

CHARACTERIZING WHICH POWERS
OF HYPERCUBES AND FOLDED HYPERCUBES
ARE DIVISOR GRAPHS

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Abstract

In this paper, we show that Q_n^k is a divisor graph, for $n = 2, 3$. For $n \geq 4$, we show that Q_n^k is a divisor graph iff $k \geq n - 1$. For folded-hypercube, we get FQ_n is a divisor graph when n is odd. But, if $n \geq 4$ is even integer, then FQ_n is not a divisor graph. For $n \geq 5$, we show that $(FQ_n)^k$ is not a divisor graph, where $2 \leq k \leq \lceil \frac{n}{2} \rceil - 1$.

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1. INTRODUCTION

A graph G is called a divisor graph if there is a bijection $f : V(G) \rightarrow S$, for some finite nonempty set S of the positive integers, such that $uv \in E(G)$ if and only if $\gcd(f(u), f(v)) = \min\{f(u), f(v)\}$ (this means $uv \in E(G)$ iff $f(u) \mid f(v)$ or $f(v) \mid f(u)$). The function f is called a divisor labeling of G .

Moreover, for a finite nonempty set S of the positive integers, the divisor graph $G(S)$ of S has S as its vertex set and two distinct vertices i and j are adjacent if $i \mid j$ or $j \mid i$. A graph G is a divisor graph if G is isomorphic to $G(S)$. While the divisor digraph $D(S)$ of S has a vertex set S and (i, j) is an arc of $D(S)$ iff i divides j . In a digraph D , a transmitter is a vertex having indegree 0, a receiver is a vertex having outdegree 0, while a vertex v is a transitive vertex if it has both positive outdegree and positive indegree, and $(u, w) \in E(D)$ whenever $(u, v) \in E(D)$ and $(v, w) \in E(D)$. An orientation D of a graph G in which every vertex is a transmitter, a receiver, or a transitive vertex is called a divisor orientation of G .

The length $g(n)$ of a longest path in the divisor graph whose divisor labeling has range $\{1, 2, \dots, n\}$ was studied in [8], [12], and [13]. The concept of a divisor graph involving finite nonempty sets of integers rather than positive integers was introduced in [14]. It was shown in [14] that odd cycles of length greater than three are not divisor graphs, while even cycles and caterpillars are. Indeed, not only caterpillars, but also all bipartite graphs are divisor graphs, as shown in [7]. Divisor graphs do not contain induced odd cycles of length greater than three, but they may contain triangles, see [7]. For instance, complete graphs are divisor graphs, see [7].

The distance between any two vertices x and y is the length of a shortest path between them. We denote this distance by $d_G(x, y)$. The diameter of a graph G is equal to $\sup\{d_G(x, y) : x, y \in V(G)\}$, denoted by d_G or $\text{diam}(G)$. The power graph of G is denoted by G^k , where the vertex set of G^k is $V(G)$ and two vertices x, y are adjacent iff $d_G(x, y) \leq k$. A complete characterization of powers of paths, cycles and caterpillars that are divisor graphs were given in [1], [2], and [3].

More results on divisor graphs can be found in [4], [5], and [7]. For undefined notions and terminology, the reader is referred to [6].

2. WHEN A POWER GRAPH OF A HYPERCUBE IS A DIVISOR GRAPH

In this section we characterize which powers of hypercubes are divisor graphs.

Definition. A *hypercube* Q_n is a graph whose vertex set $V(Q_n)$ is the set of n -bit binary strings. (Any vertex of Q_n may be labeled as $x = x_1x_2 \cdots x_n$, where $x_1x_2 \cdots x_n$ is the binary representation of x .) Two vertices $x = x_1x_2 \cdots x_n$ and $y = y_1y_2 \cdots y_n$ are adjacent if they differ in exactly one bit.

Alternatively, a hypercube can be defined recursively by $Q_n = Q_{n-1} \times K_2$, for $n \geq 2$ and $Q_1 = K_2$. In Figure 1, we have the graphs Q_1 , Q_2 , and Q_3 .

It is well known that the diameter of Q_n is n , hence Q_n^n is a complete graph. Moreover, two vertices are adjacent in Q_n^k , if they differ in k or less bits of their

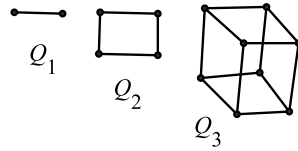


Figure 1. Q_1 , Q_2 , and Q_3 .

binary strings.

