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A BAYESIAN SIGNIFICANCE TEST OF CHANGE FOR CORRELATED OBSERVATIONS

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Abstract

This paper presents a Bayesian significance test for a change in mean when observations are not independent. Using a noninformative prior, a unconditional test based on the highest posterior density credible set is determined. From a Gibbs sampler simulation study the effect of correlation on the performance of the Bayesian significance test derived under the assumption of no correlation is examined. This paper is a generalization of earlier studies by KIM (1991) to not independent observations.

Keywords: autoregressive model, change point, HPD region sets, p-value, Gibbs sampler.

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1. Introduction

Suppose we observe the time series $(y_1, \ldots, y_n)$ as a possible generating stochastic process $\{Y_t, t \in \mathbb{Z}\}$, we consider the autoregressive model of order p $(AR(p))$

$$Y_t - \mu_t = \sum_{i=1}^{p} \phi_i (Y_{t-i} - \mu_{t-i}) + \epsilon_t,$$

where $\mu_t = \mathbb{E}(Y_t)$ and the ϵ_t's are iid Gaussian random variables with mean 0 and variance σ^2, that are $\epsilon_t\,(iid) \rightsquigarrow N(0, \sigma^2)$.

In this work, we consider the model with a change in the mean μ_t and variance σ^2 at time m, that is: $\mathbb{E}(Y_t) = \mu_1$ and $\sigma^2 = \sigma_1^2$ for $t = 1, \dots, m$ and $\mathbb{E}(Y_t) = \mu_2$ and $\sigma^2 = \sigma_2^2$ for $t = m+1, \dots, n$. Equivalently we have the model,

$$
(1) \quad \begin{cases} Y_t - \mu_1 = \sum_{i=1}^{p} \phi_i \left(Y_{t-i} - \mu_1\right) + \epsilon_t, & t = 1, \dots, m \\ Y_t - \mu_2 = \sum_{i=1}^{p} \phi_i \left(Y_{t-i} - \gamma_{t-i}\mu_1 - (1-\gamma_{t-i})\mu_2\right) + \epsilon_t, & t = m+1, \dots, m+p \\ Y_t - \mu_2 = \sum_{i=1}^{p} \phi_i (Y_{t-i} - \mu_2) + \epsilon_t, & t = m+p+1, \dots, n, \end{cases}
$$

where γ_t is the indicator function such that $\gamma_{t-i} = 1$ if $t - i \leq m$ and $\gamma_{t-i} = 0$ if $t - i > m$.

We assume that the roots of the autoregressive polynomial are outside the unit circle, i.e., the parameter vector $\phi^p = (\phi_1, \dots, \phi_p)$ lies in the stationarity region $\Phi_p = \{\phi^p : (\varphi(z) = 0) \Rightarrow |z| > 1\}$, where $\varphi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$ is the characteristic polynomial of $AR(p)$ model. The parametres μ_1, μ_2 ($\mu_1, \mu_2 \in \mathbb{R}$), $\phi_i, (i = 1, \dots, p)$, σ_1, σ_2 ($\sigma_1, \sigma_2 > 0$) are assumed to be unknown, and $m \in \{1, \dots, n-2\}$ is the change point assumed also unknown. The aim of this work is to define a decision rule to detect the existence of a change in the model parameters from the observations $(y_1, y_2, \dots, y_n)$ and to study the effect of correlation on the performance of the bayesian significance test derived under the assumption of no correlation.

