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INFINITE DIMENSIONAL UNCERTAIN DYNAMIC SYSTEMS ON BANACH SPACES AND THEIR OPTIMAL OUTPUT FEEDBACK CONTROL

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Abstract

In this paper we consider a class of partially observed semilinear dynamic systems on infinite dimensional Banach spaces subject to dynamic and measurement uncertainty. The problem is to find an output feedback control law, an operator valued function, that minimizes the maximum risk. We present a result on the existence of an optimal (output feedback) operator valued function in the presence of uncertainty in the system as well as measurement. We also consider uncertain stochastic systems and present similar results on the question of existence of optimal feedback laws.

Keywords: partially observed, uncertain systems, stochastic systems, operator valued functions, feedback operators, existence of optimal operators in the presence of uncertainty.

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1. Motivation: Physical examples

Control theory for finite as well as infinite dimensional systems with open loop controls is developed extensively in the literature. The reader is referred to the Encyclopedic presentations of Cesari [8] and Fattorini [9] including the references therein. In applications of control theory, there are many problems in physical sciences and engineering where open loop control is not feasible. For example, internet traffic control and control of nonlinear optical devices governed by nonlinear schrödinger equation. One must use feedback control based on available data from sensors which is generally noisy. So far in the literature, for construction of state feedback controls, one develops the HJB equation and from its solution, if one exists, one hopes to construct the feedback control law. This is an indirect approach presenting a very complex problem requiring solution of nonlinear HJB equations on infinite dimensional Hilbert spaces [Ahmed, 1]. Further, it does not solve the output feedback control problem which is more natural in applications, because it is very costly and almost impossible to monitor the state in infinite dimensional state space. Here in this paper, we develop a direct approach and consider infinite dimensional systems both deterministic and stochastic. We determine the feedback control law based on only available information. This involves optimization of nonlinear functionals on the space of bounded linear Operators. The feedback control law must be chosen from a specified class of operator valued functions satisfying certain constraints so as to optimize the performance of the system. The question of optimization on the space of bounded linear operators also arise in the study of inverse (or equivalently identification) problems [2]. In such problems, it is assumed that the state is observable without any measurement uncertainty. The problems considered here are also different from those of optimal controls of differential inclusions [3, 4, 5] where the controls are vector valued functions of time. One may view the controls in this paper as operator valued functions operating on the available noisy information and delivering control forces.

In this paper we consider the question of existence of optimal feedback control laws, in the presence of both system and sensor uncertainty, minimizing the maximum loss or equivalently maximizing minimum payoff. The results presented here generalize our recent results on similar topic where only measurement (sensor) uncertainty is assumed [7]. There, also necessary conditions of optimality were developed. For finite dimensional systems, similar necessary conditions of optimality were used for computation of optimal feedback laws illustrated by numerical results [6].

The rest of the paper is organized as follows. In Section 2, we present some typical notations. In Section 3, we present the mathematical model describing the system and formulate the problem considered in the paper. The basic assumptions used are given in Section 4 followed by a basic result on the existence and regularity of solutions of the feedback system. In Section 5, we present results on continuous dependence of solution on feedback operators, the operators representing perturbation of the semigroup (generator), and the process representing measurement noise. In Section 6, this result is used to prove the existence of an optimal feedback law (an operator valued function). In Section 7, we extend the previous results to infinite dimensional stochastic systems subject to both the system and measurement uncertainty.

2. Some notations

Let $\{X, Y, U\}$ denote a triple of real separable Banach spaces representing the state space, the output(measurement) space and the control space respectively. Let $I = [0, T]$ denote any closed bounded interval. For any separable reflexive Banach space Z, we let $L_1(I, Z)$ denote the space of Bochner integrable functions with values in Z, and its dual by $L_\infty(I, Z^*)$. Let Z_1, Z_2 be any pair of real Banach spaces and $\mathcal{L}(Z_1, Z_2)$ the Banach space of bounded linear operators from Z_1 to Z_2. Let $B_1(Z)$ denote the closed unit ball in any Banach space Z. An operator $S \in \mathcal{L}(Z_1, Z_2)$ is said to be compact if $S(B_1(Z_1))$ is a relatively compact subset of Z_2.

Let $B_\infty(I, \mathcal{L}(Z_1, Z_2))$ denote the space of operator valued functions which are measurable in the uniform operator topology and uniformly bounded on the interval I in the sense that

$$\sup\{\| T(t) \|_{\mathcal{L}(Z_1,Z_2)}, t \in I\} < \infty$$

for $T \in B_\infty(I, \mathcal{L}(Z_1, Z_2))$. Suppose this is furnished with the topology of strong convergence (convergence in the strong operator topology) uniformly on I in the sense that, given $T_n, T \in B_\infty(I, \mathcal{L}(Z_1, Z_2))$, $T_n \to T$ in this topology iff for every $z \in Z_1$,

$$\sup\{|T_n(t)z - T(t)z|_{Z_2}, t \in I\} \to 0$$

as $n \to \infty$. Let $\mathcal{K}(Z_1, Z_2))$ denote the class of compact operators from Z_1 to Z_2. It is well known that this is a closed linear subspace of $\mathcal{L}(Z_1, Z_2)$ in the uniform operator topology and hence a Banach space. Let Γ be a closed bounded (possibly convex) subset of $\mathcal{K}(Z_1, Z_2)$. We are interested in the set $B_\infty(I, \Gamma) \subset B_\infty(I, \mathcal{K}(Z_1, Z_2))$ endowed with the relative topology of convergence in the strong operator topology of the space $\mathcal{L}(Z_1, Z_2)$ point wise in $t \in I$.

3. System with uncertainties and problem formulation

Let X, Y, U be real Banach spaces, with X denoting the state space, Y denoting the output space, and U the space where controls take their values from. The complete system is governed by the following system of equations:

$$\dot{x} = Ax + R(t)x + F(x) + B(t)u, \qquad \text{in } X, \tag{1}$$

$$y = L(t)x + \xi \qquad \text{in } Y, \tag{2}$$

$$u = K(t)y \qquad \text{in } U, \tag{3}$$

where the first equation describes the dynamics of the system in the state space X giving the state $x(t)$ at any time $t \geq 0$, the second equation describes the (measurement) output process that observes the status of the system in a noisy environment characterized by the random process ξ and delivers the output $y(t), t \geq 0$, with values from the Banach space Y. The operator valued process R perturbing the semigroup generator is also random or uncertain and takes values from the Banach space $\mathcal{L}(X)$ of bounded linear operators in X. This represents the uncertainty in the dynamics, in the sense that the exact value of R at any given time is not known, but it is known that it takes values from a bounded set in $\mathcal{L}(X)$, for example, the closed unit ball around the origin $B_1(\mathcal{L}(X))$. We denote this class of operator valued functions by $\mathcal{V} \equiv B_\infty(I, B_1(\mathcal{L}(X)))$. In order to regulate the system (1), the third equation provides the control based on the noisy measurement process y through the operator valued function K. In general the operator A is an unbounded linear operator with domain and range in X. The operator F is a nonlinear map in X, the operator valued function B takes values from $\mathcal{L}(U, X)$, the operator L, representing the sensor (or measurement system), takes values from $\mathcal{L}(X, Y)$ and the output feedback control operator K is an operator valued function taking values from the space $\mathcal{K}(Y, U)$. Let $\mathcal{F}_{ad}$, whose precise characterization is given later, denote the class of admissible feedback operator valued functions $\{K(t), t \geq 0\}$ with values in $\mathcal{K}(Y, U)$. The process $\xi(t), t \geq 0$, represents the uncertainty in the measurement data and takes values from the Banach space Y. For most practical situations, it is reasonable to assume that the disturbance process is bounded. And so, without any loss of generality, we may assume that the process ξ is strongly measurable taking values from the closed unit ball $B_1(Y)$ centered at the origin. We denote this class of disturbance processes by $\mathcal{D}$.