The following important characterization of divisor graphs can be found in [7].

Theorem 1. *Let G be a graph. Then G is a divisor graph if and only if G has a divisor orientation. In particular, every bipartite graph is a divisor graph.*

The following easy observation can be found in [7].

Proposition 2. *Every induced subgraph of a divisor graph is a divisor graph.*

We start with the following result for characterizing which powers of hypercubes are divisor graphs.

Theorem 3. *Suppose that Q_n is a hypercube. Then Q_n^{n-1} is a divisor graph.*

Proof. At first, two vertices in Q_n^{n-1} are adjacent if they differ in $n - 1$ or less bits of their binary strings. Then according to the labeling of vertices in Q_n , each vertex of Q_n^{n-1} is adjacent to every other vertex of Q_n^{n-1} except its complement in labeling. Hence, we can partition the vertices in Q_n into $\{ a_i : 1 \leq i \leq 2^{n-1}, i \in \mathbb{N} \} \cup \{ b_i : 1 \leq i \leq 2^{n-1}, i \in \mathbb{N} \}$ such that, for each i , a_i is the complement of b_i . Now, a divisor labeling f of Q_n^{n-1} is defined as follows:

$$\begin{cases} f(a_i) = p^i q^{i-1} & \text{for } 1 \leq i \leq 2^{n-1}, i \in \mathbb{N}, \\ f(b_i) = p^{i-1} q^i & \text{for } 1 \leq i \leq 2^{n-1}, i \in \mathbb{N}, \end{cases}$$

where p and q are distinct primes. Hence, Q_n^{n-1} is a divisor graph. ■

Remark 4. This is an example of a graph for which both it and its complement graph (Q_n^{n-1} being bipartite) are divisor graphs.

Note that Q_n is a bipartite graph and hence a divisor graph. So, when $n = 2$ or 3 , Q_n^k is a divisor graph for any positive integer k . Now, we look at $n \geq 4$.

Lemma 5. Q_4^2 is not a divisor graph.

Proof. The induced subgraph on $\{0001, 0010, 0110, 1111, 1101\}$ in Q_4^2 is isomorphic to C_5 . Hence, Q_4^2 is not a divisor graph by Proposition 2. ■

We use Theorem 3 and Lemma 5 to get the following corollary.

Corollary 6. Q_4^k is not a divisor graph if and only if $k = 2$.

Lemma 7. Q_5^3 is not a divisor graph.

Proof. The induced subgraph on $\{00001, 10000, 10110, 11111, 01011\}$ in Q_5^3 is isomorphic to C_5 . Hence, Q_5^3 is not a divisor graph by Proposition 2. ■

By Theorem 3, Lemma 5, Proposition 2, and Lemma 7, we get the following corollary.

Corollary 8. Q_5^k is not a divisor graph if and only if $k = 2$ or 3.

For any positive integer n greater than three, we have the following lemma.

Lemma 9. For $n \geq 4$, $Q_n^{(n-2)}$ is not a divisor graph.

Proof. The result follows from Lemmas 5 and 7 when $n = 4$ and 5. For $n \geq 6$, we look at the induced subgraph, say G_1 , on $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ in $Q_n^{(n-2)}$, where the $a_{i'}$ s are:

$$\begin{aligned}
 a_1 &= \underbrace{0 \dots \dots \dots 00}_n, \\
 a_2 &= \underbrace{0 \dots \dots \dots 00}_{n-2} 11, \\
 a_3 &= \underbrace{0 \dots \dots \dots 0}_{n-3} 101, \\
 a_4 &= \underbrace{1 \dots \dots \dots 1}_{n-3} 000, \\
 a_5 &= \underbrace{1 \dots \dots \dots 1}_{n-3} 011, \\
 a_6 &= \underbrace{1 \dots \dots \dots 11}_{n-2} 01.
 \end{aligned}$$

The graph G_1 is represented in Figure 2. Assume to the contrary that G_1 is a divisor graph. Hence, it has a divisor orientation, say D . Suppose that $(a_1, a_2) \in E(D)$. Since $a_1a_5 \notin E(G_1)$, we must have $(a_5, a_2) \in E(D)$. We get $(a_3, a_2) \in E(D)$, because $a_3a_5 \notin E(G_1)$. We must have $(a_1, a_4), (a_3, a_6) \in E(D)$, because $a_4a_2, a_6a_2 \notin E(G_1)$.

