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ON TWO FAMILIES OF TESTS FOR NORMALITY WITH EMPIRICAL DESCRIPTION OF THEIR PERFORMANCES

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Abstract

We discuss two families of tests for normality based on characterizations of continuous distributions via order statistics and record values. Simulations of their powers show that they are competitive to widely recommended tests in the literature.

Keywords: order statistics, record values, U-statistics, normal distributions, exponential distributions, characterizations, goodness-of-fit tests, powers.

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1. Introduction

The large literature is devoted to testing and in particular tests for exponentiality and normality. Furthermore, there are many methods and techniques to construct goodness-of-fit tests. We are interested here in tests for normality which were studied, among other things, in D'Agostino nad Stephens [3], Rayner and Best [8], Thode [12], Kallenberg and Ledwina [4], Cabaña and Cabaña [1].

Goodness-of-fit tests based on characterizations of continuous distributions via expected values of two functions of order statistics or record values and U-statistics were constructed in Morris and Szynal [5, 6], respectively. Using this construction the tests for exponentiality and normality were presented in detail. An empirical description of the performances of those tests was given in Szynal and Wołyński [10, 11]. In this paper we describe how to proceed in order to test the sample $(X_1, \ldots, X_n)$ for normality using the test statistics constructed by characterization via order statistics (see Morris and Szynal [5], we call them O-tests) and record values (see Morris and Szynal [6], we call them R-tests). We should mention that the method of characterization of distribution was applied to construct tests for normality by Csörgő, Seshadri and Yalovsky [2].

For an empirical comparison of the performances of our tests (O-tests, R-tests) we use tests and alternatives choosing from Kallenberg and Ledwina [4] and Cabaña and Cabaña [1] (Tables $1a$ and $1b$).

We discuss the following omnibus tests:

SW : The Shapiro-Wilk test in [1],
AD : The Anderson-Darling test,
$D_{0.5}$: The BHEP test D_β for $\beta = 0.05$,
D_1: The BHEP test D_β for $\beta = 1$,
D_3: The BHEP test D_β for $\beta = 3$,
$SW^\star$: The Shapiro-Wilk test W in [4],
W_S : The data driven smooth test statistic,
W_{S1} : The data driven smooth modified test statistic,
W_{S2} : The data driven smooth test statistic without "adjustment",
LRk: The LaRiccia test focused on kurtosis,
LRs: The LaRiccia test focused on skewness,
KC: The Cabaña and Cabaña test K based on TEEP,
SC: The Cabaña and Cabaña test S based on TEEP,
$\tilde{K}C$: The Cabaña and Cabaña test $\tilde{K}$ based on TEEP,
$\tilde{S}C$: The Cabaña and Cabaña test $\tilde{S}$ based on TEEP.

BHEP is refered to Baringhaus and Heinze, Epps and Pulley (see [1]).
TEEP is refered to Transformed Estimated Empirical Process (see [1]).

We have selected the following alternatives (Source: Pearson *et al.* [7]).

Symmetric alternatives:

1. $SB(0, 0.5)$.
2. $Tukey(1.5)$.
5. $Tukey(0.7)$.
15. $Logistic(0.1)$.
17. $Tukey(10)$.
20. $SC(0.05, 3)$.
22. $SC(0.2, 5)$.
25. $SC(0.05, 5)$.
27. $SC(0.05, 7)$.
28. $SU(0, 1)$.

Skew alternatives:

40. $SB(1, 1)$.
41. $LO(0.2, 3)$.
44. $Weibull(2)$.
45. $LO(0.1, 3)$.
46. χ^2_{10}.
47. $LO(0.05, 3)$.
48. $LO(0.1, 5)$.
49. $SU(-1, 2)$.
50. χ^2_4.
52. $LO(0.05, 5)$.
54. $LO(0.05, 7)$.
57. $SU(1, 1)$.
58. $LN(0, 1)$.

The alternatives considered are:

$SB(\gamma, \delta)$ – Johnson's S_B distribution: the law of $\frac{\exp\left(\frac{X-\gamma}{\delta}\right)}{1+\exp\left(\frac{X-\gamma}{\delta}\right)}$, $X \sim N(0, 1)$,
$Tukey(\lambda)$ – Tukey's distribution: the law of $U^\lambda - (1-U)^\lambda$, U uniform on $[0, 1]$,
$Logistic(\theta)$ – logistic distribution: the law of $\frac{1}{\theta} \log \frac{U}{1-U}$, U uniform on $[0, 1]$,
$SC(p, \lambda)$ – scale contaminated distribution:

$$f(x) = (2\pi)^{-1/2} \left[(1-p)\exp\left(-\frac{x^2}{2}\right) + \left(\frac{p}{\lambda}\right)\exp\left(-\frac{x^2\lambda^{-2}}{2}\right)\right], \; -\infty < x < \infty,$$

$SU(\gamma, \delta)$ – Johnson's S_U distribution: the law of $\sinh\left(\frac{X-\gamma}{\delta}\right)$, $X \sim N(0, 1)$,
$LO(p, \mu)$ – location contaminated distribution:

$$f(x) = (2\pi)^{-1/2} \left[(1-p)\exp\left(-\frac{x^2}{2}\right) + p\exp\left(-\frac{(x-\mu)^2}{2}\right)\right], \; -\infty < x < \infty,$$

$Weibull(\theta)$ – Weibull distribution with parameters $(1, \theta)$,
χ^2_n – chi-squared distribution with n degrees of freedom,
$LN(\gamma, \delta)$ – lognormal distribution: the law of $\exp\left(\frac{X-\gamma}{\delta}\right)$, $X \sim N(0, 1)$.