One has a parameter set $\theta = (m, \phi^p, \mu_1, \mu_2, r_1, r_2)$, where $r_i = 1/\sigma_i^2$. The likelihood function based on the observations $y = (y_1, y_2, \dots, y_n)$ is then

$$
\begin{aligned}
l(y/\theta) \propto & r_1^{\frac{m}{2}} r_2^{\frac{n-m}{2}} \exp\left\{ \frac{r_1}{2} \left[\sum_{t=1}^{m} (y_t - \mu_1 - \sum_{i=1}^{p} \phi_i (y_{t-i} - \mu_1) \right]^2 \right\} \\
(2) \qquad & \exp\left\{ \frac{r_2}{2} \left[\sum_{t=m+1}^{m+p} (y_t - \mu_2 - \sum_{i=1}^{p} \phi_i \Big(y_{t-i} - \gamma_{t-i}\mu_1 - (1-\gamma_{t-i})\mu_2 \Big) \right]^2 \right\} \\
& \exp\left\{ \frac{r_2}{2} \left[\sum_{t=m+p+1}^{n} (y_t - \mu_2 - \sum_{i=1}^{p} \phi_i (y_{t-i} - \mu_2) \right]^2 \right\}
\end{aligned}
$$

with given $(y_{-p+1}, \dots, y_0)$.

A change point, which is generally the effect of an external event on the phenomenon of interest, may be represented by a change in the structure of the model or simply by a change of the value of some of the parameters. Since Page [9, 10] which developed a cumulative sum test to detect a location change, considerable attention has been given to this problem in a variety of settings. Hinkley [8], Sen and Srisvastava [11], Siegmund [12, 13], Worsley [14, 15] and Kim, H.-J

[5], who used likelihood ratio approaches. Worsley [14, 15] proposed a numerical method for computing the p-value of the generalized likelihood ratio test to detect a change in binomial probability and in location of an exponential family distribution. Kim, H.-J [5] considers a likelihood ratio test for a change in mean when observations are correlated. It has showed the sensitivity of the likelihood ratio statistic derived under the assumption of independence to the nonzero correlation among the observations. It is observed that the p-value deriving under noncorroleted observations underestimate/overestimate the true pvalues when we ignore positive/negative autocorrelation.

In Bayesian context, the problem of detection of change was studies by many authors. we can cite the works of Chernoff and Zacks [3], Kinder et Zacks [6], Sen and Srivastava [11] where the aim is to detect the change in the mean for normal random variables. Barbieri and Conigliani [1] adopt the Bayesian approch with weak prior information about the parameters of the models under comparison and an exact form of the likelihood function for the identification of a stationary autoregressive model for a time series and the contemporary detection of a change in its mean. Kim [7], proposed a Bayesian significance test for stationarity of a regression equation using the highest posterior density credible set. From a Monte Carlo simulation study, it has shown that the Bayesian significance test has stronger power than the Cusum and the Cusum of squares tests suggested by Brown, Durbin & Evans [2]. Ghorbanzadeh and Lounes [4] proposed a bayesian analysis of detection of a change of parametre in a sequence of independent random variables from exponential family. However In many applications, the observations are correlated in various ways. Other references related of the change-point problem when the data are correlated can be found in Kim, H.-J [5]. In this work, we propose a Bayesian test based on the HPD credible regions when the observations are correlated. The rest of paper is organized as follows, Section 2 presents the bayesian analysis and the bayesian significance test for change. Simulations results are given in In section 3. Section 4 is our conclusion.

Notations: We consider the following notations:

$$\phi^p = (\phi_1, \phi_2, \dots, \phi_p), \qquad \phi^{(j)} = (\phi_1, \dots, \phi_{j-1}, \phi_{j+1}, \dots, \phi_p),$$

$$\phi(p) = 1 - \sum_{i=1}^{p} \phi_i, \quad SS_1(m, \phi^p, \mu_1) = \sum_{t=p}^{m} \left(y_t - \mu_1 - \sum_{i=1}^{p} \phi_i (y_{t-i} - \mu_1) \right)^2,$$

$$a(m, \phi^p) = \sum_{t=m+1}^{m+p} \left(1 - \sum_{i=1}^{p} \phi_i (1 - \gamma_{t-i}) \right)^2 + (n-m-p) \left(1 - \sum_{i=1}^{p} \phi_i \right)^2,$$

