpha\int_{t+h}^{t} Me^{\omega(t-s)}\,ds.\end{aligned}$$

Using the operator-norm continuity of $\big(T(t)\big)_{t\geq 0}$ and the Lebesgue dominated convergence theorem, we get

$$\Big\|R(t+h)-R(t)\Big\| \to 0 \quad \text{as} \quad h \to 0^-.$$

Hence

$$\Big\|R(t+h)-R(t)\Big\| \to 0 \quad \text{when} \quad h \to 0 \quad \text{with} \quad t+h \in [0,b].$$

Thus $\big(R(t)\big)_{t\geq 0}$ is operator-norm continuous when $\big(T(t)\big)_{t\geq 0}$ is operator-norm continuous.

The converse is proved similarly by using the identities

$$T(t)x = -R(t)x + \int_0^t T(t-s)Q(s)x\,ds = -R(t)x + \int_0^t T(s)Q(t-s)x\,ds$$

and the continuity of $\big(Q(t)\big)_{t\geq 0}$ which follows from the property $(\mathbf{H_2})$ of $\big(B(t)\big)_{t\geq 0}$ and the continuity of $\big(R(t)\big)_{t\geq 0}$.

In fact, suppose that $\big(R(t)\big)_{t\geq 0}$ is operator-norm continuous for $t>0$. Then, we obtain

$$\begin{aligned}\Big\|T(t+h)x-T(t)x\Big\| \leq\ & \Big\|\Big[R(t+h)-R(t)\Big]x\Big\| \\ &+ \int_0^t Me^{\omega s}\Big\|\Big[Q((t-s)+h)-Q(t-s)\Big]x\Big\|\,ds \\ &+ \alpha\int_t^{t+h} Me^{\omega s}\,ds\end{aligned}$$

for every $t \in (0,b)$ and $h \in (0, b-t)$.

It follows that

$$\begin{aligned}\Big\|T(t+h)-T(t)\Big\| \leq\ & \Big\|R(t+h)-R(t)\Big\| \\ &+ \int_0^t Me^{\omega s}\Big\|Q((t-s)+h)-Q(t-s)\Big\|\,ds \\ &+ \alpha\int_t^{t+h} Me^{\omega s}\,ds.\end{aligned}$$

Therefore, by using the operator-norm continuity of $\big(R(t)\big)_{t\geq 0}$ on $(0,+\infty)$, the continuity of $\big(Q(t)\big)_{t\geq 0}$ and the Lebesgue dominated convergence theorem, we get

$$\Big\|T(t+h)-T(t)\Big\| \to 0 \quad \text{as} \quad h \to 0^+.$$

On the other hand, for every $t \in (0, b]$ and $h \in (-t, 0)$, we have

$$\begin{aligned}\Big\|T(t+h)x - T(t)x\Big\| \leq\ & \Big\|\Big[R(t+h) - R(t)\Big]x\Big\| \\ &+ \int_0^t Me^{\omega s}\Big\|\Big[Q((t-s)+h) - Q(t-s)\Big]x\Big\|\, ds \\ &+ \alpha\int_{t+h}^t Me^{\omega s}\, ds.\end{aligned}$$

It follows that

$$\begin{aligned}\Big\|T(t+h) - T(t)\Big\| \leq\ & \Big\|R(t+h) - R(t)\Big\| \\ &+ \int_0^t Me^{\omega s}\Big\|Q((t-s)+h) - Q(t-s)\Big\|\, ds \\ &+ \alpha\int_{t+h}^t Me^{\omega s}\, ds.\end{aligned}$$

Therefore, by using the operator-norm continuity of $\big(R(t)\big)_{t\geq 0}$ on $(0, +\infty)$, the continuity of $\big(Q(t)\big)_{t\geq 0}$ and the Lebesgue dominated convergence theorem, we obtain

$$\Big\|T(t+h) - T(t)\Big\| \to 0 \quad \text{as} \quad h \to 0^-.$$

Hence,

$$\Big\|T(t+h) - T(t)\Big\| \to 0 \quad \text{as} \quad h \to 0 \text{ with } \quad t+h \in [0, b].$$

Thus $\big(T(t)\big)_{t\geq 0}$ is operator-norm continuous when $\big(R(t)\big)_{t\geq 0}$ is. ∎

Definition 7. A mild solution of equation (2.1) is a function $x \in \mathcal{C}(I, X)$ such that

$$x(t) = R(t)[x_0 + g(x)] + \int_0^t R(t-s)\big[f(s, x(s)) + Cu(s)\big]\, ds \quad \text{for} \quad \text{t} \in \text{I}.$$

Definition 8. Equation (2.1) is said to be controllable on the interval I if for every $x_0,\ x_1 \in X$, there exist a control $u \in L^2(I, U)$ and a mild solution x of equation (2.1) satisfying the condition $x(b) = x_1$.

