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AN EXTENDED PROBLEM TO BERTRAND'S PARADOX

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Abstract

Bertrand's paradox is a longstanding problem within the classical interpretation of probability theory. The solutions 1/2, 1/3, and 1/4 were proposed using three different approaches to model the problem. In this article, an extended problem, of which Bertrand's paradox is a special case, is proposed and solved. For the special case, it is shown that the corresponding solution is 1/3. Moreover, the reasons of inconsistency are discussed and a proper modeling approach is determined by careful examination of the probability space.

Keywords: probability space, probability theory, problem modeling, random chords.

2010 Mathematics Subject Classification: 60A99, 60D99, 97K99, 97G99.

1. Introduction

Bertrand's paradox is a problem discussed originally by Joseph Louis Francois Bertrand (1889) and still generates considerable interest for researchers and among physicists and philosophers, in particular (e.g. Jaynes (1973) and Marinoff (1994)). Bertrand questioned the probability that a random chord in a circle has length exceeding the length of a side of an inscribed equilateral triangle and gave three different solutions based on continuous uniform distribution functions.

The first solution deals with *random endpoints*. A point is chosen at random on the circumference of the circle and a tangent is drawn at this point. The angle between the chord and the tangent is uniformly distributed between zero and π. The desired probability is obtained when this angle exceeds $\pi/3$, but is less than $2\pi/3$. Using the uniform distribution for the angle, the probability is $1/3$.

The second solution is based on a *random radius*. A point is selected at random on the radius of the circle and a chord perpendicular to the radius at the chosen point is constructed. The length of the chord is greater than the side of the triangle if $d < r/2$ where d denotes the distance of the midpoint of the chord from the center. Thus, the probability $1/2$ can be obtained based on the uniform probability density function for d.

The third approach is concerned with modeling the *random midpoint*. This approach considers a random chord by choosing a point at random in the circle and letting this point be the midpoint of the random chord. The length of the chord is longer than the side of the triangle $\sqrt{3}r$ if the chosen point falls within a concentric circle of radius $r/2$. The area of the smaller circle with radius $r/2$ is one fourth the area of the larger circle resulting in the probability of $1/4$.

These aforementioned solutions are obtained by assigning uniform probability density to (A) angles of intersections of the chord on the circumference, (B) the linear distance between centers of the chord and circle, (C) the center of the chord over the interior area of the circle. According to Jaynes (2003), these assignments lead to the probabilities of $1/3$, $1/2$, and $1/4$, respectively. Bertrand's paradox is considered a paradox because it is believed that "the uniform random choice" should uniquely determine the desired probability (Szekely 1986). Many researchers explain that the dissimilar solutions are due to the different definitions of random chords, and consequently, they have developed various principles based upon it (e.g. Gardner (1959), Basano and Ottonello (1996), Holbrook and Kim (2000), Chiu and Larson (2009)). Bertrand's paradox is not really a paradox since different solutions are based upon different methods of modeling the problem. This paper investigates the probability space of these modeling approaches to disentangle the inconsistency and settle on an appropriate approach. A probability space consists of three parts.

1) A *sample space* which is the set of all possible outcomes.
2) A set of *events* where each event is a set containing zero or more outcomes and the collection of these events is a σ-algebra.
3) A *function* assigning probabilities to the events.

The paper is organized as follows. In Section 2, the extended problem is posed and solved. In Section 3, a more general problem of Bertrand's paradox is discussed and inconsistencies of the approaches are addressed. Also, approaches are compared in light of the extended problem to examine which probability space can model the problem properly as well as the reasons for incompatible solutions. Finally, concluding remarks are provided in Section 4.

2. Extended problem

What is the probability p that a random chord through a given point N at distance d from the center of a circle with radius r is longer than $\sqrt{a}r$ for $0 \leq a \leq 4$?

First, the definition of a random chord needs to be stated clearly. A random chord can be generated by its two endpoints on the circumference, or equivalently, by its midpoint in the open disk. In other words, two points are selected on the circumference randomly and the line connecting these two points is the random chord. Alternatively, a point is chosen on the disk randomly and two points on the circumference having the same distance from it are considered as the endpoints of the random chord. These two methods of generating random chords, which are applied in the *random endpoints* and the *random midpoint* approaches, are equivalent since each chord has a unique midpoint. This is affirmed later in equations (5) and (10).