$$SS_2(m,\phi^p,\mu_1) = \sum_{t=m+1}^{n}\left(y_t-\mu_1-\sum_{i=1}^{p}\phi_i(y_{t-i}-\mu_1)\right)^2$$
$$-\frac{1}{a(m,\phi^p)}\left[\sum_{t=m+1}^{m+p}\Big(y_t-\mu_1-\sum_{i=1}^{p}\phi_i(y_{t-i}-\mu_1)\Big)\left(1-\sum_{i=1}^{p}\phi_i(1-\gamma_{t-i})\right)\right.$$
$$\left.+\sum_{t=m+p+1}^{n}\Big(y_t-\mu_1-\sum_{i=1}^{p}\phi_i(y_{t-i}-\mu_1)\Big)\left(1-\sum_{i=1}^{p}\phi_i\right)\right]^2.$$

2. Bayesian analysis

Since prior knowledge of $\theta' = (\mu_1,\mu_2,r_1,r_2)$ is often vague or diffuse, we employ a diffuse prior for θ'. Assume that the priors of the change-point m and of $\phi^{(p)}$ are given by

$$\pi(m) \propto \frac{1}{n-2};\ m\in\{1,\dots,n-2\},\quad \pi(\phi^{(p)}) \propto \text{constant in } \ \Phi_p\,.$$

The parameters m, $\phi^{(p)}$ and θ' being assumed independent. The prior distribution of θ is, therefore

$$\pi(\theta) \propto \frac{1}{r_1 r_2}\,. \tag{3}$$

The posterior distribution of θ, obtained by combination of (2) and (3) is

$$\pi(\theta/y) \propto r_1^{\frac{m}{2}-1} r_2^{\frac{n-m}{2}-1}\exp\left\{\frac{r_1}{2}\left[\sum_{t=1}^{m}(y_t-\mu_1-\sum_{i=1}^{p}\phi_i(y_{t-i}-\mu_1)\right]^2\right\}$$
$$\exp\left\{\frac{r_2}{2}\left[\sum_{t=m+1}^{m+p}(y_t-\mu_2-\sum_{i=1}^{p}\phi_i\Big(y_{t-i}-\gamma_{t-i}\mu_1-(1-\gamma_{t-i})\mu_2\Big)\right]^2\right\} \tag{4}$$
$$\exp\left\{\frac{r_2}{2}\left[\sum_{t=m+p+1}^{n}(y_t-\mu_2-\sum_{i=1}^{p}\phi_i(y_{t-i}-\mu_2)\right]^2\right\}.$$

The null hypothesis H_0 that there is no change in the parameters of model (1) is

$$\delta=\mu_2-\mu_1=0,\quad \tau=\sigma_2^2/\sigma_1^2=1.$$

For the Bayesian significance test, therefore, the posterior distributions of δ and τ are needed to obtain the confidence region, i.e., highest posterior density credible set, of δ and τ.

The followin theorem give the posterior distribution of δ and τ:

Theorem 1.

1. *Given m, ϕ^p, μ_1 and τ the conditional posterior distribution of δ is*

$$\pi(\delta|m,\phi^p,\mu_1,\tau,y) \propto \left\{1+\frac{a(m,\phi^p)\left(\delta-\widehat{\delta}(m,\phi^p,\mu_1)\right)^2}{(n-1)S_2^2(m,\phi^p,\tau)}\right\}^{-\frac{n}{2}}, \tag{5}$$

where

$$\widehat{\delta}(m,\phi^p,\mu_1) = \frac{\sum_{t=m+1}^{m+p}\left(y_t-\mu_1-\sum_{i=1}^{p}\phi_i(y_{t-i}-\mu_1)\right)\left(1-\sum_{i=1}^{p}\phi_i(1-\gamma_{t-i})\right)}{a(m,\phi^p)} + \frac{\sum_{t=m+p+1}^{np}\left(y_t-\mu_1-\sum_{i=1}^{p}\phi_i(y_{t-i}-\mu_1)\right)\left(1-\sum_{i=1}^{p}\phi_i\right)}{a(m,\phi^p)},$$

