

$(k - 1)$ -KERNELS IN STRONG k -TRANSITIVE DIGRAPHS

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Abstract

Let $D = (V(D), A(D))$ be a digraph and $k \geq 2$ be an integer. A subset N of $V(D)$ is k -independent if for every pair of vertices $u, v \in N$, we have $d(u, v) \geq k$; it is l -absorbent if for every $u \in V(D) - N$, there exists $v \in N$ such that $d(u, v) \leq l$. A (k, l) -kernel of D is a k -independent and l -absorbent subset of $V(D)$. A k -kernel is a $(k, k - 1)$ -kernel.

A digraph D is k -transitive if for any path $x_0x_1 \cdots x_k$ of length k , x_0 dominates x_k . Hernández-Cruz [*3-transitive digraphs*, Discuss. Math. Graph Theory **32** (2012) 205–219] proved that a 3-transitive digraph has a 2-kernel if and only if it has no terminal strong component isomorphic to a 3-cycle. In this paper, we generalize the result to strong k -transitive digraphs and prove that a strong k -transitive digraph with $k \geq 4$ has a $(k - 1)$ -kernel if and only if it is not isomorphic to a k -cycle.

Keywords: digraph, transitive digraph, k -transitive digraph, k -kernel.

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1. TERMINOLOGY AND INTRODUCTION

We assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1] for terminology not defined here. We only consider finite digraphs without loops or multiple arcs. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. For a vertex x in D , its *out-neighborhood* $N^+(x) = \{y \in V(D) : xy \in A(D)\}$ and its *in-neighborhood* $N^-(x) = \{y \in V(D) : yx \in A(D)\}$. For a pair X, Y of vertex sets of D , define $(X, Y) = \{xy \in A(D) : x \in X, y \in Y\}$.

Let x and y be two vertices of $V(D)$. The *distance* from x to y in D , denoted $d(x, y)$, is the minimum length of an (x, y) -path, if y is reachable from x , and

otherwise $d(x, y) = \infty$. The distance from a set X to a set Y of vertices in D is $d(X, Y) = \max\{d(x, y) : x \in X, y \in Y\}$. The *diameter* of D is $\text{diam}(D) = d(V(D), V(D))$. A digraph D is said to be *strongly connected* or just *strong*, if for every pair x, y of vertices of D , there is an (x, y) -path. Clearly, D has finite diameter if and only if D is strong.

A *cycle* is a finite sequence of distinct vertices $C = x_0x_1 \cdots x_nx_0$ such that $x_{i-1} \rightarrow x_i$ for every $1 \leq i \leq n$ and $x_n \rightarrow x_0$, whose length is $n + 1$. We denote the subpath of C from x_i to x_j by $C[x_i, x_j] = x_ix_{i+1} \cdots x_j$. Let C be a cycle of length $k \geq 2$ and let V_1, V_2, \dots, V_k be pairwise disjoint vertex sets. The *extended k -cycle* $C[V_1, V_2, \dots, V_k]$ is the digraph with vertex set $V_1 \cup V_2 \cup \cdots \cup V_k$ and arc set $\bigcup_{i=1}^k \{v_iv_{i+1} : v_i \in V_i, v_{i+1} \in V_{i+1}\}$, where subscripts are taken modulo k .

A *biorientation* of the graph G is a digraph D obtained from G by replacing each edge $\{x, y\} \in E(G)$ by either the arc xy or the arc yx or the pair of arcs xy and yx . A *complete digraph* is a biorientation of a complete graph obtained by replacing each edge $\{x, y\}$ by the arcs xy and yx . A *complete bipartite digraph* is a biorientation of a complete bipartite graph obtained by replacing each edge $\{x, y\}$ by the arcs xy and yx . A digraph is *k -transitive* if for any path $x_0x_1 \cdots x_k$ of length k , x_0 dominates x_k . A 2-transitive digraph is called a *transitive digraph*. The family of k -transitive digraphs have been studied in [4, 5, 6, 7].