Table 1*a*. (Source: Kallenberg and Ledwina [4] and Cabaña and Cabaña [1]) Estimated powers (in %) of SW, AD, $D_{0.5}$, D_1, D_3, $SW^\star$, W_S, W_{S1}, W_{S2}, LRk, LRs, KC, SC, $\tilde{K}C$ and $\tilde{S}C$.

Alt.		Tests														
Symmetric	n	SW	AD	$D_{0.5}$	D_1	D_3	$SW^\star$	W_S	W_{S1}	W_{S2}	LRk	LRs	KC	SC	$\tilde{K}C$	$\tilde{S}C$
1	20	22	34	3	27	41	44	36	34	26	56	5	42	2	41	2
$SB(0,0.5)$	50	88	92	8	87	89	99	67	93	55	99	4	92	7	93	3
2	20	9	20	2	15	25	26	20	19	14	37	2	26	2	26	2
$Tukey(1.5)$	50	58	71	4	64	67	92	34	74	26	94	2	76	3	77	1
5	20	4	22	1	7	14	12	11	9	8	12	2	13	1	13	1
$Tukey(0.7)$	50	23	40	2	34	37	62	13	45	9	72	1	45	1	47	1
15	20	13	10	14	11	6	12	10	13	11	11	10	13	13	13	13
$Logistic(0.1)$	50	24	16	20	16	11	13	13	21	12	23	12	27	20	27	20
17	20	84	93	48	71	92	82	82	87	85	85	26	59	45	62	45
$Tukey(10)$	50	99	100	76	98	100	99	99	100	99	97	18	91	64	92	65
20	20	7	11	7	7	5	19	17	19	16	6	7	8	7	7	8
$SC(0.05,3)$	50	12	7	10	8	6	31	25	38	24	10	8	13	11	13	11
22	20	23	24	22	18	10	71	65	74	65	19	15	24	22	24	23
$SC(0.2,5)$	50	46	30	36	33	18	95	92	98	92	42	21	49	36	49	36
25	20	12	14	12	10	7	36	33	37	32	10	10	13	12	13	13
$SC(0.05,5)$	50	23	14	19	14	8	62	55	66	55	19	15	25	21	26	22
27	20	16	18	16	14	8	45	42	46	42	14	13	18	16	17	17
$SC(0.05,7)$	50	34	21	29	23	12	74	70	77	70	29	20	36	31	36	31
28	20	47	44	42	42	31	43	36	47	38	44	30	46	43	47	43
$SU(0,1)$	50	81	75	72	76	65	68	61	81	61	82	40	82	67	82	65
Av.	20	23.7	29.0	16.7	22.2	23.9	39.0	35.2	38.5	33.7	29.4	12.0	26.2	16.3	26.3	16.7
Av.	50	48.8	46.6	27.6	45.3	41.3	69.5	52.9	69.3	50.3	56.7	14.1	53.6	26.1	54.2	25.5

Table 1*b*. (Source: Kallenberg and Ledwina [4] and Cabaña and Cabaña [1]) Estimated powers (in %) of SW, AD, $D_{0.5}$, D_1, D_3, $SW^\star$, W_S, W_{S1}, W_{S2}, LRk, LRs, KC, SC, $\tilde{K}C$ and $\tilde{S}C$.