$$S_2^2(m,\phi^p,\mu_1,\tau) = \frac{\tau SS_1(m,\phi^p,\mu_1)+SS_2(m,\phi^p,\mu_1)}{(n-1)},$$

which is the Student t distribution with location parameter $\widehat{\delta}(m,\phi^p,\mu_1)$, precision $\frac{a(m,\phi^p)}{S_2^2(m,\phi^p,\tau)}$, and $(n-1)$ degrees of freedom. Equivalently, the quantity

$$t(\delta) = \frac{a^{\frac{1}{2}}(m,\phi^p)\left(\delta-\widehat{\delta}(m,\phi^p,\mu_1)\right)}{S_2(m,\phi^p,\tau)} \tag{6}$$

is distributed a posteriori as a conditional standard Student t distribution with $(n-1)$ degrees of freedom.

2. *Given m, ϕ^p and μ_1, the conditional posterior distribution of τ is:*

$$\pi(\tau|m,\phi^p,\mu_1,y) \propto \tau^{\frac{m}{2}-1}\left\{\tau SS_1(m,\phi^p,\mu_1)+SS_2(m,\phi^p,\mu_1)\right\}^{-\frac{(n-1)}{2}}. \tag{7}$$

Which the quantity

$$F(\tau) = \frac{SS_1(m,\phi^p,\mu_1)/m}{SS_2(m,\phi^p,\mu_1)/(n-m-1)}\tau \tag{8}$$

is distributed a posteriori as a conditional F distribution with $(m, n-m-1)$ degrees of freedom.

3. *Given ϕ^p, μ_1 and τ the posterior conditional distribution of m is*

$$\pi(m/\phi^p,\mu_1,\tau,y) \propto a(m,\phi^p)^{-\frac{1}{2}} S\left\{\tau SS_1(m,\phi^p,\mu_1)+SS_2(m,\phi^p,\mu_1)\right\}^{-\frac{n-1}{2}}. \tag{9}$$

Proof. See Appendix A. ■

The unconditional posterior distributions of $t(\delta)$ and $F(\tau)$ are, respectively

$$\begin{aligned}\pi(t(\delta)|Y) =& \sum_m \Big\{ \int_{\phi^p} \Big[\int_{\mu_1} \Big(\int_\tau \pi(t(\delta)|m,\phi^p,\mu_1,\tau,Y)\pi(\tau|m,\phi^p,\mu_1,Y)d\tau \Big) \\ & \times \pi(\mu_1|m,\phi^p,Y)d\mu_1 \Big] \pi(\phi^p|m,Y)d\phi^p \Big\} \pi(m|Y),\end{aligned} \tag{10}$$

$$\begin{aligned}\pi(F(\tau)|Y) =& \sum_m \Big\{ \int_{\phi^p} \Big[\int_{\mu_1} \pi(F(\tau)|m,\phi^p,\mu_1,\tau,Y)\pi(\mu_1|m,\phi^p,Y)d\mu_1 \Big] \\ & \times \pi(\phi^p|m,Y)d\phi^p \Big\} \pi(m|Y).\end{aligned} \tag{11}$$

The null hypothesis H_0 can be divided into two sub-hypotheses $H_{01} : \delta = 0$ and $H_{02} : \tau = 1$, and H_o could be rejected if either of these two sub-hypotheses is rejected. One defines separately the highest posterior density credible sets of $t(\delta)$ and $F(\tau)$ based on conditional distributions since $t(\delta)$ and $F(\tau)$ are conditionally independent. These credible sets will be used to define the unconditional p-value and therefore an unconditional test.

Given m, ϕ^p, μ_1 and τ, the $(1-\alpha)$-credible set for $t(\delta)$ is defined as: $C_\delta = \left\{t(\delta)/|t(\delta)| < t_{\alpha|2}(n-1)\right\}$.