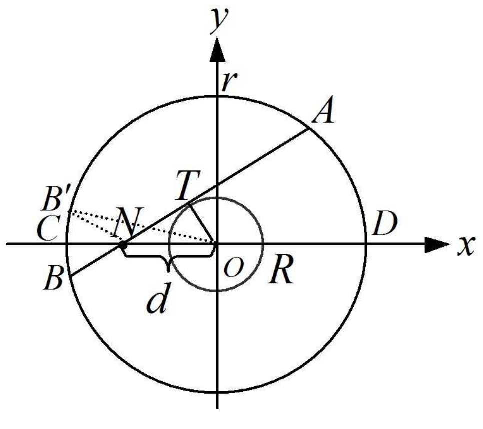

Figure 1. The extended problem of Bertrand's paradox.

In Figure 1, consider $N(x_N, y_N)$ as a random point with distance d from the origin point, O, or the center of the circle with radius r. We should find the radius of a concentric circle, R, such that if N lies inside it, all chords passing through N are longer than $\sqrt{a}r$. The radius R can be obtained for given values of a and r by considering two random points $A(x_1, y_1)$ and $B(x_2, y_2)$ located on the circumference of the circle. The distance between A and B is $\sqrt{a}r$. In addition, assume that T, the midpoint of AB, is located on the circumference of the circle with radius R. Thus,

(1)
$$\left.\begin{array}{l}(x_1 - x_2)^2 + (y_1 - y_2)^2 = ar^2 \\ ((x_1 + x_2)/2)^2 + ((y_1 + y_2)/2)^2 = R^2\end{array}\right\} \Rightarrow 4r^2 = ar^2 + 4R^2 \Rightarrow R = r\sqrt{1 - a/4}.$$

Therefore, if $0 \leq d < r\sqrt{1 - a/4}$, then all chords passing through the point N are longer than $\sqrt{a}r$; and if $r\sqrt{1 - a/4} \leq d \leq r$, then the desired probability can be obtained as follows. In Figure 1, countless random chords can pass through N, but among all points on the circumference, only points located on $\overset{\frown}{AD}$ and $\overset{\frown}{BC}$ have the potential to generate a chord longer than $\sqrt{a}r$. Note that one of the two tangent points, T, is shown in Figure 1. Thus, the desired probability is equal to $p = (2\pi r)^{-1}(2r(\angle AOD + \angle B'OC))$ where B' is the reflection of B across the horizontal axis. An exterior angle of a triangle is equal to the sum of opposite interior angles. Now,

$$\begin{aligned}\angle AOD &= \angle TNO + \angle TAO = \text{Arcsin}(R/d) + \text{Arcsin}(R/r), \\ \angle B'OC &= \angle B'NC - \angle OB'N = \angle TNO - \angle OBT \\ &= \text{Arcsin}(R/d) - \text{Arcsin}(R/r).\end{aligned} \tag{2}$$

Thus,

$$p = \pi^{-1}(\angle AOD + \angle B'OC) = 2\pi^{-1}\text{Arcsin}(R/d). \tag{3}$$

Now, let E represent the event that the length of the random chord passing through a point with distance d from the center of the circle with radius r is longer than $\sqrt{a}r$. By substituting Equation (1) into Equation (3), the probability is expressed as

$$P(E) = \begin{cases} 1, & 0 \leq d < r\sqrt{1 - a/4} \\ 2\pi^{-1}\text{Arcsin}\left(r\sqrt{1 - a/4}\Big/d\right), & r\sqrt{1 - a/4} \leq d \leq r. \end{cases} \tag{4}$$

Figure 2 illustrates the probability in Equation (4). The values of $P(E)$ are shown over the closed disk. The figure demonstrates that each point on the closed disk does not have an equal chance to be a part of a chord that is greater than $\sqrt{a}r$. In other words, the location of that point determines what percentage of chords passing through it can be greater than $\sqrt{a}r$. For example, all chords passing through the center of the circle are greater than $\sqrt{a}r$. This characteristic holds for all points of the concentric disk with radius R. Now, suppose that a point lies within disks with radii R and r. The closer this point is to the outer circumference, the less potential it has for being a chord longer than $\sqrt{a}r$.