A subset N of $V(D)$ is *k -independent* if for every pair of vertices $u, v \in N$, we have $d(u, v) \geq k$; it is *l -absorbent* if for every $u \in V(D) - N$ there exists $v \in N$ such that $d(u, v) \leq l$. A (k, l) -*kernel* of D is a k -independent and l -absorbent subset of $V(D)$. A *k -kernel* is a $(k, k - 1)$ -kernel. A 2-kernel is called *kernel*. Kernels have been widely studied. A nice survey on the subject is [2]. Chvátal proved in [3] that recognizing digraphs that have a kernel is an NP-complete problem, so finding sufficient conditions for a digraph to have a kernel or finding large families of digraphs with kernels have been a very prosperous line of investigation explored by many authors. In 1980, Kwaśnik [9] introduced the concept of (k, l) -kernels generalizing the notion of kernels. The existence of (k, l) -kernels in some digraphs has been studied. In [8], Galeana-Sánchez et al. showed that every k -transitive digraph has a k -kernel. In [5], Hernández-Cruz proved that a 3-transitive digraph has a 2-kernel if and only if it has no terminal strong component isomorphic to a 3-cycle. In this paper, we generalize the result to strong k -transitive digraphs and prove that a strong k -transitive digraph with $k \geq 4$ has a $(k - 1)$ -kernel if and only if it is not isomorphic to a k -cycle.

2. $(k - 1)$ -KERNELS IN STRONG k -TRANSITIVE DIGRAPHS

We begin with a useful lemma.

Lemma 1 [4]. *Let D be a strong k -transitive digraph with $k \geq 2$. Then $\text{diam}(D) \leq k - 1$.*

When a strong k -transitive digraph D contains a cycle of length at least k , Hernández-Cruz and Montellano-Ballesteros characterized the structure of D as follows.

Theorem 2 [7]. *Given an integer k with $k \geq 2$, let D be a strong k -transitive digraph. Suppose that D contains a cycle of length n such that $(n, k - 1) = d$ and $n \geq k + 1$. Then each of the following holds:*

- (1) *If $d = 1$, then D is a complete digraph.*
- (2) *If $d \geq 2$, then D is either a complete digraph, a complete bipartite digraph or an extended d -cycle.*

Theorem 3 [7]. *Given an integer k with $k \geq 2$, let D be a strong k -transitive digraph of order at least $k + 1$. If D contains a cycle of length k , then D is a complete digraph.*

When a strong k -transitive digraph D contains a cycle of length $k - 1$, the following lemma holds.

Lemma 4. *Let D be a strong k -transitive digraph with $k \geq 4$ and let $C = x_0x_1 \cdots x_{k-2}x_0$ be a cycle of length $k - 1$. Then for every $x \in V(D) \setminus V(C)$, $(x, V(C)) \neq \emptyset$ and $(V(C), x) \neq \emptyset$.*

Proof. Since the converse of a k -transitive digraph is still a k -transitive digraph, we only need to show $(x, V(C)) \neq \emptyset$. Again, since D is strong, there exists a path from x to C . Let $P = y_0y_1 \cdots y_s$ be a shortest path from x to C , where $s \geq 1$, $y_0 = x$ and $y_s \in V(C)$. Without loss of generality, assume that $y_s = x_0$. We prove that y_0 dominates some vertex of $V(C)$ by induction on the length s of P . It clearly holds for $s = 1$. Thus, we assume that $s \geq 2$. Note that $y_1 \cdots y_s$ is a path of length $s - 1$. By the induction hypothesis, we know that there exists a vertex $x_i \in V(C)$ such that $y_1 \rightarrow x_i$. Then $y_0y_1C[x_i, x_{i-1}]$ is a path of length k , which implies that $y_0 \rightarrow x_{i-1}$. ■

The following theorem is our main result.