Alt.		Tests														
Skew	n	SW	AD	$D_{0.5}$	D_1	D_3	$SW^\star$	W_S	W_{S1}	W_{S2}	LRk	LRs	KC	SC	$\tilde{K}C$	$\tilde{S}C$
40	20	24	28	24	27	21	31	29	17	29	6	30	8	23	10	21
$SB(1,1)$	50	69	68	69	73	55	81	72	57	71	12	76	6	65	13	57
41	20	23	25	26	29	20	31	27	19	28	6	24	9	23	9	21
$LO(0.2,3)$	50	55	65	65	69	53	60	68	52	69	6	60	7	59	10	52
44	20	13	13	15	14	10	15	15	10	16	5	17	8	16	9	15
$Weibull(2)$	50	36	32	39	36	19	41	41	29	41	7	46	11	39	15	36
45	20	26	24	28	25	13	25	24	21	26	11	25	18	28	16	28
$LO(0.1,3)$	50	57	51	61	55	32	50	58	51	58	18	57	32	61	31	60
46	20	41	40	26	24	13	25	23	18	26	9	48	22	43	22	41
χ^2_{10}	50	85	81	61	53	30	57	61	48	62	12	90	40	86	46	84
47	20	20	17	20	17	9	18	17	18	17	12	17	18	22	17	22
$LO(0.05,3)$	50	42	30	42	33	16	32	33	37	34	23	36	36	43	36	44
48	20	77	75	78	75	53	76	72	72	73	33	69	55	78	50	78
$LO(0.1,5)$	50	99	97	99	98	91	98	97	98	97	57	97	88	99	84	99
49	20	22	20	23	21	12	22	19	20	21	13	21	18	24	17	23
$SU(-1,2)$	50	45	37	47	42	24	37	43	40	42	23	43	35	46	36	46
50	20	49	48	51	48	32	53	51	38	52	10	56	26	50	27	48
χ^2_4	50	93	90	93	91	75	95	94	86	93	14	96	49	92	55	91
52	20	56	34	57	51	31	55	48	54	49	35	47	51	57	48	57
$LO(0.05,5)$	50	89	79	87	82	60	85	79	87	78	72	80	88	88	88	89
54	20	66	65	66	66	53	65	64	65	63	53	63	63	66	61	66
$LO(0.05,7)$	50	93	91	93	91	83	92	91	92	90	90	90	93	93	93	93
57	20	73	71	73	73	61	73	73	68	73	73	73	54	73	61	73
$SU(1,1)$	50	98	98	98	98	94	96	98	97	98	74	97	89	98	92	97
58	20	91	91	90	91	84	94	91	85	92	36	93	64	90	64	88
$LN(0,1)$	50	100	100	100	100	100	100	100	100	100	100	94	100	100	100	100
Av.	20	44.7	42.4	44.4	43.2	31.7	44.8	42.5	38.8	43.5	23.2	44.8	31.8	45.6	31.6	44.7
Av.	50	73.9	70.7	73.4	70.8	56.3	71.1	71.9	67.2	71.8	39.1	74.0	51.8	74.5	53.8	72.9

2. The family of O-tests

Let $(X_1, \ldots, X_n)$ be a random sample from continuous distribution function F. To verify the null hypothesis $H_0 : X \sim N(\mu, \sigma), \mu \in \mathbf{R}, \sigma \in \mathbf{R}_+ = (0, \infty)$, i.e.,

$$F(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-(t-\mu)^2/2\sigma^2} dt,$$

there are proposed (cf. Morris and Szynal [5]) the five tests $\hat{D}_n^{(r,m)}$, $\hat{D}_{n;c_1}^{(r,m)}$, $\hat{D}_{n;c_2}^{(r,m)}$, $\hat{D}_{n;c_3}^{(r,m)}$, $\hat{D}_{n;c_4}^{(r,m)}$ constructed via the characterizations of continuous distributions in terms of the expectations of two functions of order statistics, where $r > -1$, $m \in \mathbf{N}$. Write

$$\hat{\mu} = \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad \hat{\sigma^2} = S_n^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2,$$

$$Z \sim N(0,1), \quad \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \Phi(x) = \int_{-\infty}^{x} \phi(t) dt.$$

The particular procedure in a construction of the above test-statistics uses the following quantities (cf. Morris and Szynal [5]):

$$\begin{aligned} 1 + R_n^{(r,m)} &= \prod_{i=1}^{m} \left(1 - \frac{m}{n-1}\right) + \frac{m!(2m+2r+1)}{(m+1)\prod_{i=1}^{m}(n-i)} \\ &\cdot \sum_{j=2}^{m+1} \binom{m+1}{j} \binom{n-m-1}{m+1-j} \frac{j}{2m+2r+2-j} \end{aligned}$$

(cf. Morris and Szynal [5], p. 86),

$$K^{(r,m)} = \mathrm{E}^2[\phi(Z)\Phi^{m+r-1}(Z)] + \frac{1}{2}\mathrm{E}^2[Z\phi(Z)\Phi^{m+r-1}(Z)]$$

(cf. Morris and Szynal [5], p. 90, 98).

They are appear in the following test-statistics which are simple but their construction is not easy.

The test-statistics contain the quantities

$$\begin{aligned} a_{n1}^{(r,m)} &= \frac{(m+r)^2(1 - (m+r+1)^2(2m+2r+1)K^{(r,m)})}{n(m+r+1)^2(2m+2r+1)} \\ b_{n1}^{(r,m)} &= \frac{r(m+1)(m+r)(1 - (m+r+1)^2(2m+2r+1)K^{(r,m)})}{n(m+r+1)^2(2m+2r+1)} \end{aligned}$$

$$c_{n1}^{(r,m)} = \frac{r^2(m+1)^2(1-(m+r+1)^2(2m+2r+1)K^{(r,m)}+R_n^{(r,m)})}{n(m+r+1)^2(2m+2r+1)}$$

$$\Delta_{n1}^{(r,m)} = \det\begin{bmatrix} a_{n1}^{(r,m)} & b_{n1}^{(r,m)} \\ b_{n1}^{(r,m)} & c_{n1}^{(r,m)} \end{bmatrix}$$

(cf. Morris and Szynal [5], p. 90).