Where $t_{\alpha|2}(n-1))$ is the $(1-\alpha/2)$th quantile of an t-distribution with $(n-1)$ degrees of freedom. Hence, given m, ϕ^p, μ_1 and τ, the decision rule for H_0, is to reject if $t(0) \in \overline{C_\delta}$, where $\overline{C_\delta}$ is the complement of C_δ.

The unconditional p-value of H_0, therefore, is calculated from (10) to yield:

$$P_{\delta=0/y} = 2E_m E_{\phi^p} E_{\mu_1} E_\tau \Big\{ 1 - \mathcal{T}_{n-1}(|\, t(0)\,|) \Big\}, \tag{12}$$

where $\mathcal{T}_{n-1}$ is the cumulative density function of the standard Student t distribution with $(n-1)$ degrees of freedom, and the expectations E_m, E_{μ_1} and E_τ are taken with respect to m, μ_1 and τ, respectively. E_{ϕ^p} is to note expectations taken with respect to $\phi_1, \phi_2, \ldots, \phi_p$ respectively. Our test, therefore, is to reject H_{01}, if $P_{\delta=0/Y}$ falls below α. This test results in a size α test.

Likewise, the unconditional p-value of H_{02} is

$$P_{\tau=1|y} = 2E_m E_{\phi^p} E_{\mu_1} \Big\{ 1 - \mathcal{F}_{m,n-m-1}[\max(F(1), 1|F(1))] \Big\}, \tag{13}$$

where $\mathcal{F}_{m,n-m-1}$ is the cumulative density function of an F distribution with $(m, n-m-1)$ degrees of freedom, also, the test, is to reject H_{02}, if $P_{\tau=1|Y}$ falls below α.

The quantities (12) and (13) will be evaluated numerically by Gibbs Sampler algorithm using the conditional posterior distributions given in Theorem 1 and Lemma 2 of appendix B.

3. SIMULATION STUDY

Simulation has been used to study the effect of correlation on the bayesian significance test based on the highest posterior density credible set (Kim [7], Ghorbanzadeh and Lounes [4]).

We simulate samples from the model (1) for $p = 1$, $\phi_1 = 0.5$, $\mu_1 = 0.5$, $\mu_2 = 0.2$, $\sigma_1^2 = 0.5$, $\sigma_2^2 = 1.5$, $m = 34$, $y_0 = 0.5$ and for $n = 70$.

From these observations, by the application of the Gibbs sampler algorithm with 5000 repetitions, we approximate the marginal posterior densities of the change point m and the unconditional and conditional p-values for the sub-hypotheses $H_{01} : \delta = 0$ andet $H_{02} : \tau = 1$. The marginal posterior density of m and the conditional p-values given m of H_{01} and H_{02} for some values of neighborhood of the true value of m are given in Table 1.

m	$\pi\,(m/y)$	$P_{\delta=0/m,y}$	$P_{\tau=1/m,y}$
32	0.0711	0.000	0.021
33	0.153	0.000	0.017
34	0.663	0.000	0.022
35	0.014	0.000	0.095
36	0.004	0.016	0.201

Table 1. Marginal posterior density of m and the conditional p-values given m of H_{01} and H_{02}. Estimated by a Gibbs sampler algrorithm with 5000 repetitions.

Table 1 shows that the posterior mode of $\pi\,(m/y)$ detecting the true value of change $m = 34$, and the sub-hypotheses H_{01} and H_{02} are rejected respectively at significance levels $\alpha = 0.05$.

The unconditional and conditional given m and ρ p-values of H_{01} and H_{02} given in Table 2 show that the test rejects H_{01} and H_{02} respectively at significance level $\alpha = 0.01$ and $\alpha = 0.05$. Therefore, the hypothesis H_0 is rejected at significance level $\alpha = 0.01$.