The performance of the system over the time horizon $I \equiv [0, T]$ is measured by the following functional (called cost functional)

$$J(K, R, \xi) \equiv \int_I \ell(t, x(t))dt + \Phi(x(T)) \tag{4}$$

where $\ell : I \times X \longrightarrow [0, \infty]$ and $\Phi : X \longrightarrow [0, \infty]$. The cost functional depends on the choice of the control law K in the presence of dynamic uncertainty R and imperfect measurement induced by ξ. Our objective is to find an operator valued function $K \in \mathcal{F}_{ad}$ that minimizes the maximum possible cost. In other words, we want a feedback law that minimizes the maximum risk posed by system and measurement uncertainties. This problem can be formulated as min-max problem:

$$\inf_{K \in \mathcal{F}_{ad}} \sup_{(R,\xi) \in \mathcal{V} \times \mathcal{D}} J(K, R, \xi).$$

Given this pessimistic view, an element $K_o \in \mathcal{F}_{ad}$ is said to be optimal if and only if

$$(5) \quad J_o(K_o) \equiv \sup_{(R,\xi)\in\mathcal{V}\times\mathcal{D}} J(K_o,R,\xi) \leq \sup_{(R,\xi)\in\mathcal{V}\times\mathcal{D}} J(K,R,\xi) \equiv J_o(K), \ \forall\, K \in \mathcal{F}_{ad}.$$

4. Basic assumptions and preliminaries

To consider the problem as stated above, we introduce the following basic assumptions:

(A0): The Banach spaces $\{X, Y\}$ are reflexive and U is any real Banach space.

(A1): The operator A is the infinitesimal generator of a C_0-semigroup of operators $S(t), t \geq 0$, on X.

(A2): The vector field $F : X \longrightarrow X$ is uniformly Lipschitz with Lipschitz constant $C_1 > 0$.

(A3): Both B and L are measurable in the uniform operator topology, with $B \in L_1(I, \mathcal{L}(U,X))$ and $L \in B_\infty(I, \mathcal{L}(X,Y))$.

(A4): Let $\Gamma \subset \mathcal{K}(Y,U)$ be a nonempty closed bounded (possibly convex) set and denote the admissible feedback control laws by

$$\mathcal{F}_{ad} \equiv \{K \in B_\infty(I, \mathcal{K}(Y,U)) : K(t) \in \Gamma \ \ \forall\, t \in I\}.$$

(A5): The process R perturbing the semigroup is any uniformly measurable operator valued function defined on I and taking values from the closed unit ball $B_1(\mathcal{L}(X))$. This is denoted by $\mathcal{V} \equiv B_\infty(I, B_1(\mathcal{L}(X)))$.

(A6): The disturbance (noise) process $\xi : I \longrightarrow Y$, is any measurable function taking values from the closed unit ball $B_1(Y)$ of the B-space Y. We denote this family by $\mathcal{D} \equiv B_\infty(I, B_1(Y))$. This represents the uncertainty without any probabilistic structure.

Some comments on the uncertainties in dynamics $\mathcal{V}$ and measurement $\mathcal{D}$ are in order. We do not assume any probabilistic structure for these process except that they are bounded measurable process and hence locally square integrable.

(A7): The integrand $\ell : I \times X \longrightarrow (-\infty, \infty]$ is measurable in the first variable and continuous in the second argument and there exists a $p \in [1, \infty)$ such that

$$|\ell(t,x)| \leq g(t) + c_1 \parallel x \parallel_X^p, x \in X, t \geq 0$$

with $0 \leq g \in L_1(I)$ and $c_1 \geq 0$. The function Φ is also continuous on X and there exist constants $c_2, c_3 \geq 0$ such that

$$|\Phi(x)| \leq c_2 + c_3 \parallel x \parallel_X^p$$

for the same p.

Substituting the equations (2) and (3) into (1) we obtain the following uncertain feedback system

$$\dot{x} = Ax + Rx + F(x) + BKLx + BK\xi, x_0 \in X \text{ (fixed) }, K \in \mathcal{F}_{ad}, \tag{6}$$

subject to the (unstructured) disturbances $\{R, \xi\} \in \mathcal{V} \times \mathcal{D}$. Before we conclude this section we present the following standard result on the existence and regularity of solutions of the feedback system. This is used later in the paper.

Lemma 4.1. *Consider the uncertain feedback system given by* (6) *over any finite time horizon* $I \equiv [0, T]$, *and suppose the assumptions* **(A1)**–**(A6)** *hold. Then, for every initial state* $x(0) = x_0 \in X$, *and any feedback law* $K \in \mathcal{F}_{ad}$ *and disturbance* $(R, \xi) \in \mathcal{V} \times \mathcal{D}$, *the system* (6) *has a unique mild solution* $x \in C(I, X)$. *Further, the solution set*

$$\mathcal{X} \equiv \Big\{ x(K, R, \xi)(\cdot) \in C(I, X) : K \in \mathcal{F}_{ad}, R \in \mathcal{V}, \xi \in \mathcal{D} \Big\}$$

is a bounded subset of $C(I, X)$.

Proof. By definition, the mild solution of the system (6) is given by the solution (if one exists) of the following integral equation

$$\begin{aligned} x(t) \equiv S(t)x_0 &+ \int_0^t S(t-r)R(r)x(r)dr + \int_0^t S(t-r)F(x(r))dr \\ &+ \int_0^t S(t-r)(BKL)(r)x(r)dr + \int_0^t S(t-r)(BK)(r)\xi(r)dr \end{aligned} \tag{7}$$

on the Banach space X. We prove that this equation has a solution. The proof is based on Banach fixed point theorem. Define the operator $\mathcal{G}$ on $C(I, X)$ by the following expression

$$\begin{aligned} (\mathcal{G}x)(t) \equiv\ & S(t)x_0 + \int_0^t S(t-r)R(r)x(r)dr + \int_0^t S(t-r)F(x(r))dr \\ &+ \int_0^t S(t-r)(BKL)(r)x(r)dr + \int_0^t S(t-r)(BK)(r)\xi(r)dr. \end{aligned}$$

We prove that $\mathcal{G}$ has a fixed point in $C(I, X)$. By virtue of assumption (A1), it follows from the basic properties of C_0-semigroups that there exists $M \geq 1$ such that $\sup\{\| S(t) \|_{\mathcal{L}(X)}, t \in I\} \leq M$. Then using the assumptions (A2)–(A6), it is easy to verify that $\mathcal{G} : C(I, X) \longrightarrow C(I, X)$. Since $B \in L_1(I, \mathcal{L}(U, X))$ and the operator valued functions $\{R, K, L\}$ are all norm bounded, there exists an $h \in L_1^+(I)$ (dependent on M, the Lipschitz constant for F and the bounds of the operator valued functions $\{R, L, K\}$) such that for any pair $\{x, z\} \in C(I, X)$ we have

$$\sup_{0 \leq s \leq t} |\mathcal{G}x(s) - \mathcal{G}z(s)|_X \leq \int_0^t h(r)|x(r) - z(r)|_X dr, \ \forall \ t \in [0, T]. \tag{8}$$

Choosing $T_1 \in (0, T]$ sufficiently small so that $\int_0^{T_1} h(r)dr \equiv \alpha_1 < 1$, we have

$$\sup_{0 \leq s \leq T_1} |\mathcal{G}x(s) - \mathcal{G}z(s)|_X \leq \alpha_1 \sup\{|x(r) - z(r)|_X, 0 \leq r \leq T_1\}. \tag{9}$$

Thus the operator $\mathcal{G}$, restricted to $C([0, T_1], X)$, is a contraction and therefore it has a unique fixed point $x_1 \in C([0, T_1], X)$. Next, choosing $T_2 \in (T_1, T]$ and considering the restriction of $\mathcal{G}$ on the space $C([T_1, T_2], X)$ we have

$$\begin{aligned} (\mathcal{G}x)(t) \equiv S(t - T_1)x_1(T_1) &+ \int_{T_1}^t S(t - r)R(r)x(r)dr + \int_{T_1}^t S(t - r)F(x(r))dr \\ &+ \int_{T_1}^t S(t - r)(BKL)(r)x(r)dr + \int_{T_1}^t S(t - r)(BK)(r)\xi(r)dr, t \in [T_1, T]. \end{aligned} \tag{10}$$