The tests for normality are as follows

$$\begin{aligned}
\hat{D}_n^{(r,m)} &= \frac{1}{\Delta_{n1}^{(r,m)}}\left[c_{n1}^{(r,m)}\left(\frac{1}{n}\sum_{i=1}^{n}\Phi^{m+r}\left(\frac{X_i-\bar{X}_n}{S_n}\right)-\frac{1}{m+r+1}\right)^2\right.\\
&- 2b_{n1}^{(r,m)}\left(\frac{1}{n}\sum_{i=1}^{n}\Phi^{m+r}\left(\frac{X_i-\bar{X}_n}{S_n}\right)-\frac{1}{m+r+1}\right)\\
&\times\left(\frac{1}{\binom{n}{m+1}}\sum_{i=m+1}^{n}\binom{i-1}{m}\Phi^{r}\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)-\frac{m+1}{m+r+1}\right)\\
&+ \left. a_{n1}^{(r,m)}\left(\frac{1}{\binom{n}{m+1}}\sum_{i=m+1}^{n}\binom{i-1}{m}\Phi^{r}\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)-\frac{m+1}{m+r+1}\right)^2\right]\\
&= \hat{D}_{n;c_1}^{(r,m)}+\hat{D}_{n;c_2}^{(r,m)} = \hat{D}_{n;c_3}^{(r,m)}+\hat{D}_{n;c_4}^{(r,m)},
\end{aligned}$$

where

$$\begin{aligned}
\hat{D}_{n;c_1}^{(r,m)} &= \frac{1}{a_{n1}^{(r,m)}}\left[\frac{1}{n}\sum_{i=1}^{n}\Phi^{m+r}\left(\frac{X_i-\bar{X}_n}{S_n}\right)-\frac{1}{m+r+1}\right]^2\\
\hat{D}_{n;c_2}^{(r,m)} &= \frac{1}{\Delta_{n1}^{(r,m)}a_{n1}^{(r,m)}}\left[a_{n1}^{(r,m)}\frac{1}{\binom{n}{m+1}}\sum_{i=m+1}^{n}\binom{i-1}{m}\Phi^{r}\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)\right.\\
&- b_{n1}^{(r,m)}\frac{1}{n}\sum_{i=1}^{n}\Phi^{m+r}\left(\frac{X_i-\bar{X}_n}{S_n}\right)\\
&- \left.\left(a_{n1}^{(r,m)}\frac{m+1}{m+r+1}-b_{n1}^{(r,m)}\frac{1}{m+r+1}\right)\right]^2\\
\hat{D}_{n;c_3}^{(r,m)} &= \frac{1}{c_{n1}^{(r,m)}}\left[\frac{1}{\binom{n}{m+1}}\sum_{i=m+1}^{n}\binom{i-1}{m}\Phi^{r}\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)-\frac{m+1}{m+r+1}\right]^2
\end{aligned}$$

$$\hat{D}_{n;c_4}^{(r,m)} = \frac{1}{\Delta_{n1}^{(r,m)} c_{n1}^{(r,m)}} \left[c_{n1}^{(r,m)} \frac{1}{n} \sum_{i=1}^{n} \Phi^{m+r} \left(\frac{X_i - \bar{X}_n}{S_n} \right) \right.$$
$$- b_{n1}^{(r,m)} \frac{1}{\binom{n}{m+1}} \sum_{i=m+1}^{n} \binom{i-1}{m} \Phi^{r} \left(\frac{X_{i:n} - \bar{X}_n}{S_n} \right)$$
$$\left. - \left(c_{n1}^{(r,m)} \frac{1}{m+r+1} - b_{n1}^{(r,m)} \frac{m+1}{m+r+1} \right) \right]^2 .$$

3. Simulation results for powers of the O-tests

For an empirical comparison of the performances of O-tests with widely recommended tests we have choosen alternatives and tests studied in Cabaña and Cabaña [1]. Samples of size 20 and 50 are taken with $m = 1, 2, 3, 4, 5$ and $r = -0.99,\ -0.95, -0.9, -0.7, -0.5, -0.3, -0.1, 0.1, 0.5, 0.75, 1.0, 1.5, 2.0, 2.5$. For symmetric alternatives we took additionally $r = 3.5, 4.0, 4.5, 5.0$ and for skew alternatives $r = 0.3, 0.6$. Critical values were simulated using 100 000 samples and associated powers were obtained using 100 000 samples but only some results are presented here (all simulations are available at W. Wołyński).

For samples of size $n = 20$ we included simulations for some favorable omnibus tests under symmetric alternatives with Av. powers ≥ 36.5 and under skew alternatives with Av. powers ≥ 45.5 (Tables $2a$ and $2b$).

For samples of size $n = 50$ we included simulations for some favorable omnibus tests under symmetric alternatives with Av. powers ≥ 68.5 and under skew alternatives with Av. powers ≥ 72.0 (Tables $2a$ and $2b$).

4. The family of R-tests

Goodness-of-fit tests derived from characterizations of continuous distributions via record values were given, among other things, in Morris and Szynal [6].

The test statistics for exponentiality and normality were discussed in Morris and Szynal [6], Szynal [9], Szynal and Wołyński [10, 11].

The aim of Section 4 is to give empirical description of performances of tests for normality presented in Morris and Szynal [6] which we call R-tests. To do a comparison R-tests with widely recommended tests we have choosen tests and alternatives studied in Cabaña and Cabaña [1] (as it was done in Section 2 and 3).