For proving the main result of the paper we recall the notion of measure of noncompactness and the Mönch fixed-point theorem. More information on the subject can be found in [4]. In order to use the Hausdorff measure of noncompactness, we recall some properties related to this concept.

Definition 9. Let D be a bounded subset of a normed space Z. The Hausdorff measure of noncompactness (shortly MNC) is defined by

$$\beta(D) = \inf\Big\{\epsilon > 0 : D \text{ has a finite cover by balls of radius less than } \epsilon\Big\}.$$

Theorem 10. *Let D, D_1, D_2 be bounded subsets of a Banach space Z. The Hausdorff MNC has the following properties:*

(i) *If $D_1 \subset D_2$, then $\beta(D_1) \leq \beta(D_2)$, (monotonicity).*

(ii) $\beta(D) = \beta(\overline{D})$.

(iii) *$\beta(D) = 0$ if and only if D is relatively compact.*

(iv) *$\beta(\lambda D) = |\lambda|\beta(D)$ for any $\lambda \in \mathbb{R}$, (Homogeneity)*

(v) *$\beta(D_1 + D_2) \leq \beta(D_1) + \beta(D_2)$, where $D_1 + D_2 = \{d_1 + d_2 : \ d_1 \in D_1, d_2 \in D_2\}$, (subadditivity)*

(vi) *$\beta(\{a\} \cup D) = \beta(D)$ for every $a \in Z$.*

(vii) *$\beta(D) = \beta(\overline{co}(D))$, where $\overline{co}(D)$ is the closed convex hull of D.*

(viii) *For any map $G : \mathcal{D}(G) \subseteq X \to Z$ which is Lipschitz continuous with a Lipschitz constant k, we have*

$$\beta(G(D)) \leq k\beta(D),$$

for any subset $D \subseteq \mathcal{D}(G)$.

Lemma 11 ([20]). *If $(u_n)_{n\geq 1}$ is a sequence of Bochner integrable functions from I into a Banach space Z with the estimation $\|u_n(t)\| \leq \mu(t)$ for almost all $t \in I$ and every $n \geq 1$, where $\mu \in L^1(I, \mathbb{R})$, then the function*

$$\psi(t) = \beta(\{u_n(t) : n \geq 1\})$$

belongs to $L^1(I, \mathbb{R})$ and satisfies the following estimation

$$\beta\left(\left\{\int_0^b u_n(s)ds : n \geq 1\right\}\right) \leq 2\int_0^b \psi(s)ds.$$

Lemma 12. *Let Z be a Banach space and $(T_n)_{n\geq 1}$ be a sequence of bounded linear maps on Z converging pointwise to $T \in \mathcal{B}(Z)$. Then for any compact set K in Z, T_n converges to T uniformly in K, namely,*

$$\sup_{x\in K} \|T_n(x) - T(x)\| \longrightarrow 0, \text{ as } n \to +\infty.$$

Proof. By the uniform boundedness principle, we deduce that $\sup_{n\geq 1}\|T_n\| < \infty$. Let $M = \sup_{n\geq 1}\|T_n\|$ and $\epsilon > 0$ be arbitrarily given. Then there exist $a_1, a_2, \ldots, a_m$ such that $K \subset \bigcup_{i=1}^{m} B\big(a_i, \frac{\epsilon}{2(M+1)}\big)$.

For any $x \in K$, there exists $i \in \{1, \ldots, m\}$ such that $x \in B\big(a_i, \frac{\epsilon}{2(M+1)}\big)$. Since for $i = 1, 2, \ldots, m, T_n(a_i) \to T(a_i)$, then there exists $N_\epsilon > 0$ such that

$$\|T_n(a_i) - T(a_i)\| \leq \frac{\epsilon}{M+1}, \text{ for } n \geq N_\epsilon \text{ and for any } i = 1, \ldots .$$

We have

$$\begin{aligned}\|T_n(x) - T(x)\| &\leq \|T_n(x) - T_n(a_i)\| + \|T_n(a_i) - T(a_i)\| + \|T(a_i) - T(x)\| \\ &\leq \|T_n\|\|x - a_i\| + \|T\|\|a_i - x\| + \|T_n(a_i) - T(a_i)\| \\ &\leq \frac{\epsilon M}{M+1} + \|T_n(a_i) - T(a_i)\| \\ &\leq \frac{\epsilon M}{M+1} + \frac{\epsilon}{M+1} = \epsilon\,.\end{aligned}$$

Therefore, $\sup_{x\in K}\|T_n(x) - T(x)\| \to 0$ as $n \to +\infty$. ∎

To end this section, we recall a nonlinear alternative of Mönch's type for non selfmaps.