Using this equation and repeating the procedure, it is easy to verify that

$$\sup_{T_1 \leq s \leq T_2} |(\mathcal{G}x)(s) - (\mathcal{G}z)(s)|_X \leq \left(\int_{T_1}^{T_2} h(r)dr \right) \sup_{T_1 \leq s \leq T_2} |x(s) - z(s)|_X. \tag{11}$$

Since $h \in L_1^+(I)$, again we can choose T_2 sufficiently small so that $\left(\int_{T_1}^{T_2} h(r)dr\right) \equiv \alpha_2 < 1$. Thus the operator $\mathcal{G}$, restricted to $C([T_1, T_2], X)$, is a contraction and by Banach fixed point theorem it has a unique fixed point say $x_2 \in C([T_1, T_2], X)$ satisfying $x_2(T_1) = x_1(T_1)$. Since I is a compact interval, following this procedure, step by step, one can cover the entire interval in a finite number of steps. Then by concatenation of the sequence $\{x_1, x_2, \cdots x_n\}$ one has a unique solution of equation (7), say, $x \in C(I, X)$. Hence $\mathcal{G}$ has a unique fixed point, that is, $x = \mathcal{G}x$. Thus the system (6) has a unique mild solution $x \in C(I, X)$. In other words, x satisfies the integral equation

$$\begin{aligned} x(t) = S(t)x_0 &+ \int_0^t S(t - r)R(r)x(r)dr + \int_0^t S(t - r)F(x(r))dr \\ &+ \int_0^t S(t - r)(BKL)(r)x(r)dr + \int_0^t S(t - r)(BK)(r)\xi(r)dr. \end{aligned}$$

Now using the assumptions (A1)-(A5) and the above equation, one can easily verify that, independently of $R \in \mathcal{V}, K \in \mathcal{F}_{ad}, \xi \in \mathcal{D}$, there exist nonnegative constants $\tilde{C}_1, \tilde{C}_2$ (depending only on the bounds of the sets $\{\mathcal{V}, \mathcal{F}_{ad}, \mathcal{D}\}$ and the Lipschitz coefficient of F), such that

$$|x(t)|_X \leq \tilde{C}_1 + \tilde{C}_2 \int_0^t h(r)|x(r)|_X dr, t \in I.$$

Hence it follows from Gronwall inequality that the solution set $\mathcal{X}$ is a bounded subset of $C(I, X)$. This completes the brief outline of the proof. ■

5. Continuous dependence of solutions on $\{K, R, \xi\}$

For proof of existence of optimal feedback operator we need continuity of solutions with respect to the operator and vector valued processes $\{K, R, \xi\}$. Since continuity is crucially dependent on the topology of both the domain and the target spaces, it is necessary to specify the admissible topologies. For the target space $C(I, X)$, we have already the natural sup-norm topology. So we must introduce appropriate topologies on the domain spaces $\mathcal{F}_{ad} \equiv B_\infty(I, \Gamma)$, $\mathcal{V} \equiv B_\infty(I, B_1(\mathcal{L}(X))) \subset B_\infty(I, \mathcal{L}(X))$ and $\mathcal{D} \equiv B_\infty(I, B_1(Y)) \subset L_\infty(I, Y)$, where, for any vector space Z, $B_\infty(I, Z)$ denotes the class of bounded measurable functions defined on I and taking values from Z.

First let us consider the space of operator valued functions providing the feedback controls. Let $\Gamma \subset \mathcal{K}(Y, U)$ be a closed bounded (possibly convex) set and $B_\infty(I, \Gamma)$ denote the class of strongly measurable operator valued functions defined on I and taking values from Γ endowed with the topology of convergence in the strong operator topology point wise in $t \in I$. Generally this is not sufficient for the problem considered here. To proceed further, we need certain definitions and some useful notions. In particular, we need the set Γ to satisfy certain compactness property. The following result due to Mayoral [11] characterizes relatively compact subsets of $\mathcal{K}(Y, U)$.

Proposition 5.1 [Mayoral [11], Theorem 1, p. 79]**.** *If the B-space Y does not contain a copy of ℓ_1, a set $\Gamma \subset \mathcal{K}(Y, U)$ is relatively compact iff*

(i) *Γ is uniformly completely continuous (ucc) and*

(ii) *for every $y \in Y$, the y-section, $\Gamma(y) \equiv \{L(y), L \in \Gamma\}$, is relatively compact in U.*

(H1) (Admissible Feedback Operators $\mathcal{F}_{ad}$): *Since by our assumption Y is a reflexive Banach space, it does not contain a copy of ℓ_1. So we can use Mayoral's result. We assume that $\Gamma \subset \mathcal{K}(Y, U)$ satisfies the above characterization for* *relative compactness and further that it is closed so that it is compact. Then we consider the Tychonoff product topology* τ_T *on the function space* $B_\infty(I, \Gamma) \equiv \mathcal{F}_{ad}$ *which turns this into a compact Hausdorff (topological) space.*

(H2) (Admissible System Uncertainty $\mathcal{V}$): *Next, we consider the set* $\mathcal{V}$ *representing uncertainty in the system model. Since* X *is a reflexive Banach space, it is well known that the closed unit ball* $B_1(\mathcal{L}(X))$ *is compact with respect to the weak operator topology* τ_{wo}*. Using this fact we may now equip* $\mathcal{V} \equiv B_\infty(I, B_1(\mathcal{L}(X))$ *with the Tychonoff product topology and denote this by* τ_{Two}*. With respect to this topology* $\mathcal{V}$ *is a compact Hausdorff space.*

(H3) (Admissible Measurement Uncertainty $\mathcal{D}$): *Next we consider the set* $\mathcal{D} \equiv B_\infty(I, B_1(Y))$ *with* $B_1(Y)$ *denoting the closed unit ball (centered at the origin) representing the measurement uncertainty. Reflexivity of* Y *implies that* $B_1(Y)$ *is weakly compact. The set* $\mathcal{D}$ *is endowed with the Tychonoff product topology* τ_{Tw}*. With respect to this topology* $\mathcal{D}$ *is a compact Hausdorff space.*

Now we are prepared to consider the question of continuity.

Theorem 5.2. *Consider the feedback system* (6) *and suppose the assumptions* (A0)–(A6) *and* (H1)–(H3) *hold and that the operator* A *is the infinitesimal generator of a compact* C_0*-semigroup* $S(t), t > 0$*. Then the map* $(K, R, \xi) \longrightarrow x(K, R, \xi)$ *is jointly continuous from* $\mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$ *to* $C(I, X)$ *with respect to their respective topologies.*

Proof. Let $\{K^n, R^n, \xi^n\} \in \mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$ be a generalized sequence (net) and suppose $K^n \xrightarrow{\tau_T} K^o$ in $\mathcal{F}_{ad}$, $R^n \xrightarrow{\tau_{Two}} R^o$ in $\mathcal{V}$ and $\xi^n \xrightarrow{\tau_{Tw}} \xi^o$ in $\mathcal{D}$. Let $\{x_n, x_o\}$ denote the mild solutions of equation (6) corresponding to the triples $\{(K^n, R^n, \xi^n), (K^o, R^o, \xi^o)\}$ respectively. Then by the definition of mild solutions, $\{x_n, x_o\}$ are the solutions of the integral equations

$$\begin{aligned} x_n(t) &= S(t)x_0 + \int_0^t S(t-s)R^n(s)x_n(s)ds + \int_0^t S(t-s)F(x_n(s))ds \\ &+ \int_0^t S(t-s)BK^n(Lx_n + \xi^n)(s)ds, \ t \in I, \end{aligned} \tag{12}$$

$$\begin{aligned} x_o(t) &= S(t)x_0 + \int_0^t S(t-s)R^o(s)x_o(s)ds + \int_0^t S(t-s)F(x_o(s))ds \\ &+ \int_0^t S(t-s)BK^o(Lx_o + \xi^o)(s)ds, \ t \in I, \end{aligned} \tag{13}$$

respectively and they belong to $C(I,X)$. Taking the difference and rearranging the terms suitably, we have the following expression

$$x_o(t)-x_n(t)=\int_0^t S(t-r)[F(x_o(r))-F(x_n(r))]dr$$
$$+\int_0^t S(t-r)R^n(r)(x_o(r)-x_n(r))dr+\int_0^t S(t-r)[BK^nL(x_o-x_n)](r)dr$$
$$+\int_0^t S(t-r)(R^o(r)-R^n(r))x_o(r)dr$$
$$+\int_0^t S(t-r)(B(K^o-K^n)(Lx_o+\xi^o))(r)dr$$
$$+\int_0^t S(t-r)(BK^n(\xi^o-\xi^n))(r)dr,\ t\in I.$$