Theorem 5. *Let D be a strong k -transitive digraph with $k \geq 4$. Then D has a $(k - 1)$ -kernel if and only if it is not isomorphic to a k -cycle.*

Proof. The necessity is obvious. Now we show the sufficiency. Since every strong digraph contains a cycle, we consider the following four cases.

Case 1. D contains a cycle of length at least $k + 1$. By Theorem 2, D is either a complete digraph, a complete bipartite digraph or an extended d -cycle where $d = (n, k - 1) \geq 2$. By the definition of k -kernels, a t -kernel consisting of a single vertex must be a k -kernel where $k \geq t$. Clearly, if D is a complete digraph or a complete bipartite digraph, then every vertex in $V(D)$ is a 3-kernel.

Hence, D has a $(k-1)$ -kernel. Now assume that D is an extended d -cycle, denote $D = C[E_1, \dots, E_d]$, where C is a cycle of length d and every E_i is an independent set. Note that $2 \leq d \leq k-1$. It is easy to check that if $d = k-1$, then every E_i is a $(k-1)$ -kernel of D ; if $2 \leq d < k-1$, then every vertex in $V(D)$ is a $(k-1)$ -kernel.

Case 2. D contains a cycle C of length k . Let $C = x_0x_1 \cdots x_{k-1}x_0$. If $V(D) \setminus V(C) \neq \emptyset$, then Theorem 3 implies that D is a complete digraph. Recall that every complete digraph has a 2-kernel consisting of a single vertex. Hence, D has a $(k-1)$ -kernel. If $V(D) \setminus V(C) = \emptyset$, then since D is not isomorphic to a k -cycle, C contains a chord x_ix_j , where the length of $C[x_i, x_j]$ is more than or equal to 2. For any $x_l \in V(C)$, if $l \neq j+1$, then $C[x_l, x_j]$ is a path of length at most $k-2$; if $l = j+1$, then $C[x_{j+1}, x_i]x_j$ is a path of length at most $k-2$. This shows $d(x_l, x_j) \leq k-2$ and so x_j is a $(k-1)$ -kernel.

Case 3. D contains a cycle C of length $k-1$. Let $C = x_0x_1 \cdots x_{k-2}x_0$. In this case, subscripts are taken modulo $k-1$. If $V(D) \setminus V(C) = \emptyset$, then every vertex of D is a $(k-1)$ -kernel. Now assume that $V(D) \setminus V(C) \neq \emptyset$. By Lemma 4, for any $x \in V(D) \setminus V(C)$, $(x, V(C)) \neq \emptyset$ and $(V(C), x) \neq \emptyset$. Define $S_i = \{y \in V(D) \setminus V(C) : y \rightarrow x_i\}$, for every $i \in \{0, 1, \dots, k-2\}$. Clearly $\bigcup_{i=0}^{k-2} S_i = V(D) \setminus V(C)$.

Suppose that C is an induced cycle. We first claim that if there exist $x \in V(D) \setminus V(C)$ and $x_i, x_j \in V(C)$ such that $x_i \rightarrow x \rightarrow x_j$, where $i = j$ or $C[x_i, x_j]$ is a path of length at least three, then x_j is a $(k-1)$ -kernel. Assume, without loss of generality, that $i = 0$. Then $j = 0$ or $3 \leq j \leq k-2$. It is obvious that for any $y \in V(D) - S_{j+1}$, $d(y, x_j) \leq k-2$. Let $z \in S_{j+1}$ be a vertex different from x . If $j = 0$, then $zC[x_1, x_0]x$ is a path of length k , which implies that $z \rightarrow x$ and so $d(z, x_0) = 2 \leq k-2$ as $k \geq 4$. Hence, x_0 is a $(k-1)$ -kernel. Now assume that $3 \leq j \leq k-2$. Then $zC[x_{j+1}, x_0]xx_j$ is a path of length at most $k-2$. Hence, x_j is a $(k-1)$ -kernel and the claim holds.