Now, assume that $(a_4, a_6) \in E(D)$, and since $(a_1, a_4) \in E(D)$, we get $(a_1, a_6) \in E(D)$, which is a contradiction, since $a_1a_6 \notin E(G_1)$. Otherwise, if $(a_6, a_4) \in E(D)$, and since $(a_3, a_6) \in E(D)$, then we get $(a_3, a_4) \in E(D)$. This leads to a contradiction, since $a_4a_3 \notin E(G_1)$. Similarly, we argue the case $(a_2, a_1) \in E(D)$. ■

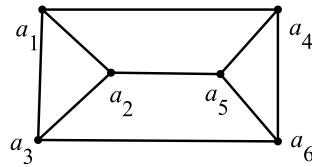


Figure 2. G_1 .

Theorem 10. *If $n \geq 5$ and $2 \leq k \leq n - 3$, then Q_n^k is not a divisor graph.*

Proof. For any positive integer k with $2 \leq k \leq n - 3$, we have the following nested induced subgraphs: $Q_n^k \supseteq \dots \supseteq Q_{k+3}^k \supseteq Q_{k+2}^k$. But, Q_{k+2}^k is not a divisor graph, by previous lemma. Hence, Q_n^k is not a divisor graph, by Proposition 2. ■

Using the previous theorem, Lemma 9, and Theorem 3, we get the following corollary.

Corollary 11. *Suppose $n \geq 4$ and $k \geq 2$. Then Q_n^k is a divisor graph if and only if $k \geq n - 1$.*

3. CHARACTERIZING WHEN POWERS OF FOLDED-HYPERCUBES ARE DIVISOR GRAPHS

In this section we characterize when powers of folded-hypercubes are divisor graphs. First we give the definition of the folded-hypercube.

Definition. A *folded-hypercube*, denoted by FQ_n is a graph whose vertex set is $V(Q_n)$ and two vertices x and y are adjacent if $d_{Q_n}(x, y) = 1$ or n , i.e., x and y differ in exactly one or n bits.

A folded-hypercube is a standard hypercube with some extra edges between the vertices with complementary binary strings. In addition, if n is odd then FQ_n is bipartite, see [9]. Hence, FQ_n is a divisor graph if n is odd or $n = 2$. While for $n \geq 4$ is even, we get the following lemma.

Lemma 12. *If n is an even integer with $n \geq 4$, then FQ_n is not a divisor graph.*

Proof. Consider the set $A = \{a_1, a_2, \dots, a_{n+1}\}$ in $V(FQ_n)$, where

$$a_1 = 0 \underbrace{1 \dots \dots \dots 1}_{n-1},$$

$$a_2 = 00 \underbrace{1 \dots \dots \dots 11}_{n-2},$$

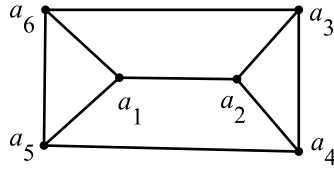


Figure 3. The induced subgraph on S .

Proof. Consider the induced subgraph on the set $S = \{a_1, a_2, a_3, a_4, a_5\}$ in $(FQ_n)^3$, where the a_i 's are:

$$\begin{aligned}
 a_1 &= \underbrace{0 \dots \dots \dots 0}_{n-7} 1000001, \\
 a_2 &= \underbrace{0 \dots \dots \dots 0}_{n-7} 0100001, \\
 a_3 &= \underbrace{0 \dots \dots \dots 0}_{n-7} 0100010, \\
 a_4 &= \underbrace{0 \dots \dots \dots 0}_{n-7} 0010010, \\
 a_5 &= \underbrace{0 \dots \dots \dots 0}_{n-7} 1010000.
 \end{aligned}$$

For $n \geq 7$, the induced subgraph on the set S in $(FQ_n)^3$ is isomorphic to C_5 . The result follows by Proposition 2. ■

For $k = 4, 5$ with $n \geq 2k + 1$, we get that $(FQ_n)^k$ is not a divisor graph. The proof is similar to that of Lemma 13. For $k \geq 6$, we proceed as follows.