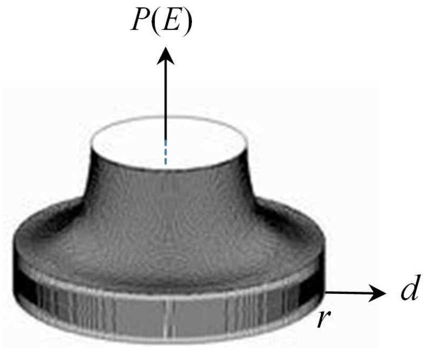

Figure 2. The probabilities in Equation (4) over the closed disk.

The question deliberated by Bertrand can be answered by substituting $d = r$ and $a = 3$ into Equation (4). In other words, this special case of the extended problem gives the desired probability for points on the circumference. Since all points on the circumference have similar probability of 1/3, as shown in Figure 2 and Equation (4), the solution to Bertrand's problem is 1/3.

3. Comparisons of modeling approaches

In this section, a generalization of Bertrand's Paradox is developed in light of the extended problem. The generalization is helpful to comprehend its special case, and to identify the appropriate probability space for representing the problem. The discussions and comparisons in this section also provide insight about the merits and the deficiencies of the approaches.

What is the probability that a random chord in a circle with radius r is longer than $\sqrt{a}r$ for $0 \leq a \leq 4$?

Bertrand's paradox is a special case of this problem when $a = 3$. Applying the random endpoints approach leads to $p = 2\pi^{-1}\text{Arcsin}(\sqrt{1 - a/4})$ for this generalization. This solution is obtained later in Equation (7). We obtained $p = r\sqrt{1 - a/4}/r = \sqrt{1 - a/4}$ and $p = \pi(r\sqrt{1 - a/4})^2/\pi r^2 = 1 - a/4$ as the desired probabilities based on the random radius approach and the random midpoint approach, respectively. All three solutions are obtained according to the similar probability spaces assumed in Bertrand's paradox.

Like Bertrand's Paradox, the generalization of Bertrand's Paradox is a perplexing problem unless the components of the probability space are clearly delineated. The different answers are due to different probability spaces. Thus, at least one component of the probability space, namely the sample space, event, or function, is not consistent among the approaches. Note that the discrepancy of solutions is not due to the vague definition of random chords. In fact, it is caused by an inappropriate sample space that cannot represent random chords or a function that does not assign probabilities to the events adequately. Table 1 provides the principal elements of the probability space for the three approaches used in the generalization of Bertrand's paradox.

Now, let us investigate the probability space of the modeling approaches in the following subsections.

	Random Endpoints	**Random Radius**	**Random Midpoint**
Sample Space	Circumference of the circle.	A radius of the circle.	The closed disk.
Event	Given either endpoints of a chord as the vertex of an inscribed isosceles triangle with sides of $\sqrt{a}r$, the other endpoint lies on the intercepted arc; or equivalently, the Euclidean distance between the two endpoints is greater than $\sqrt{a}r$.	The Eucliden distance between the midpoint of a chord and the center of the circle is less than $r\sqrt{1-a/4}$.	The midpoint of a chord lies on a concentric disk with radius $r\sqrt{1-a/4}$.
Function	Uniform distribution on the circumference of the circle.	Uniform distribution along the radius.	Uniform distribution over the closed disk.
Solution	$2\pi^{-1}\text{Arcsin}(\sqrt{1-a/4})$	$\sqrt{1-a/4}$	$1-a/4$

Table 1. Probability space elements for the three approaches.

3.1. Random endpoints probability space

First, the sample space needs to be investigated in order to verify if the set covers all possible outcomes. Here, these outcomes correspond to random chords.

Clearly, in the random endpoints approach, two random points on the circumference can be considered as the endpoints of a random chord. Thus, the sample space generates all possible random chords. Also, the event based on Euclidean distance between two endpoints is a well defined criterion to find out whether the length of the chord is larger than $\sqrt{a}r$. The following equation is based on the sample space and the event defined in the endpoints approach where

$$\begin{aligned} P_{d_{AB}}(d_{AB} > \sqrt{a}r) &= P_{x_1,x_2,y_1,y_2}((x_1 - x_2)^2 + (y_1 - y_2)^2 > ar^2) \\ &= P_{x_1,x_2,y_1,y_2}(x_1x_2 + y_1y_2 < (1 - a/2)r^2). \end{aligned} \tag{5}$$

Using polar coordinates, Equation (5) gives

$$P_{0\le\alpha,\beta\le 2\pi}(\cos(\beta - \alpha) < 1 - a/2). \tag{6}$$

If we associate probabilities to the event correctly, the desired probability of $2\pi^{-1}\text{Arcsin}(\sqrt{1 - a/4})$ will be obtained as follows. The hatched area in Figure 3 illustrates the feasible region of Equation (6).