Introduce the sequence of functions $\{e_{i,n}, i=1,2,3\}_{n\in N}$ as follows:

$$e_{1,n}(t)\equiv\int_0^t S(t-r)(R^o(r)-R^n(r))x_o(r)dr$$
$$e_{2,n}(t)\equiv\int_0^t S(t-r)(B(K^o-K^n)(Lx_o+\xi^o))(r)dr$$
$$e_{3,n}(t)\equiv\int_0^t S(t-r)((BK^n)(\xi^o-\xi^n))(r)dr.$$

Clearly, these are elements of $C(I,X)$. Using these functions in the above expression and taking norms, it is easy to verify that

$$|x_o(t)-x_n(t)|_X\le\eta_n(t)+\int_0^t h(r)|x_o(r)-x_n(r)|_X dr,\ t\in I, \tag{14}$$

where $\eta_n(t)$ is given by

$$\eta_n(t)\equiv|e_{1,n}(t)|_X+|e_{2,n}(t)|_X+|e_{3,n}(t)|_X$$

and the function h is given by

$$h(t)\equiv M\{1+C_1+C_2\parallel B(t)\parallel\}$$

with the constants $\{M,C_1,C_2\}$ as given below. By assumption (A1), $\sup\{\parallel S(t)\parallel_{\mathcal{L}(X)}, t\in I\}\equiv M<\infty$, by assumption (A2) there exists a constant $C_1>0$, denoting the Lipschitz coefficient of F, while by assumption (A3) and (A4) there exists a constant C_2 such that $\sup\{\parallel (K^nL)(t)\parallel_{\mathcal{L}(X,U)}, t\in I, n\in N\}\le C_2<\infty$.

Thus, by virtue of the assumptions (A1)–(A4), $h \in L_1^+(I)$. Hence it follows from Gronwall inequality applied to (14) that

$$|x_o(t) - x_n(t)|_X \leq \eta_n(t) + \exp(\parallel h \parallel_{L_1}) \int_0^t h(r)\eta_n(r)dr, \ t \in I. \tag{15}$$

We show that the expression on righthand side of the above inequality converges to zero uniformly on I. First note that the integrand, defining $e_{1,n}$, is dominated by $2M|x_o(t)|_X \leq 2M \parallel x_0 \parallel_{C(I,X)}, t \in I$. Since $R^n \to R^o$ in the Tychonoff product topology τ_{Two}, it is clear that $(R^o - R^n)x_o$ converges weakly to zero in X for each $t \in I$. Thus by the compactness of the semigroup $S(t), t > 0$, we have $e_{1,n}(t) \to 0$ strongly in X uniformly on I. Consider the second term $e_{2,n}$. Since $K^n \to K^o$ in the Tychonoff product topology τ_T on $\mathcal{F}_{ad}$ and $(Lx_o + \xi^o) \in B_\infty(I, Y)$, and B is Bochner integrable, the integrand, defining $e_{2,n}(t)$, converges to zero almost everywhere strongly in X and further it's norm is dominated by an integrable function. Thus, by Lebesgue dominated convergence theorem, we conclude that

$$\sup\{|e_{2,n}(t)|_X, t \in I\} \to 0.$$

Considering the third term, $e_{3,n}$, we note that, by Proposition 5.1, Γ is norm bounded and uniformly completely continuous (equivalently, weak-norm equicontinuous) and $K^n(t) \in \Gamma$ for all $t \in I$, and $\xi^n \to \xi^o$ in the Tychonoff product topology τ_{Tw} and consequently by (H1) and (H3) the integrand converges (along a subsequence if necessary) to zero almost every where. It is important to note that it is the weak-norm equicontinuity of the set Γ and Tychonoff product topology that permits such conclusion. Since the integrand is also dominated by an integrable function, again it follows from Lebesgue dominated convergence theorem that

$$\sup\{|e_{3,n}(t)|_X, t \in I\} \to 0.$$

In view of these facts $\eta_n(t) \to 0$ uniformly on I. Thus it follows from (15) that $x_n(t) \stackrel{s}{\longrightarrow} x_o(t)$ in X uniformly on I. This completes the proof. ■

Now we prove that the cost functional J given by the expression (4) is jointly continuous on $\mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$.

Corollary 5.3. *Suppose the assumptions of Theorem* 5.2 *hold and the functions* ℓ *and* Φ *satisfy the assumption* (A7)*. Then, the functional* $(K, R, \xi) \longrightarrow J(K, R, \xi)$ *is jointly continuous on* $\mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$ *with respect to the topology* $\tau_T \times \tau_{Two} \times \tau_w$.

Proof. Let $\{K^n, R^n, \xi^n\}$ be a generalized sequence from the set $\mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$ converging to $\{K^o, R^o, \xi^o\}$. Let $x_n \in C(I, X)$, $x_o \in C(I, X)$, denote the corresponding mild solutions of the evolution equation (6). Then by assumption (A7), it follows from Theorem 5.2 that, along a subsequence if necessary,

$\ell(t, x_n(t)) \rightarrow \ell(t, x_o(t))$ a.e; and that it is dominated by an integrable function since the sequence of solutions $\{x_n\}$ are uniformly bounded [see Lemma 4.1]. Since $\{x_n, x_o\} \in C(I, X)$ and the function Φ is continuous and bounded on bounded sets, $\Phi(x_n(T)) \rightarrow \Phi(x_o(T))$. Thus letting $n \rightarrow \infty$, it follows from dominated convergence theorem that

$$\lim_{n\to\infty} J(K^n, R^n, \xi^n) = \lim_{n\to\infty} \int_I \ell(t, x_n(t))dt + \Phi(x_n(T))$$
$$= \int_I \ell(t, x_o(t))dt + \Phi(x_o(T)) \equiv J(K^o, R^o, \xi^o).$$

This proves the joint continuity as stated. ∎

Remark 5.4. Note that the assumption on compactness of the semigroup is used only to prove that $e_{1,n}(t) \rightarrow 0$ strongly in X uniformly on I. In view of this compactness property, it is tempting to bypass the Mayoral's result and use weak operator topology instead (for the admissible feedback operators). If the Banach space U is reflexive, we know that the closed unit ball $B_1(\mathcal{L}(Y, U))$ is compact in the weak operator topology. One may be tempted to use this topology combined with the Tychonoff product topology for $\mathcal{F}_{ad}$. Unfortunately, this does not seem to be possible. Indeed, considering $e_{3,n}$ given by

$$e_{3,n}(t) \equiv \int_0^t S(t-r)(BK^n(r)(\xi^o(r) - \xi^n(r))dr,$$

we note the interaction of two weak topologies, the weak operator topology on $B_1(\mathcal{L}(Y, U))$ and the weak topology on $B_1(Y)$. Despite the compactness of the semigroup, we cannot conclude even point wise convergence of $e_{3,n}(t)$ strongly in X. However, if the measurement is perfect, that is $\xi \equiv 0$, then the weak operator topology is sufficient.

Now we are prepared to prove the existence of an optimal operator valued function $K_o \in \mathcal{F}_{ad}$ that solves the min-max problem in the sense of (5). For this we need the notions of upper and lower semi-continuity of multi functions.