The construction of R-tests presented in Morris and Szynal [6] is not easy but the test-statistics have simple forms. We use here the quantities and the test statistics introduced in Morris and Szynal [6].

Table 2*a*. Powers of 5% O-tests under symmetric alternatives based on 100 000 samples with Av. ≥ 36.5 for $n = 20$ or Av. ≥ 68.5 for $n = 50$.

m		1	1	2	2	3	4	4	4	4	5	5	5
Tests		$\hat{D}_{n;c_1}^{(r,m)}$	$\hat{D}_{n}^{(r,m)}$	$\hat{D}_{n;c_1}^{(r,m)}$	$\hat{D}_{n}^{(r,m)}$	$\hat{D}_{n}^{(r,m)}$	$\hat{D}_{n;c_1}^{(r,m)}$	$\hat{D}_{n}^{(r,m)}$	$\hat{D}_{n;c_1}^{(r,m)}$	$\hat{D}_{n}^{(r,m)}$	$\hat{D}_{n;c_1}^{(r,m)}$	$\hat{D}_{n}^{(r,m)}$	$\hat{D}_{n;c_1}^{(r,m)}$
Alt. Symm.	n/r	3.5	3.5	2.5	2.5	1.5	0.5	0.5	0.75	0.75	-0.7	-0.7	-0.5
1	20	44	44	44	44	43	44	43	44	43	45	44	44
$SB(0, 0.5)$	50	93	93	93	94	94	93	94	91	92	94	94	93
2	20	31	31	31	31	30	31	30	30	30	31	30	31
$Tukey(1.5)$	50	81	81	81	82	82	81	81	80	79	82	82	81
5	20	18	19	19	19	18	19	18	19	18	19	18	18
$Tukey(0.7)$	50	57	58	57	58	58	57	57	56	56	57	57	57
15	20	11	10	11	10	10	11	11	11	11	11	11	11
$Logistic(0.1)$	50	20	19	20	20	20	20	20	20	20	20	20	20
17	20	61	62	61	62	62	61	63	60	61	62	63	61
$Tukey(10)$	50	95	97	95	97	97	95	97	94	96	96	98	95
20	20	17	17	17	17	17	17	17	18	18	17	17	17
$SC(0.05, 3)$	50	33	33	33	33	33	33	33	34	34	33	33	33
22	20	66	65	65	65	65	66	66	66	66	65	65	66
$SC(0.2, 5)$	50	97	97	97	97	97	97	97	97	97	97	97	97
25	20	34	33	33	33	33	33	33	34	34	33	33	33
$SC(0.05, 5)$	50	62	62	62	62	62	62	62	62	62	61	62	62
27	20	43	43	43	43	43	43	43	44	44	43	43	43
$SC(0.05, 7)$	50	74	74	74	74	74	74	74	74	74	73	74	74
28	20	40	39	40	39	39	40	39	40	39	39	40	40
$SU(0, 1)$	50	78	77	78	77	78	78	77	78	78	77	77	77
Av.	20	36.4	36.4	36.4	36.4	36.2	36.4	36.3	36.6	36.4	36.5	36.5	36.4
Av.	50	68.9	69.2	69.0	69.4	69.4	68.9	69.2	68.4	68.8	68.9	69.2	68.9

Table 2*b*. Powers of 5% O-tests under skew alternatives based on 100 000 samples with Av. ≥ 45.5 for $n = 20$ or Av. ≥ 72.0 for $n = 50$.

m		1	1	2	2	2	2	2	2	2	2
Tests		$\hat{D}^{(r,m)}_{n;c_1}$	$\hat{D}^{(r,m)}_{n;c_1}$	$\hat{D}^{(r,m)}_{n;c_4}$	$\hat{D}^{(r,m)}_{n;c_4}$	$\hat{D}^{(r,m)}_{n;c_1}$	$\hat{D}^{(r,m)}_{n;c_4}$	$\hat{D}^{(r,m)}_{n;c_1}$	$\hat{D}^{(r,m)}_{n;c_4}$	$\hat{D}^{(r,m)}_{n}$	$\hat{D}^{(r,m)}_{n;c_4}$
Alt. Skew	n/r	0.1	0.3	-0.99	-0.95	-0.9	-0.9	-0.7	-0.7	-0.7	-0.5
40	20	28	27	34	34	29	33	27	31	27	29
$SB(1,1)$	50	68	63	55	52	69	50	64	42	63	35
41	20	29	29	34	34	29	33	29	33	29	31
$LO(0.2,3)$	50	69	67	63	60	69	58	67	50	66	43
44	20	17	17	22	22	17	22	17	22	17	22
$Weibull(2)$	50	41	39	48	47	41	46	39	42	39	39
45	20	28	30	33	34	29	34	30	36	31	37
$LO(0.1,3)$	50	60	63	64	64	60	64	63	65	63	64
46	20	27	28	33	34	27	33	28	33	28	33
χ^2_{10}	50	62	61	66	65	62	64	60	60	60	57
47	20	19	21	22	23	20	24	21	25	22	27
$LO(0.05,3)$	50	36	39	41	41	36	42	40	45	40	47
48	20	76	77	75	76	76	76	78	77	78	77
$LO(0.1,5)$	50	97	98	77	76	97	74	98	67	98	61
49	20	23	24	26	27	23	27	24	28	24	29
$SU(-1,2)$	50	44	47	45	46	44	46	47	48	47	49
50	20	54	53	59	59	54	59	53	56	53	55
χ^2_4	50	93	91	78	76	93	74	91	69	91	64
52	20	53	55	53	54	53	55	55	57	55	58
$LO(0.05,5)$	50	79	83	69	70	80	72	83	74	83	76
54	20	65	65	63	64	65	64	65	64	65	64
$LO(0.05,7)$	50	90	92	41	42	90	44	92	48	92	50
57	20	73	69	66	65	73	64	69	59	69	54
$SU(1,1)$	50	96	95	95	94	96	94	95	93	96	91
58	20	91	91	85	85	91	84	91	82	91	80
$LN(0,1)$	50	100	100	21	21	100	21	100	22	100	23
Av.	20	44.8	45.2	46.6	46.9	44.9	46.9	45.2	46.3	45.2	45.8
Av.	50	72.0	72.0	58.6	57.9	72.0	57.6	72.2	55.7	72.0	54.0