Theorem 13 [14] (Mönch, 1980)**.** *Let G be an open neighborhood of the origin in a Banach space Z. Suppose that $F : \overline{G} \to Z$ is a continuous map which satisfies the following conditions:*

(i) *$D \subset \overline{G}$ countable and $D \subseteq \overline{co}(\{0\} \cup F(D)) \implies \overline{D}$ is compact.*

(ii) *$F(x) \neq \lambda x$ for all $x \in \partial G$ and $\lambda > 1$, (Leray-Schauder condition).*

Then F has a fixed point.

We now state the following nonlinear alternative of Mönch's type for selfmaps, which we shall use in the proof of the controllability of equation (2.1).

Theorem 14 [14] (Mönch, 1980)**.** *Let $\mathcal{K}$ be a closed and convex subset of a Banach space Z and $0 \in \mathcal{K}$. Assume that $F : \mathcal{K} \to \mathcal{K}$ is a continuous map which satisfies Mönch's condition, namely, let $D \subseteq \mathcal{K}$ be countable and $D \subseteq \overline{co}(\{0\} \cup F(D))$, then $\overline{D}$ is compact. Then F has a fixed point.*

Corollary 15. *Let $\mathcal{K}$ be a closed, convex and bounded subset of a Banach space Z and 0 be an interior point of $\mathcal{K}$. Assume that $F : \mathcal{K} \to \mathcal{K}$ is a continuous map which satisfies Mönch's condition. Then F has a fixed point.*

We observe that in the statements of Mönch's Theorem for non selfmaps in [20] and [17], the key Leray-Schauder boundary condition is missing.

3. CONTROLLABILITY RESULT

We shall consider furthermore the following hypotheses.

($\mathbf{H_3}$) The linear operator $W : L^2(I, U) \to X$ satisfies the following condition:

(i) W defined by

$$Wu = \int_0^b R(b-s)Cu(s)\,ds,$$

is surjective so that it induces an isomorphism between $L^2(I,U)/_{\mathrm{Ker}W}$ and X again denoted by W with inverse W^{-1} taking values in $L^2(I,U)/_{\mathrm{Ker}W}$, (see e.g., [16]).

(ii) There exists a function $L_W \in L^1(I, \mathbb{R}^+)$ such that for any bounded set $Q \subset X$ we have

$$\beta((W^{-1}Q)(t)) \leq L_W(t)\beta(Q),$$

where β is the Hausdorff MNC.

($\mathbf{H_4}$) The function $f : I \times X \longrightarrow X$ satisfies the following conditions:

(i) $f(\cdot, x)$ is measurable for $x \in X$ and $f(t, \cdot)$ is continuous for a.e $t \in I$.
(ii) There exist a function $L_f \in L^1(I, \mathbb{R}^+)$ and a nondecreasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,x)\| \leq L_f(t)\phi(\|x\|) \text{ for } x \in X, \quad t \in I \text{ and } \liminf_{r\to+\infty} \frac{\phi(r)}{r} = 0.$$

(iii) There exists a function $h \in L^1(I, \mathbb{R}^+)$ such that for any bounded set $D \subset X$,

$$\beta(f(t,D)) \leq h(t)\beta(D) \quad \text{for} \quad \text{a.e } t \in I,$$

where β is the Hausdorff MNC.

($\mathbf{H_5}$) $g : \mathcal{C}(I, X) \longrightarrow X$ is continuous, compact and satisfies $\liminf_{r\to+\infty} \frac{g_r}{r} = 0$, where

$$g_r = \sup\Big\{\|g(x)\| : \|x\| \leq r\Big\}.$$

Theorem 16. *Suppose the equation* (2.2) *has a resolvent operator* $\big(R(t)\big)_{t\geq 0}$ *which is continuous in the operator-norm topology for* $t>0$ *and hypotheses* $(\mathbf{H_3})$–$(\mathbf{H_5})$ *are satisfied. Then equation* (2.1) *is controllable on I, provided that*

$$\gamma = \Big(1 + 2M_1M_2\|L_W\|_{L^1}\Big)\Big(2M_1\|h\|_{L^1}\Big) < 1,$$

where $M_1 = \sup_{0\leq t\leq b}\|R(t)\|$ *and* M_2 *is such that* $\|C\| = M_2$.