To examine the effect of correlation on the Bayesian significance test based on the HPD credible set derived under the assumption of no correlation, we simulated

$P_{\delta=0/y}$	$P_{\tau=1/y}$	$P_{\delta=0/m,\rho,y}$	$P_{\tau=1/m,\rho,y}$
0.0028	0.0272	0.0000	0.0327

Table 2. The unconditional and conditional p-values given m and ρ of H_{01} and H_{02} estimated by a Gibbs sampler algrorithm with 5000 repetitions.

observations from model (1) with $\mu_1 = 0,5$, $\mu_2 = 1,0$, $\sigma_1^2 = 0,5$, $\sigma_2^2 = 1,0$ and $m = 34$ for values of ρ from $-0,7$ to $0,7$ with step equal to $0,2$. It is observed that the p-value taken under the assumption of non corelation $P_{\delta=0/\rho=0,y}$ and $P_{\tau=1/\rho=0,y}$ significantly underestimate the true p-value $P_{\delta=0/y}$ and $P_{\tau=1/y}$ when the correlation is positive, and overestimate it when the correlation is negative (Table 3).

ρ	$P_{\delta=0/y}$	$P_{\tau=1/y}$	$P_{\delta=0/\rho=0,y}$	$P_{\tau=1/\rho=0,y}$
-0.7	8.10^{-13}	0,035	$5,4.10^{-3}$	0,177
-0.5	5.10^{-11}	0,028	$7,9.10^{-5}$	0,253
-0.3	$2,7.10^{-10}$	$0,027$	$1,4.10^{-6}$	0,146
0.0	$9,9.10^{-7}$	0,026	$2,9.10^{-7}$	0,042
0.3	$3,9.10^{-3}$	0,040	$6,5.10^{-9}$	0,014
0.5	$0,0358$	0,040	$2,6.10^{-8}$	$9,5.10^{-3}$
0.7	$0,131$	0,053	$1.0.10^{-8}$	$9,0.10^{-3}$

Table 3. The unconditional and conditional p-values given $\rho = 0$ of H_{01} and H_{02} for different values of ρ estimated by a Gibbs sampler algrorithm with 5000 repetitions.

4. Conclusion

In this paper, we developed a bayesian significance test of change in parameters when the observations are correlated. by numerical studies, we have showed that the bayesian significance test based on the HPD region is sensitive to the correlation in the data.

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Appendice

A. Proof of Theorem

Derivation of the posterior distribution of δ, τ and m:

Transforming the parameter set $\Theta = (m, \phi^{(p)}, \mu_1, \mu_2, r_1, r_2)$ into $\Phi = (m, \phi^{(p)}, \mu_1, \delta, \tau)$, we can form the posterior distribution of Φ; that is,

$$\pi(\Phi/y) = \int_{r_2} \pi(m, \phi^{(p)}, \mu_1, \delta + \mu_1, \tau r_2, r_2/y)|r_2|dr_2$$

$$(14)\quad \begin{aligned} &\propto \tau^{\frac{m}{2}-1} \left\{ \tau \sum_{t=1}^{m} \left[y_t - \mu_1 - \sum_{i=1}^{p} \phi_i (y_{t-i} - \mu_1) \right]^2 \right. \\ &+ \sum_{t=m+1}^{m+p} \left[y_t - \delta - \mu_1 - \sum_{i=1}^{p} \phi_i \Big(y_{t-i} - \gamma_{t-i}\mu_1 - (1-\gamma_{t-i})(\delta + \mu_1) \Big) \right]^2 \\ &\left. + \sum_{t=m+p+1}^{n} \left[y_t - \delta - \mu_1 - \sum_{i=1}^{p} \phi_i (y_{t-i} - \delta - \mu_1) \right]^2 \right\}^{-\frac{n}{2}} \end{aligned}$$

(15)

$$\propto \tau^{\frac{m}{2}-1} \left\{ \tau SS_1(m, \phi^p, \mu_1) + SS_2(m, \phi^p, \mu_1) + a(m, \phi^p) \left(\delta - \widehat{\delta}(m, \phi^p, \mu_1) \right)^2 \right\}^{-\frac{n}{2}}.$$

(i) By application of Bayes theorem, the posterior conditional distribution of δ is obtained as given in (5).