Definition 5.5. Let Z_1, Z_2 be any pair of topological spaces. A multi function $G : Z_1 \longrightarrow 2^{Z_2} \setminus \emptyset$ is upper semi-continuous if for every closed set $C \subset Z_2$, the preimage $G^{-1}(C) \equiv \{x \in Z_1 : G(x) \cap C \neq \emptyset\}$ is closed. And it is lower semi-continuous if for every open set $D \subset Z_2$ the preimage $G^{-1}(D) \equiv \{x \in Z_1 : G(x) \cap D \neq \emptyset\}$ is open.

For details on multi-functions see the Handbook by Hu and Papageorgiou [9].

6. Existence of optimal feedback operators

In this section we consider the question of existence of optimal feedback operators.

Theorem 6.1. *Consider the feedback system* (6). *Suppose the assumptions of Theorem* 5.2 *and Corollary* 5.3 *hold. Then there exists an optimal feedback operator valued function $K_o \in \mathcal{F}_{ad}$ in the sense that*

$$J_o(K_o) \equiv \sup_{R\in\mathcal{V},\xi\in\mathcal{D}} J(K_o,R,\xi) \le \sup_{R\in\mathcal{V},\xi\in\mathcal{D}} J(K,R,\xi) \equiv J_o(K) \;\forall\; K \in \mathcal{F}_{ad}.$$

Proof. For each given $K \in \mathcal{F}_{ad}$ the set

$$\Pi(K) \equiv \{(R,\xi) \in \mathcal{V}\times\mathcal{D} : J(K,R,\xi) = \sup_{(Q,\eta)\in\mathcal{V}\times\mathcal{D}} J(K,Q,\eta)\}.$$

For a given $K \in \mathcal{F}_{ad}$, this is the set of points in $\mathcal{V}\times\mathcal{D}$ at which the functional $J(K,\cdot,\cdot)$ attains its maximum. The functional $J_o : \mathcal{F}_{ad} \longrightarrow \overline{R}$ is defined as

$$J_o(K) = J(K,\Pi(K)).$$

This is the maximal risk functional. The problem considered in the paper is: find $K^o \in \mathcal{F}_{ad}$ so that $J_o(K^o) \le J_o(K)$ for all $K \in \mathcal{F}_{ad}$. In other words, find an operator K^o that minimizes the maximum risk. By virtue of joint continuity of J in all its arguments (Corollary 5.3) it is clear that, for each fixed $K \in \mathcal{F}_{ad}$, the functional $(R,\xi) \longrightarrow J(K,R,\xi)$ is $\tau_{Two}\times\tau_{Tw}$-continuous on $\mathcal{V}\times\mathcal{D}$. Since this set is $\tau_{Two}\times\tau_{Tw}$ compact, the set $\Pi(K)$ is well defined and so $\Pi(K)\neq\emptyset$. In general this is a multifunction $\Pi : \mathcal{F}_{ad} \longrightarrow 2^{\mathcal{V}\times\mathcal{D}}\setminus\emptyset$. Define, for each $K \in \mathcal{F}_{ad}$,

$$J_o(K) \equiv J(K,\Pi(K)).$$

One may describe this functional as the measure of maximal risk. So by our definition of optimality, an element $K_o \in \mathcal{F}_{ad}$ is optimal if, and only if,

$$J_o(K_o) \le J_o(K) \;\forall\; K \in \mathcal{F}_{ad}$$

which minimizes the maximal risk. We show that such an element exists. Since $\mathcal{F}_{ad}$ is compact in the Tychnoff product topology τ_T, it suffices to verify that $K \longrightarrow J_o(K)$ is τ_T continuous. Consider the net $\{K_\alpha, \alpha \in \triangle\} \subset \mathcal{F}_{ad}$ and $K_o \in \mathcal{F}_{ad}$ such that $K_\alpha \xrightarrow{\tau_T} K_o$. By definition of J_o, it is clear that $J_o(K_\alpha) = J(K_\alpha,\Pi(K_\alpha))$ for $\alpha\in\triangle$. Thus there exists a net $(R_\alpha,\xi_\alpha) \in \Pi(K_\alpha)$ such that $J_o(K_\alpha) = J(K_\alpha,R_\alpha,\xi_\alpha)$. Since $\mathcal{F}_{ad}\times\mathcal{V}\times\mathcal{D}$ is $\tau_T\times\tau_{Two}\times\tau_{Tw}$ compact, there exists a subnet, relabeled as the original net, and a triple $(K_o,R_o,\xi_o) \in \mathcal{F}_{ad}\times\mathcal{V}\times\mathcal{D}$ such that

$$(K_\alpha,R_\alpha,\xi_\alpha) \longrightarrow (K_o,R_o,\xi_o)$$

with respect to the product topology $\tau_T \times \tau_{Two} \times \tau_{Tw}$. Again, by virtue of joint continuity of J (Corollary 5.3), we have $\lim_{\alpha\in\triangle} J(K_\alpha, R_\alpha, \xi_\alpha) = J(K_o, R_o, \xi_o)$. Thus, to complete the proof of continuity of the functional $K \longrightarrow J_o(K)$, we must show that $(R_o, \xi_o) \in \Pi(K_o)$. For this it suffices to verify that the graph $\mathcal{G}_r(\Pi)$ of the multifunction Π is closed. It is well known that an upper semi-continuous multifunction from a Hausdorff topological space to a regular topological space has closed graph [10, Proposition 2.17]. Thus, as both $\mathcal{F}_{ad}$ and $\mathcal{V}\times\mathcal{D}$ are Hausdorff regular, it suffices to show that the multifunction Π is upper semi-continuous. More precisely, we show that $K \longrightarrow \Pi(K)$ is upper semi continuous (usc) with respect to the given topologies on the domain space $\mathcal{F}_{ad}$ and target space $\mathcal{V} \times \mathcal{D}$. According to the Definition 5.5, we must verify that, for any closed set $\mathcal{C} \subset \mathcal{V} \times \mathcal{D}$, the set

$$\Pi^{-1}(\mathcal{C}) \equiv \{K \in \mathcal{F}_{ad} : \Pi(K) \cap \mathcal{C} \neq \emptyset\}$$

is closed. Let $\{K_\alpha\} \in \Pi^{-1}(\mathcal{C}) \subset \mathcal{F}_{ad}$ be any net (generalized sequence) and note that it follows from the definition of Π that

$$J(K_\alpha, \Pi(K_\alpha)) \geq J(K_\alpha, R, \xi) \;\; \forall \;\; (R, \xi) \in \mathcal{V} \times \mathcal{D}. \tag{16}$$

Hence, for any net $\{R_\alpha, \xi_\alpha\} \in \Pi(K_\alpha) \cap \mathcal{C}$, we have

$$J(K_\alpha, R_\alpha, \xi_\alpha) \geq J(K_\alpha, R, \xi) \;\; \forall \;\; (R, \xi) \in \mathcal{V} \times \mathcal{D}, \alpha \in \triangle. \tag{17}$$

Since $\mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$ is compact in the product topology (because of compactness in their respective topologies), there exists a subnet, relabeled as the original net, $(K_\alpha, R_\alpha, \xi_\alpha)$ and an element $(K_o, R_o, \xi_o) \in \mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$ such that

$$(K_\alpha, R_\alpha, \xi_\alpha) \longrightarrow (K_o, R_o, \xi_o)$$

in the product topology. The limit is unique because the given topologies are Hausdorff. Taking the limit in (17), it follows from Corollary 5.3, asserting joint continuity of J, that

$$J(K_o, R_o, \xi_o) \geq J(K_o, R, \xi) \;\; \forall \;\; (R, \xi) \in \mathcal{V} \times \mathcal{D}. \tag{18}$$