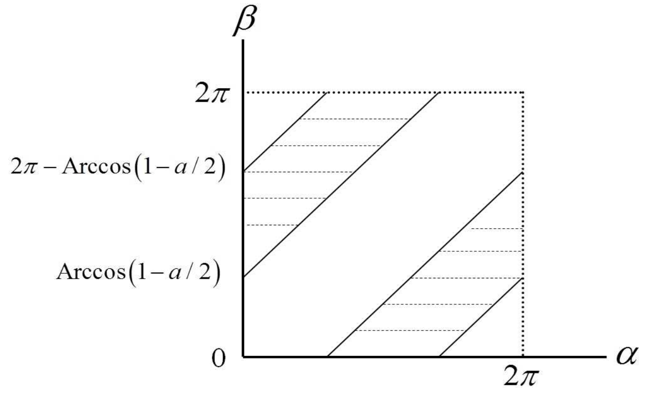

Figure 3. The feasible region of $P_{0\le\alpha,\beta\le 2\pi}(\cos(\beta - \alpha) < 1 - a/2)$.

The area of the isosceles trapezoids is equal to $4\pi(\pi-\text{Arccos}(1 - a/2))$ and dividing by $(2\pi)^2$ gives $\text{Arccos}(a/2 - 1)/\pi = 2\pi^{-1}\text{Arcsin}(\sqrt{1 - a/4})$. Note that $\phi = \text{Arcsin}(\sqrt{1 - a/4}))$ is plugged into $\cos(2\phi) = 1 - 2\sin^2(\phi)$ to obtain

$$P_{0\le\alpha,\beta\le 2\pi}(\cos(\beta - \alpha) < 1 - a/2) = 2\pi^{-1}\text{Arcsin}(\sqrt{1 - a/4}). \tag{7}$$

The probability can also be obtained by substituting $d = r$ in Equation (4). Alternatively, assume that an isosceles triangle with sides of $\sqrt{a}r$ is inscribed in a circle with radius r. The measure of the inscribed angle is $2\text{Arcsin}(\sqrt{(2r)^2 - (\sqrt{a}r)^2}/2r) = \text{Arcsin}(\sqrt{1 - a/4})$. Also, the measure of the intercepted arc is twice that of the inscribed angle. Thus, the desired probability is $2(2\text{Arcsin}(\sqrt{1 - a/4}))/2\pi = 2\pi^{-1}\text{Arcsin}(\sqrt{1 - a/4})$.

3.2. Random radius probability space

Now, consider the random radius approach. It should be asked whether the sample space consisting of radii of the circle could represent all random chords. The points constructing the chosen radius are midpoints of chords. Figure 4 shows that countless random chords can pass through a given point N located on the radius. The figure indicates that the perpendicular chord to the radius is one realization of countless random chords passing through N.

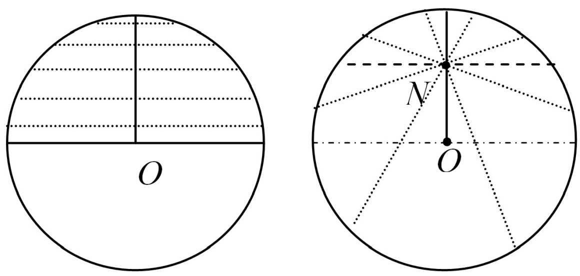

Figure 4. A given point N on a radius with length r.

The points on the radius are loci of projected endpoints of parallel chords from the circumference. In other words, a point on the radius can only represent a chord perpendicular to the radius. Consequently, the chosen radius cannot be considered as a proper sample space to represent random chords. Therefore, in the random radius approach, the probability of $\sqrt{1 - a/4}$ is obtained based upon the assumption that chords are perpendicular to the chosen radius in a semicircle. As a result, we should infer that $\sqrt{1 - a/4}$ of random chords that are perpendicular to the chosen radius are larger than $\sqrt{a}r$. If chords are drawn randomly in the semicircle, the following probability is obtained. Suppose two random endpoints lie in quadrants 1 and 2. If the random endpoints approach is applied with consideration of the circumference of the upper semicircle as the sample space, the probability that the length of a random chord is greater than $\sqrt{a}r$ is illustrated in Figure 5.