Proof. Note that $M_1 < +\infty$ according to the exponential growth of the resolvent operator $\big(R(t)\big)_{t\geq 0}$. Using $(\mathbf{H_3})$ we define the control u_x by

$$u_x(t) \;=\; W^{-1}\left(x_1 - R(b)[x_0 + g(x)] - \int_0^b R(b-s)f(s,x(s))\,ds\right)(t) \quad \text{for} \;\; t\in I,$$

for an arbitrarily given function $x\in\mathcal{C}(I,X)$.

Using this control, we shall show that the operator $K:\mathcal{C}(I,X)\to\mathcal{C}(I,X)$ defined by

$$(Kx)(t) = R(t)[x_0 + g(x)] + \int_0^t R(t-s)\Big[f(s,x(s)) + Cu_x(s)\Big]\,ds,$$

has a fixed point x which is just a mild solution of the equation (2.1). Observe that $(Kx)(b) = x_1$ and so the control u_x steers the integrodifferential equation from x_0 to x_1 in time b. This means that equation (2.1) is controllable on I.

For each positive r, let $B_r = \{x\in\mathcal{C}(I,X) : \|x\|_\infty \leq r\}$. We shall prove the above theorem through the following steps.

Step 1. We claim that there exists $r>0$ such that $K(B_r)\subset B_r$.
Suppose on the contrary that this is not true. Then for each positive r, there exists a function $x_r\in B_r$, such that $K(x_r)\notin B_r$, i.e., $\|(Kx_r)(\tau)\| > r$, for some $\tau = \tau(r)\in I$. Now

$$(*) \qquad \frac{\|(Kx_r)(\tau)\|}{r} > 1 \quad \text{that implies that} \qquad \liminf_{r\to+\infty}\frac{\|(Kx_r)(\tau)\|}{r} \geq 1.$$

On the other hand, let $M_3 = \|W^{-1}\|$. We have

$$\begin{aligned}\Big\|(Kx_r)(\tau)\Big\| \;\leq\; & M_1\|x_0\| + M_1\|g(x_r)\| + M_1\int_0^b \Big\|f(s,x_r(s))\Big\|\,ds \\ & +\; bM_1M_2M_3\Big(\|x_1\| + M_1\|x_0\|\Big)\end{aligned}$$

$$
\begin{aligned}
&+ \ bM_1M_2M_3\Big(M_1\|g(x_r)\| + M_1\int_0^b \Big\|f(s,x_r(s))\Big\|\,ds\Big)\\
&\le \ M_1\|x_0\| + M_1 g_r + M_1\phi(r)\|L_f\|_{L^1}\\
&+ \ bM_1M_2M_3\Big(\|x_1\| + M_1\|x_0\|\Big)\\
&+ \ bM_1M_2M_3\Big(M_1 g_r + M_1\phi(r)\|L_f\|_{L^1}\Big)\\
&\le \ w_r := \Big(1 + bM_1M_2M_3\Big)M_1\|x_0\| + \Big(1 + bM_1M_2M_3\Big)M_1 g_r\\
&+ \ \Big(1 + bM_1M_2M_3\Big)M_1\|L_f\|_{L^1}\phi(r) + bM_1M_2M_3\|x_1\|.
\end{aligned}
$$

Since

$$\liminf_{r\to+\infty}\frac{w_r}{r} = 0 = \liminf_{r\to+\infty}\frac{g_r}{r}$$

then we get

$$\liminf_{r\to+\infty}\frac{\|(Kx_r)(\tau)\|}{r} = 0.$$

This is clearly a contradiction to $(*)$. Consequently, there exists $r > 0$ such that $K(B_r) \subset B_r$.

Step 2. The operator K is continuous on B_r. To see this, let $(x_n)_{n\ge1} \subset B_r$ be such that $x_n \to x$ in B_r. Set

$$(K_1x)(t) := R(t)[x_0 + g(x)] \quad \text{and}$$

$$(K_2x)(t) := \int_0^t R(t-s)\big[f(s,x(s)) + Cu_x(s)\big]ds \quad \text{for} \quad t \in I,$$

then $K = K_1 + K_2$.

Therefore, since g is continuous, we obtain

$$\|K_1x_n - K_1x\| \le M_1\|g(x_n) - g(x)\| \to 0, \text{as } n \to +\infty.$$

For the proof of the continuity of K_2, we set

$$F_n(s) = f(s,x_n(s)) \ \text{ for every } n \text{ and a.e. } s, \text{ and } \ F(s) = f(s,x(s)) \ \text{ for a.e. } s.$$

Therefore, by $(\mathbf{H_4})$–(i), $F_n(s) \to F(s)$ and by $(\mathbf{H_4})$–(ii),

$$\Big\|F_n(s)\Big\| = \Big\|f(s,x_n(s))\Big\| \le L_f(s)\phi(\|x_n\|) \le \phi(r)L_f(s)$$

for every n and a.e. s.