Since this holds for all $(R, \xi) \in \mathcal{V} \times \mathcal{D}$, it follows from this inequality and the definition of the multi function Π that $(R_o, \xi_o) \in \Pi(K_o)$. On the other hand, since $\mathcal{C}$ is closed, the limit of any sequence from it must belong to it and hence $(R_o, \xi_o) \in \mathcal{C}$ and therefore $(R_o, \xi_o) \in \Pi(K_o) \cap \mathcal{C}$. Thus $K_o \in \Pi^{-1}(\mathcal{C})$ proving the closure as required. Hence $K \longrightarrow \Pi(K) \subset 2^{\mathcal{V}\times\mathcal{D}} \setminus \emptyset$ is an upper semi-continuous multi function and therefore, by Proposition 2.17 [10], the graph $\mathcal{G}_r(\Pi)$ is closed. Thus we conclude that $J(K_o, R_o, \xi_o) = J(K_o, \Pi(K_o)) = J_o(K_o)$ proving the continuity, $\lim_{\alpha\in\triangle} J_o(K_\alpha) \longrightarrow J_o(K_o)$, as required. Since $\mathcal{F}_{ad}$ is τ_T compact and J_o is continuous in this topology, it attains its minimum on it. This proves the existence of an optimal operator valued function. ■

Remark 6.2. In view of the above theorem there exists a $K_o \in \mathcal{F}_{ad}$ such that $J_o(K_o) \leq J_o(K)$ for all $K \in \mathcal{F}_{ad}$. Since $J_o(K) \equiv J(K, \Pi(K))$ for any $K \in \mathcal{F}_{ad}$, we have

$$J(K_o, R, \xi) \leq J(K_o, \Pi(K_o)) \leq J(K, \Pi(K)) \ \forall\ K \in \mathcal{F}_{ad}\ \forall\ (R, \xi) \in \mathcal{V} \times \mathcal{D}.$$

Clearly, the right side inequality says that the optimal feedback operator minimizes the maximum risk (maximum potential cost), while the left side inequality tells that the cost in all other situations will never exceed the pessimistic (conservative) cost. This is precisely what is desired in the presence of uncertainty in the system model and measurement (sensor) errors.

7. Extension to stochastic systems

The results presented in the preceding sections can be readily extended to cover stochastic systems governed by evolution equations of the form

$$dx = Axdt + R(t)xdt + F(x)dt + BK(Lx + \xi)dt + \sigma(x)dW, x(0) = x_0 \tag{19}$$

where W is an H-Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in I}, P)$ with $\{\mathcal{F}_t\}_{t \in I}$, denoting the family of completed nondecreasing subsigma algebras of the sigma algebra $\mathcal{F}$, and $\sigma : X \longrightarrow \mathcal{L}(H, X)$. The objective functional is now given by the expected value of the cost functional

$$J(K, R, \xi) \equiv \mathbf{E}\left\{\int_0^T \ell(t, x(t))dt + \Phi(x(T))\right\}, \tag{20}$$

where x is the mild solution of the evolution equation (19) corresponding to the triple $(K, R, \xi) \in \mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$. The problem is to find a feedback law $K \in \mathcal{F}_{ad}$ that minimizes the maximum possible cost in the presence of dynamic uncertainty represented by $R \in \mathcal{V}$ and the noisy output induced by the presence of $\xi \in \mathcal{D}$ in the measurement. Recall that we do not assume any probabilistic structure on $\mathcal{V} \times \mathcal{D}$. Again the problem is to find a $K_o \in \mathcal{F}_{ad}$ so that

$$J_o(K_o) \leq J_o(K),\ \forall\ K \in \mathcal{F}_{ad}, \tag{21}$$

where $J_o(K) \equiv \sup\{J(K, R, \xi) : (R, \xi) \in \mathcal{V} \times \mathcal{D}\}$. Thus, fundamentally, the stochastic problem is very similar to the deterministic one. Clearly, the question of existence of an optimal feedback law is intimately connected with the question of continuity of the functional (20) on the product space $\mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$. Since we are now dealing with stochastic systems we need some fundamental changes in the setting. We assume that all the spaces X, Y, U (state space, output space,control space) are Hilbert spaces. Further, we assume that H is a separable Hilbert space and that $W \equiv \{W(t), t \in I\}$, is the H-Brownian motion having the incremental covariance Q. That is, for every $h \in H$, $\{(W(t), h), t \in I\}$ is a real valued Gaussian random process with mean zero and variance $t(Qh, h)$. In case Q is the identity operator, W is called the cylindrical Brownian motion (Wiener process) and if Q is positive nuclear, it follows from the well known Minlos-Sazanov theorem that, for each $t \in I$, the probability measure induced by $W(t)$ is supported on H. Now we introduce the basic assumptions:

(B1): The operator A, with domain and range in X, is the infinitesimal generator of a compact C_0-semigroup $S(t), t > 0$, on the Hilbert space X with $S(0) = I_d$. The operator valued function $B \in L_2(I, \mathcal{L}(U, X))$.

(B2): There exists a constant $C_1 > 0$ such that the drift F satisfies the following growth and Lipschitz conditions

$$|F(x)|_X^2 \le C_1^2(1 + |x|_X^2), \quad |F(x_1) - F(x_2)|_X^2 \le C_1^2|x_1 - x_2|_X^2 \ \forall\ x, x_1, x_2 \in X.$$

(B3): There exists a constant $C_Q > 0$ such that the diffusion σ satisfies the following growth and Lipschitz conditions:

$$Tr(\sigma(x)Q\sigma^*(x)) \le C_Q^2(1 + |x|_X^2),\ \forall\ x \in X,$$
$$Tr(\sigma(x_1) - \sigma(x_2))Q(\sigma(x_1) - \sigma(x_2))^* \le C_Q^2(|x_1 - x_2|_X^2)\ \forall\ x_1, x_2 \in X.$$

(B4): The properties **(H1)–(H3)** for the admissible feedback operators $\mathcal{F}_{ad}$ and the uncertainty sets $\{\mathcal{V}, \mathcal{D}\}$ remain the same with the Banach spaces $\{X, Y, U\}$ now having Hilbertian structure.

For any real Hilbert space Z, let $B_{\infty,2}^a(I, Z)$ denote the space of Z valued $\mathcal{F}_t$-adapted random processes $\{x(t), t \in I\}$ having finite second moments. This is endowed with the norm topology $\| x \|_{B_{\infty,2}^a(I,Z)} \equiv \left(\sup\{\mathbf{E}|x(t)|_Z^2, t \in I\}\right)^{1/2}$. Being a closed subspace of the Banach space $L_\infty^a(I, L_2(\Omega, Z))$, it is a Banach space. Note that, for each $t \in I$, $x(t) \in L_2^{\mathcal{F}_t}(\Omega, Z)$, that is, $x(t)$ is $\mathcal{F}_t$ measurable and $\mathbf{E}|x(t)|_Z^2 < \infty$. For convenience, we denote the space $L_2^{\mathcal{F}_0}(\Omega, Z)$ by simply $L_2^0(\Omega, Z)$.

Now we are prepared to prove the existence and regularity of mild solutions of the stochastic system (19).

Theorem 7.1. *Under the assumptions* (B1)–(B4), *for every initial state* $x_0 \in L_2^0(\Omega, X)$ *and* $(K, R, \xi) \in \mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$, *the system* (19) *has a unique* $\mathcal{F}_t$*-adapted mild solution* $x \in B_{\infty,2}^a(I, X)$ *having continuous modification. Further, the solution set* $\mathcal{S} \equiv \{x(K, R, \xi) : (K, R, \xi) \in \mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}\}$ *is a bounded subset of* $B_{\infty,2}^a(I, X)$.