For any $x \in S_i$, by Lemma 4, there exists $x_l \in V(C)$ such that $x_l \rightarrow x$. If $l \in \{0, 1, \dots, k-2\} \setminus \{i-1, i-2\}$, then by the above claim, D has a $(k-1)$ -kernel. If $l = i-1$, then $x_{i-1}xC[x_i, x_{i-1}]$ is a cycle of length k . By Case 2, D has a $(k-1)$ -kernel. Hence, we may assume $l = i-2$. Indeed, we may assume that for any vertex $z \in V(D) \setminus V(C)$, there exists $i \in \{0, 1, \dots, k-2\}$ such that $x_{i-1} \rightarrow z \rightarrow x_{i+1}$ and z has no other neighbor on C .

Now we show that D is an extended $(k-1)$ -cycle. Let $E_i = \{x \in V(D) \setminus V(C) : x_{i-1} \rightarrow x \rightarrow x_{i+1}\}$, for every $i \in \{0, 1, \dots, k-2\}$. Clearly, $\bigcup_{i=0}^{k-2} E_i = V(D) \setminus V(C)$. We first prove that every vertex of E_i dominates every vertex of E_{i+1} . For any $x \in E_i$ and $y \in E_{i+1}$, we have $xC[x_{i+1}, x_i]y$ is a path of length k . Hence $x \rightarrow y$. Let $x, x' \in V(D) \setminus V(C)$ such that $x' \rightarrow x$. Then there exist $i, j \in \{0, 1, \dots, k-2\}$ such that $x_{j-1} \rightarrow x \rightarrow x_{j+1}$ and $x_{i-1} \rightarrow x' \rightarrow x_{i+1}$. Then $x'xC[x_{j+1}, x_j]$ is a path of length k , which implies that $x' \rightarrow x_j$. Hence, we have

$j = i + 1$. It follows that $D = C[E_0, E_1, \dots, E_{k-2}]$. It is easy to check that every $E_i, i \in \{0, 1, \dots, k - 2\}$ is a $(k - 1)$ -kernel.

Suppose that C is not an induced cycle. Then there exists a chord x_jx_i in C . Now we will show that x_i is a $(k - 1)$ -kernel. It is obvious that for any $y \in V(D) - S_{i+1}, d(y, x_i) \leq k - 2$. Let $z \in S_{i+1}$. Note that $zC[x_{i+1}, x_j]x_i$ is a path of length at most $k - 2$. Hence, $d(z, x_i) \leq k - 2$ and x_i is a $(k - 1)$ -kernel.

Case 4. There exists no cycle of length more than or equal to $k - 1$ in D . Let x be a vertex of maximum out-degree in D . If $d^+(x) = 1$, then the out-degree of every vertex in D is one. Since D is strong and there exists no cycle of length more than or equal to $k - 1, D$ is a cycle of length at most $k - 2$. Hence, every vertex of D is a $(k - 1)$ -kernel. Now assume that $d^+(x) \geq 2$. If x is a $(k - 1)$ -kernel, then we are done; if not, there exists $z \in V(D) \setminus \{x\}$ such that $d(z, x) \geq k - 1$. Combining this with Lemma 1, we have $d(z, x) = k - 1$. Denote $W_s = \{y : d(y, x) = s\}$, for $s \in \{0, 1, \dots, k - 1\}$. Observe that $(W_j, W_1 \cup \dots \cup W_i) = \emptyset$ when $j \geq i + 2$.

Claim 1. $N^+(x) \subseteq W_1 \cup \dots \cup W_{k-3}$.

Proof. Since D has no loops, $N^+(x) \cap W_0 = \emptyset$. By the definition of W_s , every vertex of W_s can reach x in s steps. If $N^+(x) \cap (W_{k-1} \cup W_{k-2}) \neq \emptyset$, say $v \in N^+(x) \cap (W_{k-1} \cup W_{k-2})$, then let P be the shortest path from v to x . Then Pv is a cycle of length k or $k - 1$, a contradiction to the hypothesis of Case 4. Hence, $N^+(x) \subseteq W_1 \cup \dots \cup W_{k-3}$. □

Claim 2. Every vertex of $N^+(x)$ is contained in the shortest path from any vertex of W_{k-1} to x .