It follows from the Lebesgue dominated convergence theorem that

$$\int_0^t \Big\| F_n(s) - F(s) \Big\| ds \longrightarrow 0, \ as\ n \to +\infty, \ t \in I.$$

Moreover, we have

$$\Big\| K_2 x_n - K_2 x \Big\| \leq M_1 \int_0^t \Big\| F_n(s) - F(s) \Big\| ds + M_1 M_2 b^{\frac{1}{2}} \Big\| u_{x_n} - u_x \Big\|_{L^2(I,U)},$$

where

$$\Big\| u_{x_n} - u_x \Big\|_{L^2(I,U)} \leq M_3 \left(M_1 \Big\| g(x_n) - g(x) \Big\| + M_1 \int_0^b \Big\| F_n(s) - F(s) \Big\| ds \right).$$

Thus it follows that $\|K_2 x_n - K_2 x\| \longrightarrow 0$ as $n \to +\infty$, showing that K_2 is continuous on B_r. Hence K is continuous on B_r.

Step 3. The Mönch condition holds.
Suppose that $D \subseteq B_r$ is countable and $D \subseteq \overline{co}\Big(\{0\} \cup K(D)\Big)$. We have to show that D is relatively compact. To this end, it suffices to show that $\beta(D) = 0$, where β is the Hausdorff MNC. Since D is countable, we can describe it as $D = \{x_n\}_{n \geq 1}$. Therefore, $K(D) = \{Kx_n\}_{n \geq 1}(t)$ and its relative compactness implies that D is also relatively compact. So we have to prove that $K(D)$ is equibounded and equicontinuous on I in order to use Ascoli-Arzela's theorem. We show that $K(D)$ is equicontinuous. Let $y \in K(D)$, and $0 \leq t_1 < t_2 \leq b$. There exists $x \in D$ such that $y = Kx$ and

$$\begin{aligned} \Big\| y(t_2) - y(t_1) \Big\| \leq & \Big\| R(t_2)x_0 - R(t_1)x_0 \Big\| + \Big\| R(t_2)g(x) - R(t_1)g(x) \Big\| \\ & + \Big\| \int_0^{t_2} R(t_2 - s) f(s, x(s)) ds - \int_0^{t_1} R(t_1 - s) f(s, x(s)) ds \Big\| \\ & + \Big\| \int_0^{t_2} R(t_2 - s) C u_x(s) ds - \int_0^{t_1} R(t_1 - s) C u_x(s) ds \Big\|. \end{aligned}$$

Firstly assume that $t_1 > 0$. By (ii) of definition 2, the first term on the right hand side tends to 0 as $|t_2 - t_1| \to 0$. That is

$$\Big\| R(t_2)x_0 - R(t_1)x_0 \Big\| \to 0 \ \text{ as } \ |t_2 - t_1| \to 0.$$

Moreover, we have

$$\begin{aligned} \Big\| R(t_2)g(x) - R(t_1)g(x) \Big\| \leq & \Big\| R(t_2) - R(t_1) \Big\| \Big\| g(x) \Big\| \\ \leq & \Big\| R(t_2) - R(t_1) \Big\| g_r, \end{aligned}$$

where $g_r = \sup\{\|g(x)\| : \|x\| \leq r\}$. Moreover, $\big\|R(t_2)-R(t_1)\big\| \to 0$ as $|t_2-t_1| \to 0$, by the continuity of $\big(R(t)\big)_{t\geq 0}$ for $t>0$ in the operator-norm topology.

Now let $t_1 = 0$. Since $\overline{g(B_r)}$ is compact, then we have $\big\|R(h)g(x) - g(x)\big\| \leq \sup_{y\in\overline{g(B_r)}} \big\|R(h)y - y\big\| \to 0$, as $h \to 0^+$, by Lemma 12.

Therefore, $\big\|R(t_2)g(x) - R(t_1)g(x)\big\| \to 0$ as $t_2 \to t_1$.