Lemma 15. *Suppose $i = 3$ or 4 , $k = 2i$, $n_i = 4i + 1$, and $n \geq n_i$. Then $(FQ_n)^k$ is not a divisor graph.*

Proof. Consider the induced subgraph on the set $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ in $(FQ_n)^k$, where the a_i 's are:

$$\begin{aligned}
 a_1 &= \underbrace{0 \dots \dots 0}_{n-n_i} \underbrace{10 \dots 00}_i \underbrace{10 \dots 00}_i \underbrace{1 \dots 11}_i \underbrace{0 \dots 00}_{i-1}, \\
 a_2 &= \underbrace{0 \dots \dots 0}_{n-n_i} \underbrace{1 \dots \dots 10}_{i+2} \underbrace{\dots \dots 01}_i \underbrace{\dots \dots 11}_{2i-1}, \\
 a_3 &= \underbrace{0 \dots \dots 00}_{n-n_i} \underbrace{\dots 01}_{i-1} \underbrace{\dots 100}_{i+1} \underbrace{110 \dots 0}_{i-2} \underbrace{110 \dots 0}_{i-3}, \\
 a_4 &= \underbrace{0 \dots \dots 01}_{n-n_i} \underbrace{00 \dots 0}_{i-2} \underbrace{11100 \dots 0}_{i-2} \underbrace{1100 \dots 10}_{2i-4}, \text{ for } i = 3,
 \end{aligned}$$

$$\begin{aligned}
 a_4 &= \underbrace{0\dots\dots 0}_{n-n_i} \underbrace{100\dots 0}_{i-2} \underbrace{11100\dots 0}_{i-2} \underbrace{11001\dots 11}_{2i-4}, \text{ for } i = 4, \\
 a_5 &= \underbrace{0\dots\dots 0}_{n-n_i} \underbrace{00\dots 0}_{i-1} \underbrace{1110011\dots 1}_{i-2} \underbrace{001\dots 100}_{2i-5}, \\
 a_6 &= \underbrace{0\dots\dots 0}_{n-n_i} \underbrace{001\dots\dots 1}_{i+2} \underbrace{00\dots\dots 0}_{2i-4} \underbrace{100\dots 01}_{i-1}.
 \end{aligned}$$

The induced subgraph on the set S in $(FQ_{n_i})^k$ is isomorphic to the graph in Figure 3. This induced subgraph is isomorphic to G_1 and the result follows by Lemma 9. For $n \geq 1+n_i$, the induced subgraph on the set $S_1 = \{a_2, a_3, a_4, a_5, a_6\}$ in $(FQ_n)^k$ is isomorphic to C_5 . The result follows by Proposition 2. ■

Lemma 16. *Suppose $i = 3$ or 4 , $k = 2i + 1$, $n_i = 4i + 3$, and $n \geq n_i$. Then $(FQ_n)^k$ is not a divisor graph.*

Proof. Consider the induced subgraph on the set $S = \{a_1, a_2, a_3, a_4, a_5\}$ in $(FQ_n)^k$, where the $a_{i's}$ are:

$$\begin{aligned}
 a_1 &= \underbrace{0\dots\dots 0}_{n-n_i} \underbrace{10\dots 00}_{i} \underbrace{10\dots 00}_{i} \underbrace{1\dots 11}_{i} \underbrace{0\dots 01}_{i-1}, \\
 a_2 &= \underbrace{0\dots\dots 0}_{n-n_i} \underbrace{101\dots 1}_{i} \underbrace{0\dots\dots 0}_{i} \underbrace{1\dots\dots 11}_{2i-1} 00, \\
 a_3 &= \underbrace{0\dots\dots 0}_{n-n_i} \underbrace{00\dots 0}_{i-1} \underbrace{1\dots 1}_{i+1} \underbrace{00110\dots 0}_{i-2} \underbrace{110\dots 01}_{i-2}, \\
 a_4 &= \underbrace{0\dots\dots 0}_{n-n_i} \underbrace{001\dots\dots 1}_{i+2} \underbrace{0\dots\dots 00}_{2i-4} \underbrace{10\dots 0}_{i-1} 111, \\
 a_5 &= \underbrace{0\dots\dots 0}_{n-n_i} \underbrace{00\dots 0}_{i-1} \underbrace{111001\dots 11}_{i-2} \underbrace{001\dots 11}_{2i-5} 0011.
 \end{aligned}$$