Proof. Let $z' \in W_{k-1}$ and P' be a shortest path from z' to x . If there exists $v \in N^+(x)$ such that $v \notin V(P')$, then $P'v$ is a path of length k , which implies that $z' \rightarrow v$, a contradiction to $v \in W_1 \cup \dots \cup W_{k-3}$ and $(W_{k-1}, W_1 \cup \dots \cup W_{k-3}) = \emptyset$. Hence, $v \in V(P')$ and furthermore $N^+(x) \subseteq V(P')$. The proof of the claim is complete. □

By Claim 2, $|N^+(x) \cap W_s| \leq 1$, for every $s \in \{1, 2, \dots, k - 3\}$. Let $r = \min\{j : W_j \cap N^+(x) \neq \emptyset\}$. Denote $N^+(x) \cap W_r = \{w\}$. Now we show that w is a $(k - 1)$ -kernel. For any $u \in W_{k-1}$, by Claim 2, we can conclude that $d(u, w) \leq k - 2$. For any $u \in W_0 \cup W_1 \cup \dots \cup W_{k-3}$, since $d(u, x) \leq k - 3$, we have $d(u, w) \leq k - 2$. If $d(u, w) \leq k - 2$ for any $u \in W_{k-2}$, then w is a $(k - 1)$ -kernel. Since $\text{diam}(D) \leq k - 1$, we assume that $d(u, w) = k - 1$. Let $R = u_{k-2}u_{k-3} \dots u_1u_0$ be a shortest path from u to x , where $u_{k-2} = u, u_0 = x$ and $u_j \in W_j$ for $j = 1, 2, \dots, k - 3$, and let $Q = x_{k-1}x_{k-2} \dots x_0$ be a shortest path from z to x , where $x_{k-1} = z, x_0 = x$ and $x_j \in W_j$ for $j = 1, 2, \dots, k - 2$. By Claim 2 and the choice of r , we have $N^+(x) \subseteq \{x_r, \dots, x_{k-3}\}$ and $w = x_r$. Again since $d^+(x) \geq 2$, there exists $x_p \in N^+(x) \cap \{x_{r+1}, \dots, x_{k-3}\}$. It is obvious that $u \notin V(Q)$ and $w \notin V(R)$, otherwise $d(u, w) \leq k - 2$, a contradiction. If there exists

a vertex $x_i \in \{x_{r+1}, \dots, x_{k-3}\} \cap V(R)$, then $x_i = u_i$ and $R[u_{k-2}, u_i]P[x_{i-1}, x_r]$ is a path of length $k-2-r$. Recalling $1 \leq r \leq k-3$, we have $1 \leq k-2-r \leq k-3$. So, $R[u_{k-2}, u_i]P[x_{i-1}, x_r]$ is a path of length at most $k-3$, a contradiction to $d(u, w) = k-1$. Hence, $\{x_{r+1}, \dots, x_{k-3}\} \cap V(R) = \emptyset$. Combining this with $x_p \in N^+(x) \cap \{x_{r+1}, \dots, x_{k-3}\}$, we have $Rx_p x_{p-1}$ is a path of length k . Hence, $u \rightarrow x_{p-1}$, a contradiction to $(W_{k-2}, W_1 \cup \dots \cup W_{k-4}) = \emptyset$, as $p-1 \leq k-4$. So, we have shown that w is a $(k-1)$ -kernel. ■

For 3-transitive digraphs, Hernández-Cruz [5] proved the following theorem.

Theorem 6 [5]. *Let D be a 3-transitive digraph. Then D has a 2-kernel if and only if it has no terminal strong component isomorphic to a 3-cycle.*

In view of this result and Theorem 5, we propose the following conjecture.

Conjecture 7. *Let D be a k -transitive digraph. Then D has a $(k-1)$ -kernel if and only if it has no terminal strong component isomorphic to a k -cycle.*

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