Now we have

$$\begin{aligned}
&\left\|\int_0^{t_2} R(t_2-s)f(s,x(s))ds - \int_0^{t_1} R(t_1-s)f(s,x(s))ds\right\| \\
&\leq M_1\int_{t_1}^{t_2}\Big\|f(s,x(s))\Big\|ds \\
&+ \int_0^{t_1}\Big\|R(t_2-s)-R(t_1-s)\Big\|\Big\|f(s,x(s))\Big\|ds \\
&\leq M_1\phi(r)\int_{t_1}^{t_2} L_f(s)ds \\
&+ \phi(r)\int_0^{t_1}\Big\|R(t_2-s)-R(t_1-s)\Big\|L_f(s)ds\,.
\end{aligned}$$

The right hand side tends to 0 as $t_2 \to t_1$ by the Lebesgue dominated convergence theorem, showing that the family $\big\{\int_0^t R(t-s)f(s,x(s))ds,\ x\in D\big\}$ is equicontinuous. Moreover,

$$\begin{aligned}
&\left\|\int_0^{t_2} R(t_2-s)Cu_x(s)ds - \int_0^{t_1} R(t_1-s)Cu_x(s)ds\right\| \\
&\leq M_1\int_{t_1}^{t_2}\Big\|Cu_x(s)\Big\|ds \\
&+ \int_0^{t_1}\Big\|R(t_2-s)-R(t_1-s)\Big\|\Big\|Cu_x(s)\Big\|ds \\
&\leq M_2M_3\Big[\|x_1\| + M_1\Big(\|x_0\|+g_r\Big) + M_1\phi(r)\|L_f\|_{L^1}\Big] \\
&\Big[M_1(t_2-t_1) + \int_0^{t_1}\Big\|R(t_2-s)-R(t_1-s)\Big\|ds\Big]
\end{aligned}$$

and the right hand side tends to 0 as $t_2 \to t_1$. Therefore, the family

$$\left\{\int_0^t R(t-s)Cu_x(s)\,ds\,;\ x\in D\right\}$$

is equicontinuous and the set $K(D)$ is equicontinuous on I.

We prove that $K(D)$ is equibounded. To do this, we show that for all $t\in[0,b]$, the set $\big\{K(x)(t);\, x\in D\big\}$ is relatively compact. We achieve this using the measure of noncompactness. For $t = 0$, the set

$$\Big\{(Kx)(0);\ x \in D\Big\} = \Big\{x_0 + g(x);\ x \in D\Big\} = x_0 + g(D)$$

is relatively compact in X. Since g is compact, then $\overline{g(D)}$ is compact also.

For $t \in (0, b]$, we have

$$\beta\Big(\{(K_1 x_n)(t)\}_{n\geq 1}\Big) \leq \beta\Big(\{R(t)(x_0 + g(x_n))\}_{n\geq 1}\Big) = 0,$$

by the compactness of g. Also, by $(\mathbf{H_3})$–(ii) we obtain

$$\begin{aligned}
&\beta\Big(\{u_{x_n(t)}\}_{n\geq 1}(t)\Big)\\
&= \beta\Big(W^{-1}\Big\{x_1 - R(b)\Big(x_0 + g(x_n)\Big) - \int_0^b R(t-s)f(s,x_n(s))\,ds\Big\}_{n\geq 1}(t)\Big)\\
&\leq L_W(t)\beta\Big(\Big\{x_1 - R(b)\Big(x_0 + g(x_n)\Big) - \int_0^b R(t-s)f(s,x_n(s))\,ds\Big\}_{n\geq 1}(t)\Big)\\
&\leq L_W(t)\beta\Big(\Big\{x_1 - R(b)\Big(x_0 + g(x_n)\Big)\Big\}_{n\geq 1}(t)\Big)\\
&+ L_W(t)\beta\Big(\Big\{\int_0^b R(t-s)f(s,x_n(s))\,ds\Big\}_{n\geq 1}(t)\Big).
\end{aligned}$$

By Lemma 11 and $(\mathbf{H_4})$–(iii), we deduce that

$$\begin{aligned}
\beta\Big(\{u_{x_n(t)}\}_{n\geq 1}(t)\Big) &\leq 2M_1 L_W(t)\left(\int_0^b h(s)\,ds\right)\beta\Big(\{x_n(s)\}_{n\geq 1}(t)\Big)\\
&\leq 2M_1 L_W(t)\left(\int_0^b h(s)\,ds\right)\beta\Big(D(t)\Big).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\beta\Big(\{(K_2 x_n)(t)\}_{n\geq 1}\Big)\\
&= \beta\left(\left\{\int_0^t R(t-s)f(s,x_n(s))\,ds + \int_0^t R(t-s)Cu_{x_n}(s)\,ds\right\}_{n\geq 1}(t)\right)\\
&\leq 2M_1\left(\int_0^b h(s)\,ds\right)\beta\Big(D(t)\Big)\\
&+ 2M_1 M_2\left(\int_0^b L_W(s)\,ds\right)2M_1\left(\int_0^b h(s)\,ds\right)\beta\Big(D(t)\Big)\\
&\leq 2M_1\|h\|_{L^1}\beta\Big(D(t)\Big) + 2M_1 M_2\|L_W\|_{L^1}2M_1\|h\|_{L^1}\beta\Big(D(t)\Big)\\
&\leq \Big(1 + 2M_1 M_2\|L_W\|_{L^1}\Big)\Big(2M_1\|h\|_{L^1}\Big)\beta\Big(D(t)\Big).
\end{aligned}$$