Figure 5 displays the feasible region for $0 \leq a \leq 2$. Similarly, the region can be found for $2 \leq a \leq 4$ by reducing the hatched region toward the illustrated direction. Equation (8) gives the desired probability as

(8)

$$P_{\substack{0\le\alpha\le\pi/2,\\ \pi/2\le\beta\le\pi}}(\cos(\beta-\alpha)<1-a/2)=\begin{cases}1-2(\pi^{-1}\text{Arccos}(1-a/2))^2, & 0\le a\le 2\\ 2(\pi^{-1}\text{Arccos}(a/2-1))^2, & 2\le a\le 4.\end{cases}$$

If the sample space changes to the circle instead of the upper semicircle, Equation (4) holds assuming $d=r$. In the random radius approach, since the radius does not generate all random chords, it cannot be considered a suitable sample space.

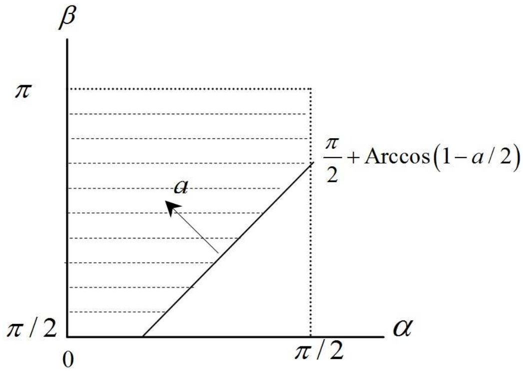

Figure 5. The feasible region of $P_{0\le\alpha\le\pi/2,\pi/2\le\beta\le\pi}(\cos(\beta-\alpha)<1-a/2)$.

As another example, the following situation demonstrates the importance of establishing a correct probability space to arrive at a valid solution. Let us consider the sample space of the random radius approach for the random endpoints. Assume $x_2=-x_1$ and $y_2=y_1$, or $\beta=\pi-\alpha$ in polar coordinates, for the random endpoints approach. It stipulates that one of the endpoints is selected randomly on the circumference of the upper semicircle, but the other endpoint is the first point reflected over the vertical axis. The following solution can be obtained using Equation (6) and the perpendicularity assumption

$$\begin{aligned}P_{\substack{0\le\alpha\le\pi/2,\\ \pi/2\le\beta\le\pi}}(\cos(\beta-\alpha)<1-a/2) &= P_{0\le\alpha\le\pi/2}(-\cos(2\alpha)<1-a/2)\\ &= 2\pi^{-1}\text{Arcsin}(\sqrt{1-a/4}).\end{aligned} \tag{9}$$

Note that in Equation (9) the feasible region on the circumference is identical to that of the random endpoints approach when the first random point is set at $(-r,0)$. Although the answer is identical to Equation (7), the solution is not acceptable since the sample space does not represent random chords appositely.

3.3. Random midpoint probability space

Now, consider the random midpoint approach leading to $1 - a/4$. Again, the sample space needs to be checked to see if it can represent all random chords. The sample space consists of all points forming the closed disk. Suppose in Figure 4, point N is the midpoint of a chord, but this time N is not limited to the chosen radius and can be located anywhere on the closed disk. Since a chord has a unique midpoint, any point on the closed disk corresponds to two endpoints on the circumference and, as a result, it represents a random chord. The endpoints of chords are distributed on the circumference randomly similar to the random endpoints. Thus, the sample space is well posed and represents all possible random chords. All points in a concentric open disk with radius $r\sqrt{1-a/4}$ produce the desired events. A random chord is greater than $\sqrt{a}r$ if the midpoint of the chord, namely T in Figure 1, lies inside of the concentric open disk with radius $r\sqrt{1-a/4}$. This indicates that the event is also defined properly. The following equation shows that the sample space and event are well established and are in accordance with the random endpoints

$$\begin{aligned}
&P_{OT}(OT < r\sqrt{1-a/4}) \\
&= P_{x_1,x_2,y_1,y_2}(((x_1+x_2)/2)^2 + ((y_1+y_2)/2)^2 < r^2(1-a/4)) \\
&= P_{x_1,x_2,y_1,y_2}(x_1x_2 + y_1y_2 < (1-a/2)r^2) \\
&= P_{0\le\alpha,\beta\le 2\pi}(\cos(\beta-\alpha) < 1-a/2)\,.
\end{aligned} \tag{10}$$