(ii) By integration with respect of δ, we obtained the joint posterior distribution of m, ϕ^p, μ_1 and τ:

$$\pi(m, \phi^p, \mu_1, \tau/y) \propto a(m, \phi^p)^{-\frac{1}{2}} S \left\{ \tau SS_1(m, \phi^p, \mu_1) + SS_2(m, \phi^p, \mu_1) \right\}^{-\frac{n-1}{2}}.$$

(iii) By application of Bayes theorem, given m, ϕ^p and μ_1 the posterior conditional distribution of τ is given as in (7).

(iv) By application of Bayes theorem, given ϕ^p, μ_1 and τ the posterior conditional distribution of m as in (9).

B. Conditional posterior distribution of μ_1 and ϕ^p

Lemma 2. (1) *Given m, ϕ^p and τ the conditional posterior distribution of μ_1 is*

$$(16)\quad \pi\left(\mu_1 | m, \phi^p, \tau, Y\right) \propto \left\{ 1 + \frac{b(m, \phi^p, \tau)\Big(\mu_1 - \widehat{\mu_1}(m, \phi^p, \tau)\Big)^2}{(n-2)S_3^2(m, \phi^p, \tau)} \right\}^{-\frac{(n-2)}{2}},$$

where

$$\widehat{\mu_1}(m, \phi^p, \tau) = \frac{c(m, \phi^p, \tau)}{b(m, \phi^p, \tau)},$$

$$S_3^2(m,\phi^p,\tau) = \frac{1}{(n-2)}\left[d(m,\phi^p) - \frac{c^2(m,\phi^p,\tau)}{b(m,\phi^p,\tau)}\right],$$

with

$$\begin{aligned} b(m,\phi^p,\tau) =\ & \phi^2(p)(n-m+\tau m) \\ & - \frac{\phi(p)}{a(m,\phi^p)}\left[(n-m-p)\phi(p) + \sum_{t=m+1}^{m+p}(1-\sum_1^p \phi_i(1-\gamma_{t-i}))\right]^2, \end{aligned}$$

$$\begin{aligned} d(m,\phi^p) =\ & \tau\sum_1^m (y_t - \sum_{i=1}^p \phi_i y_{t-i})^2 + \sum_{m+1}^n (y_t - \sum_{i=1}^p \phi_i y_{t-i})^2 \\ & - \frac{1}{a(m,\phi^p)}\left[\sum_{m+1}^{m+p}(y_t - \sum_{i=1}^p \phi_i y_{t-i})(1-\sum_{i=1}^p \phi_i(1-\gamma_{t-i}))\right. \\ & \left. + \sum_{m+p+1}^n (y_t - \sum_{i=1}^p \phi_i y_{t_i})\phi(p)\right]^2, \end{aligned}$$

$$\begin{aligned} c(m,\phi^p,\tau) =\ & \tau\phi(p)\sum_1^m (y_t - \sum_{i=1}^p \phi_i y_{t-i}) + \phi(p)\sum_{m+1}^n (y_t - \sum_{i=1}^p \phi_i y_{t-i}) \\ & - \frac{\phi(p)}{a(m,\phi^p)}\left[(n-m-p)\phi(p) + \sum_{m+1}^{m+p}(1-\sum_{i=1}^p \phi_i(1-\gamma_{t-i}))\right] \\ & \times\left[\sum_{m+1}^{m+p}(y_t - \sum_{i=1}^p \phi_i y_{t-i})(1-\sum_{i=1}^p \phi_i(1-\gamma_{t-i})) + \sum_{m+p+1}^n (y_t - \sum_{i=1}^p \phi_i y_{t_i})\phi(p)\right], \end{aligned}$$

which is the Student t distribution with location parameter $\widehat{\mu_1}(m,\phi^p,\tau)$, precision $\frac{b(m,\phi^p,\tau)}{S_3^2(m,\phi^p,\tau)}$, and $(n-2)$ degrees of freedom.