Finally

$$\begin{aligned}\beta\Big(K(D)(t)\Big) &\leq \beta\Big(K_1(D)(t)\Big) + \beta\Big(K_2(D)(t)\Big)\\ &\leq \Big(1 + 2M_1M_2\|L_W\|_{L^1}\Big)\Big(2M_1\|h\|_{L^1}\Big)\beta\Big(D(t)\Big).\end{aligned}$$

It means that $\beta\big(K(D(t))\big) \leq \gamma\beta\big(D(t)\big)$. From Mönch's condition, we obtain

$$\beta\Big(D(t)\Big) \leq \beta\Big(\overline{co}\Big(\{0\} \cup K(D(t))\Big)\Big) = \beta\Big(K(D(t))\Big) \leq \gamma\beta\Big(D(t)\Big).$$

This implies that $\beta\big(D(t)\big) = 0$, since $\gamma < 1$ and therefore, $\beta\big(K(D)(t)\big) = 0$. This shows that $\overline{K(D)(t)}$ is compact, that is $\overline{\{K(x)(t);\, x \in D\}}$ is compact as desired. So $K(D)$ is equicontinuous and equibounded and therefore, by Ascoli-Arzela's Theorem, we deduce that $K(D)$ is relatively compact.

But

$$\beta\Big(D\Big) \leq \beta\Big(\overline{co}\Big(\{0\} \cup K(D)\Big)\Big) = \beta\Big(K(D)\Big) = 0.$$

This implies that $\overline{D}$ is compact in X as desired. Thus D is relatively compact and the Mönch condition is satisfied. Therefore, by Corollary 15, K has a fixed point x in B_r, which is a mild solution of equation (2.1) and satisfies $x(b) = x_1$. The proof is complete. ∎

4. Example

We now illustrate our main result by the following example.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Consider the following nonlinear integrodifferential equation.

$$(4.1)\quad \begin{cases} \frac{\partial v(t,\xi)}{\partial t} = \Delta v(t,\xi) + \displaystyle\int_0^t \zeta(t-s)\Delta v(s,\xi)\,ds + \tfrac{e^{-t}}{k+e^t}\sin(v(t,\xi)) + a\omega(t,\xi) \\ \qquad\qquad\qquad \text{for } t \in [0,b] = I \text{ and } \xi \in \Omega \\ v = 0 \text{ on } \partial\Omega \\ v(0,\xi) = v_0(\xi) + \displaystyle\int_\Omega \int_0^b \rho(t,\xi)\log\Big(1 + |v(t,\eta)|^{\frac{1}{2}}\Big)\,dt\,d\eta \quad \text{for } \ \xi \in \Omega, \end{cases}$$

where $a > 0,\ k \geq 1,\ \omega : I \times \Omega \to \Omega$ is continuous in t and $\omega(t,\xi) = 0$ for all $\xi \in \partial\Omega$, $\rho \in \mathcal{C}(I \times \overline{\Omega})$ and $\rho(t,\xi) = 0$ for all $\xi \in \partial\Omega$, and $\zeta \in W^{1,1}(\mathbb{R}^+,\mathbb{R})$.

Let $X = U = \mathcal{C}_0(\overline{\Omega})$, the space of all continuous functions from $\overline{\Omega}$ to $\mathbb{R}$ vanishing on the boundary. We define $A : \mathcal{D}(A) \subset X \to X$ by:

$$\begin{cases} \mathcal{D}(A) = \left\{ v \in \mathcal{C}_0(\overline{\Omega}) \cap H_0^1(\Omega);\ \Delta v \in \mathcal{C}_0(\overline{\Omega}) \right\} \\ Av = \Delta v, \end{cases} \tag{4.2}$$

for each $v \in \mathcal{D}(A)$.