To verify $H_0\colon X \sim N(\mu,\sigma)$ we use the notation of Section 2 and the following quantities and test statistics:

$$a_n^{(r,k)} = \frac{1}{\binom{n}{k}} \sum_{j=1}^{k-1} \binom{k}{j}\binom{n-k}{k-j} \left[2\frac{\Gamma(2r+2)}{k^r(k-j)^r} B_{\frac{k-j}{2k-j}}(r+1,r+1) \right.$$

$$\left. + j\frac{\Gamma(2r+1)}{(2k-j)^{2r+1}} - \frac{\Gamma^2(r+1)}{k^{2r}} \right] + \frac{1}{\binom{n}{k}} \left[\frac{\Gamma(2r+1)-\Gamma^2(r+1)}{k^{2r}} \right]$$

$$b_n^{(r,k)} = \frac{1}{\binom{n}{k}} \sum_{j=1}^{k-1} \binom{k}{j}\binom{n-k}{k-j} \left[2(2k-j)\frac{\Gamma(2r+2)+\Gamma(2r+3)}{k^{r+1}(k-j)^{r+1}} B_{\frac{k-j}{2k-j}}(r+2,r+2) \right.$$

$$\left. + j\frac{\Gamma(2r+2)}{(2k-j)^{2r+2}} - \frac{\Gamma(r+1)\Gamma(r+2)}{k^{2r+1}} \right] + \frac{1}{\binom{n}{k}} \left[\frac{\Gamma(2r+2)-\Gamma(r+1)\Gamma(r+2)}{k^{2r+1}} \right]$$

$$c_n^{(r,k)} = \frac{1}{\binom{n}{k}} \sum_{j=1}^{k-1} \binom{k}{j}\binom{n-k}{k-j} \left[2\frac{\Gamma(2r+4)}{k^{r+1}(k-j)^{r+1}} B_{\frac{k-j}{2k-j}}(r+2,r+2) \right.$$

$$\left. + j\frac{\Gamma(2r+3)}{(2k-j)^{2r+3}} - \frac{\Gamma^2(r+2)}{k^{2r+2}} \right] + \frac{1}{\binom{n}{k}} \left[\frac{\Gamma(2r+3)-\Gamma^2(r+2)}{k^{2r+2}} \right],$$

where

$$B_x(\alpha,\beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt, \quad 0<x<1; \quad \alpha,\beta>0,$$

denotes the incomplete beta function.

$$E_1^{(r,k)} = \mathrm{E}\left[\phi(Z)(1-\Phi(Z))^{k-2}\log^{r-1}\frac{1}{1-\Phi(Z)} \right], \quad Z \sim N(0,1),$$

$$E_2^{(r,k)} = \mathrm{E}\left[Z\phi(Z)(1-\Phi(Z))^{k-2}\log^{r-1}\frac{1}{1-\Phi(Z)} \right].$$

$$s_n^{(r,k)} = \frac{k^2r^2}{n}\left[(E_1^{(r,k)})^2 + \frac{1}{2}(E_2^{(r,k)})^2 \right]$$

$$t_n^{(r,k)} = \frac{k^2r(r+1)}{n}\left[E_1^{(r,k)}E_1^{(r+1,k)} + \frac{1}{2}E_2^{(r,k)}E_2^{(r+1,k)} \right]$$

$$u_n^{(r,k)} = \frac{k^2(r+1)^2}{n}\left[(E_1^{(r+1,k)})^2 + \frac{1}{2}(E_2^{(r+1,k)})^2 \right]$$

$$
\begin{aligned}
a_{n1}^{(r,k)} &= a_n^{(r,k)} - s_n^{(r,k)} \\
b_{n1}^{(r,k)} &= b_n^{(r,k)} - t_n^{(r,k)} \\
c_{n1}^{(r,k)} &= c_n^{(r,k)} - u_n^{(r,k)}
\end{aligned}
$$

$$
\Delta_{n1}^{(r,k)} = \det \begin{bmatrix} a_{n1}^{(r,k)} & b_{n1}^{(r,k)} \\ b_{n1}^{(r,k)} & c_{n1}^{(r,k)} \end{bmatrix}.
$$