(2) *For $j = 1,\ldots,p$, given m, $\phi^{(j)}$, μ_1, δ and τ, the conditional posterior distribution of ϕ_j is:*

$$\begin{aligned} \pi\left(\phi_j|m,\phi^{(j)},\mu_1,\delta,\tau,Y\right) \propto & \left\{1 + \frac{e(m,\mu_1,\delta,\tau)}{(n-1)S_4^2(m,\phi^{(j)},\mu_1,\delta,\tau)}\right. \\ & \left.\times\left[\phi_j - \widehat{\phi_j}(m,\phi^{(j)},\mu_1,\delta,\tau)\right]^2\right\}^{-\frac{(n-2)}{2}}, \end{aligned} \tag{17}$$

where

$$\widehat{\phi}_j(m,\phi^{(j)},\mu_1,\delta,\tau)=\frac{e_1(m,\phi^{(j)},\mu_1,\delta,\tau)}{e(m,\mu_1,\delta,\tau)},$$

with

$$\begin{aligned}e_1(m,\phi^{(j)},\mu_1,\delta,\tau)=\tau\sum_{t=1}^{m}\Big(y_{t-j}-\mu_1\Big)\Big(y_t-\mu_1-\sum_{i\neq j}(y_{t-i}-\mu_1)\Big)\\ +\sum_{t=m+1}^{m+p}\Big(y_{t-j}-\gamma_{t-j}\mu_1-(1-\gamma_{t-j})(\delta+\mu_1)\Big)\\ \times\Big(y_t-\mu_1-\delta-\sum_{i\neq j}(y_{t-i}-\gamma_{t-i}\mu_1-(1-\gamma_{t-i}(\delta+\mu_1)\Big)\\ +\sum_{m+p+1}^{n}\Big(y_{t-j}-\delta-\mu_1\Big)\Big(y_t-\delta-\mu_1-\sum_{i\neq j}\phi_i(y_{t-i}-\delta-\mu_1)\Big),\end{aligned}$$

and

$$\begin{aligned}e(m,\mu_1,\delta,\tau)=\ \tau\sum_{i=1}^{m}(y_{t-j}-\mu_1)^2+\sum_{m+1}^{m+p}(y_{t-j}-\gamma_{t-j}\mu_1\\ -(1-\gamma_{t-j})(\delta+\mu_1))^2+\sum_{m+p+1}^{n}(y_{t-j}-\delta-\mu_1)^2,\end{aligned}$$

$$\begin{aligned}S_4^2(m,\phi^{(j)},\mu_1,\delta,\tau)=\ \frac{1}{n-1}\Bigg\{\Bigg(y_t-\mu_1-\sum_{i\neq j}(y_{t-i}-\mu_1)\Bigg)^2\\ +\Bigg(y_t-\mu_1-\delta-\sum_{i\neq j}(y_{t-i}-\gamma_{t-i}\mu_1-(1-\gamma_{t-i})(\delta+\mu_1)\Bigg)^2\\ +\Bigg(y_t-\delta-\mu_1-\sum_{i\neq j}\phi_i(y_{t-i}-\delta-\mu_1)\Bigg)^2\Bigg\},\end{aligned}$$

which is the Student t distribution with location parameter $\widehat{\phi}_j(m,\mu_1,\delta,\phi^{(j)})$*, precision* $\frac{e(m,\mu_1,\delta,\tau)}{S_4^2(m,\phi^{(j)},\mu_1,\delta,\tau)}$*, and* $(n-1)$ *degrees of freedom.*

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