For $n \geq n_i$, the induced subgraph on the set S in $(FQ_n)^k$ is isomorphic to C_5 . The result follows by Proposition 2. ■

Theorem 17. *Suppose $i \geq 5$, $k = 2i$, $n_i = 4i + 1$, and $n \geq n_i$. Then $(FQ_n)^k$ is not a divisor graph.*

Proof. Consider the induced subgraph on the set $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ in $(FQ_n)^k$, where the $a_{i's}$ are:

$$\begin{aligned}
 a_1 &= \underbrace{0\dots 0}_{n-n_i} \underbrace{10\dots\dots 0}_{i} \underbrace{10\dots\dots 00}_{i} \underbrace{1\dots\dots 1}_{i} \underbrace{00\dots\dots 00}_{i-1}, \\
 a_2 &= \underbrace{0\dots 0}_{n-n_i} \underbrace{11\dots\dots 1}_{i} \underbrace{10\dots\dots 00}_{i} \underbrace{1\dots\dots 1}_{i} \underbrace{11\dots\dots 11}_{i-1}, \\
 a_3 &= \underbrace{0\dots 0}_{n-n_i} \underbrace{0\dots 0}_{\lceil \frac{i}{2} \rceil + 1} \underbrace{1\dots 1}_{\lceil \frac{i}{2} \rceil} \underbrace{11\dots 1}_{\lfloor \frac{i}{2} \rfloor} \underbrace{0\dots 0}_{\lceil \frac{i}{2} \rceil} \underbrace{1\dots 1}_{\lfloor \frac{i}{2} \rfloor} \underbrace{0\dots 0}_{\lceil \frac{i}{2} \rceil} \underbrace{1\dots 1}_{\lfloor \frac{i}{2} \rfloor} \underbrace{00\dots 00}_{\lceil \frac{i-1}{2} \rceil} \underbrace{1}_{\lfloor \frac{i-1}{2} \rfloor - 1},
 \end{aligned}$$

for i even,

$$a_4 = \underbrace{0 \dots 0}_{n-n_i} \underbrace{0 \dots 0}_{\lfloor \frac{i}{2} \rfloor} \underbrace{1 \dots 1}_{\lfloor \frac{i}{2} \rfloor + 1} \underbrace{0 \dots 0}_{\lfloor \frac{i}{2} \rfloor} \underbrace{1 \dots 1}_{\lfloor \frac{i}{2} \rfloor} \underbrace{0 \dots 0}_{\lfloor \frac{i}{2} \rfloor - 2} \underbrace{1 \dots 1}_{\lfloor \frac{i}{2} \rfloor} \underbrace{0 \dots 0}_{\lfloor \frac{i-1}{2} \rfloor} \underbrace{1 \dots 1}_{\lfloor \frac{i-1}{2} \rfloor + 1} 00 111,$$

$$a_5 = \underbrace{0 \dots 0}_{n-n_i} \underbrace{0 \dots 0}_{\lfloor \frac{i}{2} \rfloor} \underbrace{1 \dots 1}_{\lfloor \frac{i}{2} \rfloor} \underbrace{0 \dots 0}_{\lfloor \frac{i}{2} \rfloor} \underbrace{1 \dots 1}_{\lfloor \frac{i}{2} \rfloor} \underbrace{0 \dots 0}_{\lfloor \frac{i}{2} \rfloor} \underbrace{1 \dots 1}_{\lfloor \frac{i}{2} \rfloor - 1} \underbrace{0 \dots 0}_{\lfloor \frac{i-1}{2} \rfloor} \underbrace{1 \dots 1}_{\lfloor \frac{i-1}{2} \rfloor + 1} 00 11.$$

For $n \geq n_i$, the induced subgraph on the set S in $F(Q_n)^k$ is isomorphic to C_5 . The result follows by Proposition 2. ■

Previous results characterize which powers of folded-hypercubes are divisor graphs. We collect these results in the following theorem.

Theorem 19. $(FQ_n)^k$ is not a divisor graph iff $2 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$, where $n \geq 5$.

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