Proof. The proof is very similar to that of Lemma 4.1. We present a brief outline. For any $x_0 \in L_2^0(\Omega, X)$ and $(K, R, \xi) \in \mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$, define the operator $\mathcal{G}$ on $B^a_{\infty,2}(I, X)$ by

$$\begin{aligned}(\mathcal{G}x)(t) \equiv S(t)x_0 &+ \int_0^t S(t-r)R(r)x(r)dr + \int_0^t S(t-r)F(x(r))dr \\ &+ \int_0^t S(t-r)(BKL)(r)x(r)dr + \int_0^t S(t-r)(BK)(r)\xi(r)dr \\ &+ \int_0^t S(t-r)\sigma(x(r))dW(r), t \in I.\end{aligned} \tag{22}$$

Clearly, the mild solution of the stochastic evolution equation (19) is given by the fixed point (if one exists) of the operator $\mathcal{G}$. In view of (B4), the uncertainty sets are bounded. So there exists $\gamma > 0$ such that $\sup\{\parallel \mathbf{\Xi} \parallel_{\mathcal{L}(Y,U)}, \mathbf{\Xi} \in \Gamma\} \leq \gamma$, and since the elements of $\mathcal{V}$ and $\mathcal{D}$ are functions with values in the unit balls $B_1(\mathcal{L}(X))$ and $B_1(Y)$ respectively, their norms are bounded above by 1. By assumption (A3), $L \in B_\infty(I, \mathcal{L}(X, Y))$ and so there exists a number $\ell > 0$ such that $\sup\{\parallel L(t) \parallel_{\mathcal{L}(X,Y)}, t \in I\} \leq \ell$. Using the growth conditions given in (B2) and (B3) and carrying out some laborious but straight forward computations one can easily verify that there exist constants $C_3, C_4 \geq 0$ such that

$$\begin{aligned}\parallel \mathcal{G}x \parallel^2_{B^a_{\infty,2}(I,X)} &\equiv \sup\{\mathbf{E}|(\mathcal{G}x)(t)|^2_X, t \in I\} \leq C_3 + C_4 \sup\{\mathbf{E}|x(t)|^2_X, t \in I\} \\ &\equiv C_3 + C_4 \parallel x \parallel^2_{B^a_{\infty,2}(I,X)}\end{aligned} \tag{23}$$

for every $x \in B^a_{\infty,2}(I, X)$ with $x(0) = x_0$, where the constants C_3, C_4 are given by

$$\begin{aligned}C_3 &= 2^6 M^2 \{\mathbf{E}|x_0|^2_X + (C_1 T)^2 + \gamma^2 T \parallel h \parallel^2_{L_2(I)} + C_Q^2 T\} \\ C_4 &\equiv 2^6 M^2 \{T^2 + (C_1 T)^2 + (\gamma\ell)^2 T \parallel h \parallel^2_{L_2(I)} + C_Q^2 T\}.\end{aligned} \tag{24}$$

Clearly, it follows from the inequality (23) that the operator $\mathcal{G}$ maps $B^a_{\infty,2}(I, X)$ into $B^a_{\infty,2}(I, X)$. We show that $\mathcal{G}$ has a fixed point in $B^a_{\infty,2}(I, X)$. For any $\tau \in [0, T]$, define the closed interval $I_\tau \equiv [0, \tau]$. Carrying out similar computations, one can verify that, for every $x, y \in B^a_{\infty,2}(I_\tau, X)$ satisfying $x(0) = y(0) = x_0$, and every $\tau \in [0, T]$,

$$\parallel \mathcal{G}x - \mathcal{G}y \parallel^2_{B^a_{\infty,2}(I_\tau,X)} \leq C(\tau) \parallel x - y \parallel^2_{B^a_{\infty,2}(I_\tau,X)} \tag{25}$$

where the constant $C(\tau)$ is given by

$$C(\tau) \equiv 2^3 M^2 \left(\tau^2 + (C_1\tau)^2 + (\gamma\ell)^2 \tau \int_0^\tau |h(r)|^2 dr + C_Q^2 \tau\right).$$

It is clear that $\tau \longrightarrow C(\tau)$ is a continuous and monotone increasing function of its argument starting from $C(0) = 0$. Thus, choosing τ_1 sufficiently small so that $C(\tau_1) < 1$, we find that $\mathcal{G}$ is a contraction on the Banach space $B^a_{\infty,2}(I_{\tau_1}, X)$ and hence by Banach fixed point theorem, the operator $\mathcal{G}$ has a unique fixed point $x_1 \in B^a_{\infty,2}(I_{\tau_1}, X) \equiv B^a_{\infty,2}(I_1, X)$. It follows from Da Prato-Kwapien-Zabczyk factorization technique [12, p128] that x_1 has a continuous modification which we denote by x_1 again. Next, consider the interval $I_2 \equiv [\tau_1, \tau_2]$ for $\tau_2 \in (\tau_1, T]$ and $x, y \in B^a_{\infty,2}(I_2, X)$ with $x(\tau_1) = y(\tau_1) = x_1(\tau_1)$. Carrying out similar computations, one can verify that

$$\text{(26)} \qquad \| \mathcal{G}x - \mathcal{G}y \|^2_{B^a_{\infty,2}(I_2,X)} \leq C(\tau_2 - \tau_1) \| x - y \|^2_{B^a_{\infty,2}(I_2,X)} .$$

Thus we can choose $\tau_2 \in (\tau_1, T]$ so that $C(\tau_2 - \tau_1) < 1$. This implies that the operator $\mathcal{G}$, restricted to the B-space $B^a_{\infty,2}(I_2, X)$, is a contraction and hence has a unique fixed point say $x_2 \in B^a_{\infty,2}(I_2, X)$ having continuous modification. Repeating this procedure, one can cover the compact interval I in a finite number of steps and generate a finite sequence $\{x_1, x_2, \cdots x_n\}$. By concatenation of this sequence, we can then construct a unique mild solution $x \in B^a_{\infty,2}(I, X)$ for the SDE (19). The reader can easily verify the last statement on boundedness of the solution set $\mathcal{S}$ by use of Gronwall inequality and the facts that the set $\mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$ is bounded, F has linear growth and $B \in L_2(I, \mathcal{L}(U, X))$. This completes the outline of our proof. ■

In the following theorem we prove continuity of the solution with respect to the triple $\{K, R, \xi\}$. This is used later to prove the existence of an optimal feedback operator.

Theorem 7.2. *Under the assumptions of Theorem* 7.1, *the solution map*

$$(K, R, \xi) \longrightarrow x(K, R, \xi)$$

from $\mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$ *to* $B^a_{\infty,2}(I, X)$ *is continuous with respect to the product topology* $\tau_T \times \tau_{Two} \times \tau_{Tw}$ *on* $\mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$, *and the norm topology on* $B^a_{\infty,2}(I, X)$.

Proof. The proof is similar to that of Theorem 5.2. We present a brief outline. Take any generalized sequence (net) $(K^n, R^n, \xi^n) \in \mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$ and suppose that it converges to (K^o, R^o, ξ^o) with respect to the product topology $\tau_T \times \tau_{Two} \times \tau_{Tw}$. Let x_n denote the mild solution of the stochastic evolution equation (19) corresponding to the sequence $(K^n, R^n, \xi^n) \in \mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$ and x_o the mild solution corresponding to (K^o, R^o, ξ^o). We show that $x_n \longrightarrow x_o$ in the norm topology of

$B^a_{\infty,2}(I,X)$. Using the corresponding integral equations:

$$\begin{aligned} x_n(t) = S(t)x_0 &+ \int_0^t S(t-r)F(x_n(r))dr + \int_0^t S(t-r)R^n(r)x_n(r)dr \\ &+ \int_0^t S(t-r)B(r)K^n(r)L(r)x_n(r)dr \\ &+ \int_0^t S(t-r)\sigma(x_n(r))dW(r),\ t \in I, \end{aligned}$$

and

$$\begin{aligned} x_o(t) = S(t)x_0 &+ \int_0^t S(t-r)F(x_o(r))dr + \int_0^t S(t-r)R^o(r)x_o(r)dr \\ &+ \int_0^t S(t-r)B(r)K^o(r)L(r)x_o(r)dr \\ &+ \int_0^t S(t-r)\sigma(x_o(r))dW(r),\ t \in I, \end{aligned}$$

and subtracting one from the other we obtain a similar identity as in the deterministic case with an additional term for the stochastic integral as follows:

$$\begin{aligned} x_o(t) - x_n(t) = &\int_0^t S(t-r)[F(x_o(r)) - F(x_n(r))]dr \\ &+ \int_0^t S(t-r)R^n(r)(x_o(r) - x_n(r))dr \\ &+ \int_0^t S(t-r)[BK^nL(x_o - x_n)](r)dr \\ &+ \int_0^t S(t-r)[\sigma(x_o(r)) - \sigma(x_n(r))]dW(r) \\ &+ \int_0^t S(t-r)(R^o(r) - R^n(r))x_o(r)dr \\ &+ \int_0^t S(t-r)(B(K^o - K^n)(Lx_o + \xi^o))(r)dr \\ &+ \int_0^t S(t-r)(BK^n(\xi^o - \xi^n))(r)dr,\ t \in I. \end{aligned} \tag{27}$$