Theorem 17 (Theorem 4.1.4, p. 82 of [19]). *If Ω has a $\mathcal{C}^1$-boundary, then the operator A defined above is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{C}_0(\overline{\Omega})$.*

By Theorem 17, A generates a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ of contractions on $\mathcal{C}_0(\overline{\Omega})$. Moreover, $\big(T(t)\big)_{t\geq 0}$ generated by A is compact for $t > 0$ and operator-norm continuous for $t > 0$. Then by Theorem 6, the corresponding resolvent operator is operator-norm continuous. Define

$$x(t)(\xi) = v(t, \xi), \quad x'(t)(\xi) = \frac{\partial v(t, \xi)}{\partial t}.$$

Let $f(t, x)(\xi) = \frac{e^{-t}}{k+e^t} \sin(x(t)(\xi))$ for $t \in I$, $\xi \in \Omega$ and let $C : X \to X$ be defined by $Cu = a\omega$.

Let $(B(t)x)(\xi) = \zeta(t)\Delta x(t)(\xi)$ for $t \in I$, $x \in \mathcal{D}(A)$, $\xi \in \Omega$ and let $g : \mathcal{C}(I, X) \to X$ be defined by

$$g(x)(\xi) = \int_\Omega \int_0^b \rho(t, \xi) \log\left(1 + |x(t)(\eta)|^{\frac{1}{2}}\right) dt\, d\eta \quad \text{for } \xi \in \overline{\Omega} \text{ and } \ x \in \mathcal{C}(I, X).$$

Equation (4.1) can be transformed into the following form

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t B(t-s)x(s)ds + f(t, x(t)) + Cu(t) \quad \text{for t} \in \text{I} = [0, \text{b}], \\ x(0) = x_0 + g(x). \end{cases} \tag{4.3}$$

Then f is Lipschitz continuous with respect to its second variable and we get

$$\left\| f(t, x) \right\| \leq \frac{e^{-t}}{k + e^t} \ \text{ for } \ (t, x) \in I \times \overline{\Omega}.$$

Consequently, f satisfies $(\mathbf{H_4})$–(i), $(\mathbf{H_4})$–(ii) and $(\mathbf{H_4})$–(iii), with $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\phi(x) = 1$.

Moreover,

$$\Big\|g(x)\Big\|_{\mathcal{C}_0(\overline{\Omega})} \leq \Big(b\,\mathrm{mes}(\Omega)\Big) M_\rho \Big(\|x\|\Big)^{\frac{1}{2}},$$

where $M_\rho = \max_{(t,\xi)\in I\times\overline{\Omega}} |\rho(t,\xi)|$.

It is clear that for $g_r = \sup\big\{\|g(x)\| : \|x\|_\infty \leq r\big\}$, we have $\lim_{r\to+\infty} \frac{g_r}{r} = 0$.

Lemma 18. *The map $g : \mathcal{C}(I, \mathcal{C}_0(\overline{\Omega})) \to \mathcal{C}_0(\overline{\Omega})$ defined by*

$$g(x)(\xi) = \int_\Omega \int_0^b \rho(t,\xi) \log\Big(1 + |x(t)(\eta)|^{\frac{1}{2}}\Big)\, dt\, d\eta \quad \text{for } \xi \in \overline{\Omega} \text{ and } x \in \mathcal{C}(I, X),$$

is compact.

Proof. Let $E \subset \mathcal{C}(I, \mathcal{C}_0(\overline{\Omega}))$ be bounded. Then, by computing as above, we have

$$\Big\|g(x)\Big\|_{\mathcal{C}_0(\overline{\Omega})} \leq \Big(b\,\mathrm{mes}(\Omega)\Big) M_\rho \Big(\|x\|\Big)^{\frac{1}{2}}, \quad \text{for all } \ x \in E.$$

So $g(E)$ is bounded.

Now since ρ is uniformly continuous on $I \times \overline{\Omega}$, it follows that $g(E)$ is equicontinuous on $\overline{\Omega}$. Therefore, by Ascoli-Arzela's theorem, $g(E)$ is relatively compact in $\mathcal{C}_0(\overline{\Omega})$. Hence, g is compact. ■

By Lemma 18 g is compact and therefore, it satisfies $(\mathbf{H_5})$. Now for $\xi \in \Omega$, the operator W is given by

$$(Wu)(\xi) = \int_0^1 R(1-s)\, a\, \omega(s,\xi)\, ds.$$

Assuming that W satisfies $(\mathbf{H_3})$, all conditions of Theorem 16 hold and equation (4.3) is controllable.

Conclusion

The paper contains the controllability result of some partial functional integrodifferential differential equation with nonlocal initial condition in Banach spaces by using the Hausdorff Measure of Noncompactness and the Mönch fixed point theorem. The result shows that without compactness of the resolvent operator for the associated linear homogeneous part, the Mönch fixed point theorem can effectively be used to obtain controllability results under some sufficient conditions.

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