R-tests are as follows

$$
\begin{aligned}
\hat{T}_n^{(r,k)} &= \frac{1}{\Delta_{n1}^{(r,k)}} \Bigg[c_{n1}^{(r,k)} \left(\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^r \frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)} - \frac{\Gamma(r+1)}{k^r} \right)^2 \\
&- 2b_{n1}^{(r,k)} \left(\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^r \frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)} - \frac{\Gamma(r+1)}{k^r} \right) \\
&\times \left(\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r+1} \frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)} - \frac{\Gamma(r+2)}{k^{r+1}} \right) \\
&+ a_{n1}^{(r,k)} \left(\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r+1} \frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)} - \frac{\Gamma(r+2)}{k^{r+1}} \right)^2 \Bigg] \\
&= \hat{T}_{n;c_1}^{(r,k)} + \hat{T}_{n;c_2}^{(r,k)} = \hat{T}_{n;c_3}^{(r,k)} + \hat{T}_{n;c_4}^{(r,k)},
\end{aligned}
$$

where

$$
\hat{T}_{n;c_1}^{(r,k)} = \frac{1}{a_{n1}^{(r,k)}} \left[\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^r \frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)} - \frac{\Gamma(r+1)}{k^r} \right]^2
$$

$$
\begin{aligned}
\hat{T}_{n;c_2}^{(r,k)} &= \frac{1}{\Delta_{n1}^{(r,k)} a_{n1}^{(r,k)}} \Bigg[a_{n1}^{(r,k)} \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r+1} \frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)} \\
&- b_{n1}^{(r,k)} \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^r \frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)} \\
&- \left(a_{n1}^{(r,k)} \frac{\Gamma(r+2)}{k^{r+1}} - b_{n1}^{(r,k)} \frac{\Gamma(r+1)}{k^r} \right) \Bigg]^2
\end{aligned}
$$

$$\hat{T}_{n;c_3}^{(r,k)} = \frac{1}{c_{n1}^{(r,k)}} \left[\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r+1} \frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)} - \frac{\Gamma(r+2)}{k^{r+1}} \right]^2$$

$$\begin{aligned}\hat{T}_{n;c_4}^{(r,k)} &= \frac{1}{\Delta_{n1}^{(r,k)} c_{n1}^{(r,k)}} \left[c_{n1}^{(r,k)} \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r} \frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)} \right. \\ &- b_{n1}^{(r,k)} \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r+1} \frac{1}{1-\Phi\left(\frac{X_{i:n}-\bar{X}_n}{S_n}\right)} \\ &- \left. \left(c_{n1}^{(r,k)} \frac{\Gamma(r+1)}{k^r} - b_{n1}^{(r,k)} \frac{\Gamma(r+2)}{k^{r+1}} \right) \right]^2 .\end{aligned}$$

5. Simulation results for powers of the R-tests

Similarly as for the O-tests to compare performances of the R-tests with widely recommended tests we have choosen alternatives and tests discussed in Cabaña and Cabaña [1]. Samples of size $n = 20$ and $n = 50$ are taken with $k = 1, 2, 3, 4, 5$ and $r = -0.499, -0.45, -0.4, -0.1, 0.1, 0.3, 0.5, 0.7, 0.9, 1.0, 1.1, 1.3, 1.5, 1.7, 2.0$. Critical values were simulated using 100 000 replications and associated powers were obtained using 100 000 replications but only some results are presented here.

For samples of size $n = 20$ we included simulations for some favorable omnibus tests under symmetric alternatives with Av. powers ≥ 38.0 and under skew alternatives with Av. powers ≥ 46.0 (Tables $3a$ and $3b$).

For samples of size $n = 50$ we included simulations for some favorable omnibus tests under symmetric alternatives with Av. powers ≥ 71.0 and under skew alternatives with Av. powers ≥ 74.0 (Tables $3a$ and $3b$).

6. Final comments

We have presented two families of tests (O-tests and R-tests) for normality using some parameters m, r for O-tests and k, r for R-tests in their domains. We conclude that our tests for normality perform very well and they can be recommended to use them in the statistical inference. From Tables $2a$, $2b$, $3a$ and $3b$ one can see that R-tests perform a bit better than O-tests. Our best tests are collected in Table 4.

Table 3*a*. Powers of 5% R-tests under symmetric alternatives based on 100 000 samples with Av. ≥ 38.0 for $n = 20$ or Av. ≥ 71.0 for $n = 50$.