Using the above expression and the assumptions (B1)–(B4), and taking the expected value of the X-norm square, and carrying out similar computations, we obtain the following expression

$$\mathbf{E}|x_o(t) - x_n(t)|_X^2 \le \int_0^t \tilde{h}(r)\ \mathbf{E}|x_o(r) - x_n(r)|_X^2 dr + \ \mathbf{E}|\eta_n(t)|_X^2,\ t \in I, \tag{28}$$

where the functions $\tilde{h}$ and η_n are given by

$$\tilde{h}(t) = 2^4 M^2 \{C_1^2 t + t + (\gamma\ell)^2 \parallel B(t) \parallel^2_{\mathcal{L}(U,X)} + C_Q^2\}$$

and

$$\eta_n(t) \equiv 2^2\{|e_{1,n}(t)|_X + |e_{2,n}(t)|_X + |e_{3,n}(t)|_X\} \tag{29}$$

respectively. Since the interval I is finite and, by assumption (B1), $B \in L_2(I, \mathcal{L}(U,X))$, it is clear that $\tilde{h} \in L_1^+(I)$. Note that the functions $\{e_{i,n}, i = 1,2,3\}$ are given by the same expressions as displayed immediately following the equation (5). In this case, however, the first two are stochastic. Now it follows from Gronwall inequality applied to (28) that

$$\mathbf{E}|x_o(t) - x_n(t)|_X^2 \leq \mathbf{E}|\eta_n(t)|_X^2 + \exp(\parallel \tilde{h} \parallel_{L_1}) \int_0^t \tilde{h}(r)\ \mathbf{E}|\eta_n(r)|^2 dr,\ t \in I. \tag{30}$$

We show that $\mathbf{E}|\eta_n(t)|^2 \longrightarrow 0$ for each $t \in I$ and even uniformly on I. Consider the first term $e_{1,n}$ given by

$$e_{1,n}(t) \equiv \int_0^t S(t-r)(R^o(r) - R^n(r))x_o(r)dr,\ t \in I.$$

Since $R^n \stackrel{\tau_{Tw\varrho}}{\longrightarrow} R^o$ in $\mathcal{V}$ and $x_o(t)$ is contained in a bounded subset of $L_2^{\mathcal{F}_t}(\Omega, X)$, it is clear that $(R^o(t) - R^n(t))x_o(t) \longrightarrow 0$ weakly in X for each $t \in I$, P-a.s. On the other hand, by compactness of the semigroup, the integrand converges to zero strongly in X for each $t \in I$, P-a.s. Further, since $x_o \in B^a_{\infty,2}(I,X)$, the integrand is dominated by the square integrable process $\{2M|x_o(t)|_X, t \in I\}$. Thus it follows from Lebesgue dominated convergence theorem that the integral $\mathbf{E}|e_{1,n}(t)|_X^2 \longrightarrow 0$ uniformly on I. Consider now the second term $e_{2,n}$ given by

$$e_{2,n}(t) \equiv \int_0^t S(t-r)(B(K^o - K^n)(Lx_o + \xi^o))(r)dr,\ t \in I.$$

Since $\xi^o \in \mathcal{D}$ and $sup\{\parallel L(t) \parallel_{\mathcal{L}(X,Y)}, t \in I\} \leq \ell$, the process y_o, given by $y_o(t) \equiv L(t)x_o(t) + \xi^o(t), t \in I$, is in $B^a_{\infty,2}(I,Y)$ and

$$\mathbf{E}|y_o(t)|_X^2 \leq 2(\ell^2 E|x_o(t)|_X^2 + 1),\ \forall\ t \in I.$$

By virtue of compactness of the set $\mathcal{F}_{ad}$ (see **(H1)**) in the Tychonoff product topology τ_T, $K^n(t) \longrightarrow K^o(t)$ in $\Gamma \subset \mathcal{K}(Y,U)$ in the uniform operator topology for each $t \in I$. Further, since $B \in L_2(I, \mathcal{L}(U,X))$ and $y_o \in B^a_{\infty,2}(I,Y)$, the integrand is dominated by a square integrable process ζ given by

$$\zeta(r) \equiv 2M\gamma \parallel B(r) \parallel_{\mathcal{L}(U,X)} |y_o(r)|_Y,\ r \in I.$$

Thus, again by dominated convergence theorem, we conclude that $\mathbf{E}|e_{2,n}(t)|_X^2 \to 0$ uniformly on I. Consider next the third term $e_{3,n}$ given by

$$e_{3,n}(t) \equiv \int_0^t S(t-r)((BK^n)(\xi^o - \xi^n))(r)dr, \ t \in I.$$

Since the set $\mathcal{D}$ is compact in the Tychonoff product topology, $\xi^n(t) \xrightarrow{\tau_w} \xi^o(t)$ in Y for each $t \in I$. By Mayoral's result (see Proposition 5.1) the set Γ is uniformly completely continuous (equicompact). Thus $K^n(t)(\xi^o(t) - \xi^n(t)) \xrightarrow{s} 0$ in U for each $t \in I$. Hence the integrand converges to zero for a.e $t \in I$. Further, the integrand is dominated by the square integrable function $\vartheta(r) \equiv 2M\gamma \parallel B(r) \parallel_{\mathcal{L}(U,X)}, r \in I$. Thus by dominated convergence theorem we conclude that $\mathbf{E}|e_{3,n}(t)|_X^2 = |e_{3,n}(t)|_X^2 \to 0$ uniformly on I. Put together the above results, we conclude that $\mathbf{E}|\eta_n(t)|^2 \to 0$ uniformly on I. Finally, taking the limit on each side of the inequality (30), we conclude that $x_n \longrightarrow x_o$ in the norm topology of $B^a_{\infty,2}(I, X)$. This proves the continuity as stated and completes the outline of our proof. ■

Corollary 7.3. *Suppose the assumptions of Theorem* 7.2 *hold and the functions* ℓ *and* Φ *satisfy the following assumptions:*

(B5): *For each* $x \in X$, $\ell(t, x)$ *is measurable in* t *on* I, *and, for almost all* $t \in I$, *it is continuous in* x *on* X *and there exist* $c_1, c_2 \geq 0$ *such that* $|\ell(t, x)| \leq c_1 + c_2|x|_X^2$.

(B6): $\Phi(x)$ *is continuous in* $x \in X$ *and there exist constants* $c_3, c_4 \geq 0$ *such that* $|\Phi(x)| \leq c_3 + c_4|x|_X^2$.

Then, the functional $(K, R, \xi) \longrightarrow J(K, R, \xi)$ *is jointly continuous on* $\mathcal{F}_{ad} \times \mathcal{V} \times \mathcal{D}$ *with respect to the topology* $\tau_T \times \tau_{Two} \times \tau_w$.

Proof. The proof is similar to that of Corollary 5.3. In the proof, we use Theorem 7.2 in place of Theorem 5.2. ■

Using the above result and Theorem 6.1 we arrive at the following result asserting the existence of optimal output feedback operator for the stochastic system.

Theorem 7.4. *Suppose the assumptions of Corollary* 7.3 *hold. Then there exists an optimal output feedback law* $K_o \in \mathcal{F}_{ad}$ *for the stochastic system* (19) *with the cost functional* (20) (*in place of* (4)) *solving the problem* (21).

Proof. The proof is based on Corollary 7.3 (in place of Corollary 5.3) and it is identical to that of Theorem 6.1 with the cost functional J_o as defined by the expressions (20)–(21).

7.1. Some open problems

For computation of the optimal policy (operator valued function), one needs to develop necessary and, possibly, sufficient conditions for optimality as in [7]. This involves the characterization of topological duals of the spaces $\mathcal{L}(X)$ and $\mathcal{L}(Y,U)$. Also, one needs the associated duality maps and their regularity properties. These are interesting challenging problems of topology and analysis. We hope to consider this problem in future publications.

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