In order to calculate the desired probability, the random midpoint approach uses the desired area which is the area of the disk with radius $r\sqrt{1-a/4}$ divided by the total area of the closed disk with radius r. This formula assumes that the random midpoints are distributed uniformly over the closed disk. Is it appropriate to assign a uniform distribution to the points of the closed disk? If this assumption is true, we would expect a flat cylinder in Figure 2 and a uniform distribution function in Equation (4). Points constructing the closed disk are not merely midpoints of random chords, but they are also part of other chords. In other words, a point on the disk can be a midpoint of a chord or part of other chords which indicates that countless chords can pass through the point and only one of the chords has equidistance from the circumference. The extended problem has helped to clarify this statement. If, in the random midpoint approach, the function assigns probabilities to the events properly, then the desired probability of $2\pi^{-1}\text{Arcsin}(\sqrt{1-a/4})$ is obtained as shown in Equation (7).

There is one explanation for the solution $1 - a/4$. It should not be assumed that $1 - a/4$ of random chords are larger than $\sqrt{a}r$. Rather, it should be said that $1-a/4$ of points on the closed disk are part of random chords that are larger than $\sqrt{a}r$ since the midpoints of all these chords construct the open disk with radius $r\sqrt{1-a/4}$. The other $a/4$ points are part of random chords that may or may not be larger than $\sqrt{a}r$. This possibility can be determined by Equation (4).

4. Concluding remarks

Bertrand obtained three different solutions to his proposed problem using different probability spaces. An extended problem of Bertrand's paradox is developed. Also, comments are provided for the generalization of Bertrand's paradox according to the probability space elements. Like Bertrand's Paradox, the generalization of Bertrand's Paradox can be considered a perplexing problem due to defining an inappropriate sample space and probability function in the random radius approach and the random midpoint approach.

In the random radius approach, the sample space is not defined properly for the problem. The major flaw for this approach is that chords are not drawn completely at random in the semicircle. The perpendicularity constraint for the chosen radius is an additional assumption preventing the sample space from generating all random chords. The issue with the random midpoint approach is that although the sample space and the event are valid, the function does not suitably assign probabilities to the event. As a result, it causes deficiency in constructing a proper probability space.

The authors advocate the random endpoints approach leading to the probability of $2\pi^{-1}\text{Arcsin}(\sqrt{1-a/4})$ or $1/3$ for Bertrand's paradox since its probability space is credible and aptly models the problem. In a broad perspective, this paradox serves as a warning to researchers to be careful with assumptions when modeling problems.

References

[1] L. Basano and P. Ottonello, *The ambiguity of random choices: probability paradoxes in some physical processes*, Amer. J. Phys. **64** (1996) 34–39. doi:10.1119/1.18289

[2] J.L.F. Bertrand, Calcul des Probabilities (Gauthier-Villars, Paris, 1889).

[3] S.S. Chiu and R.C. Larson, *Bertrand's paradox revisited: more lessons about that ambiguous word, random*, Journal of Industrial and Systems Engineering **3** (2009) 1–26.

[4] M. Gardiner, *Mathematical games: problems involving questions of probability and ambiguity*, Scientific American **201** (1959) 174–182.

[5] J. Holbrook and S.S. Kim, *Bertrand's paradox revisited*, Mathematical Intelligencer **22** (2000) 16–19.

[6] E.T. Jaynes, Probability Theory: The Logic of Science (Cambridge University Press, New York, 2003).

[7] E.T. Jaynes, *Well-posed problem*, Foundations of Physics **3** (1973) 477–493.

[8] L. Marinoff, *A resolution of Bertrand's paradox*, Philosophy of Science **61** (1994) 1–24. doi:10.1086/289777

[9] G.J. Szekely, Paradoxes in Probability Theory and Mathematical Statistics (Kluwer Academic Publishers, New York, 1986).

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