k		1	2	2	2	3	3	3	3	4
Tests		$\hat{T}_{n;c_4}^{(r,k)}$	$\hat{T}_{n;c_4}^{(r,k)}$	$\hat{T}_{n;c_4}^{(r,k)}$	$\hat{T}_{n;c_2}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$	$\hat{T}_{n}^{(r,k)}$
Alt. Symm.	n/r	0.3	0.3	0.5	0.7	0.9	1.0	1.1	1.3	1.5
1	20	43	47	53	41	46	50	51	49	51
$SB(0, 0.5)$	50	95	93	94	94	95	95	95	93	94
2	20	28	37	39	25	31	34	34	32	36
$Tukey(1.5)$	50	83	87	86	81	82	81	80	76	79
5	20	15	25	24	13	18	19	19	19	22
$Tukey(0.7)$	50	57	70	64	52	54	53	52	48	53
15	20	13	9	10	14	10	11	10	10	9
$Logistic(0.1)$	50	25	18	19	24	20	20	20	19	17
17	20	58	63	66	68	76	78	78	78	78
$Tukey(10)$	50	91	98	97	95	99	99	99	99	99
20	20	21	17	17	20	18	17	17	16	15
$SC(0.05, 3)$	50	42	36	34	38	33	33	32	32	28
22	20	71	65	65	71	71	70	70	69	64
$SC(0.2, 5)$	50	98	97	97	98	97	97	97	97	96
25	20	38	34	33	36	34	34	33	33	30
$SC(0.05, 5)$	50	69	64	63	65	62	62	62	61	58
27	20	46	43	43	45	44	44	43	42	41
$SC(0.05, 7)$	50	78	75	75	76	74	74	74	73	72
28	20	45	37	39	47	43	43	42	41	37
$SU(0, 1)$	50	82	78	77	83	79	79	80	80	77
Av.	20	37.9	37.5	38.9	38.0	39.0	40.1	39.8	38.9	38.3
Av.	50	72.0	71.7	70.5	70.7	69.6	69.4	69.1	67.8	67.3

Table 3*b*. Powers of 5% R-tests under skew alternatives based on 100 000 samples with Av. ≥ 46.0 for $n = 20$ or Av. ≥ 74.0 for $n = 50$.

k		1	2	2	2	2	2	2	2	2
Tests		$\hat{T}_n^{(r,k)}$	$\hat{T}_{n;c_3}^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_{n;c_3}^{(r,k)}$	$\hat{T}_{n;c_1}^{(r,k)}$	$\hat{T}_{n;c_1}^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$	$\hat{T}_n^{(r,k)}$
Alt. Skew	n/r	2.0	0.1	0.1	0.3	1.1	1.3	1.3	1.5	1.7
40	20	26	36	29	31	35	31	31	31	29
$SB(1,1)$	50	61	72	65	62	73	63	70	65	60
41	20	28	33	29	31	32	31	31	32	33
$LO(0.2,3)$	50	61	69	63	63	69	65	68	69	68
44	20	17	23	19	22	23	22	19	20	19
$Weibull(2)$	50	39	48	41	43	48	44	40	38	36
45	20	32	33	32	35	33	35	32	34	35
$LO(0.1,3)$	50	65	64	64	67	64	68	62	64	66
46	20	29	35	31	34	34	34	30	30	31
χ^2_{10}	50	60	68	63	64	68	65	61	59	58
47	20	24	24	24	26	23	25	23	25	26
$LO(0.05,3)$	50	50	41	45	45	40	46	45	46	47
48	20	80	72	78	74	71	74	78	81	82
$LO(0.1,5)$	50	99	97	99	98	97	98	99	99	99
49	20	26	27	27	28	27	28	26	27	28
$SU(-1,2)$	50	50	49	50	52	49	53	50	50	51
50	20	53	61	56	58	61	57	56	56	54
χ^2_4	50	90	94	92	91	94	92	92	90	89
52	20	59	51	58	56	51	57	58	59	60
$LO(0.05,5)$	50	90	79	88	84	79	85	89	89	89
54	20	66	60	65	64	60	64	66	66	66
$LO(0.05,7)$	50	93	88	92	91	88	91	93	93	93
57	20	70	63	66	56	62	56	69	67	64
$SU(1,1)$	50	97	96	97	94	96	94	98	97	97
58	20	90	90	92	90	89	90	92	91	90
$LN(0,1)$	50	100	97	100	100	97	100	100	100	100
Av.	20	46.1	46.7	46.7	46.6	46.2	46.5	47.0	47.6	47.4
Av.	50	73.4	73.9	73.8	73.5	73.9	73.9	74.1	73.7	73.4

Table 4. Our recommended tests.

n	O-tests symmetric	O-tests skew	R-tests symmetric	R-tests skew
20	$\hat{D}^{(0.75,4)}_{20;c_1}$ Av.=36.6	$\hat{D}^{(-0.95,2)}_{20;c_4}$ Av.=46.9	$\hat{T}^{(1.0,3)}_{20}$ Av.=40.1	$\hat{T}^{(1.5,2)}_{20}$ Av.=47.6
50	$\hat{D}^{(2.5,2)}_{50}$ Av.=69.4	$\hat{D}^{(-0.7,2)}_{50;c_1}$ Av.=72.2	$\hat{T}^{(0.3,1)}_{50;c_4}$ Av.=72.0	$\hat{T}^{(1.3,2)}_{50}$ Av.=74.1

There is a chance to improve performances of those tests choosing other values of the mentioned parameters than we did. But it is almost sure that one can get better tests enlarging the domains of those parameters. This can be done deriving new tests via the discussed characterization conditions. Some successful attempts in this direction for exponentiality were done in Szynal [9] and Szynal and Wolynski [10].

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