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GLOBAL APPROXIMATIONS FOR THE γ-ORDER LOGNORMAL DISTRIBUTION

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Abstract

A generalized form of the usual Lognormal distribution, denoted with $\mathcal{LN}_\gamma$, is introduced through the γ-order Normal distribution $\mathcal{N}_\gamma$, with its p.d.f. defined into $(0, +\infty)$. The study of the c.d.f. of $\mathcal{LN}_\gamma$ is focused on a heuristic method that provides global approximations with two anchor points, at zero and at infinity. Also evaluations are provided while certain bounds are obtained.

Keywords: cumulative distribution function, γ-order Lognormal distribution, global Padé approximation.

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1. Introduction

The p-variate γ-order Normal distribution, denoted by $\mathcal{N}_\gamma^p(\mu, \Sigma)$, is an multivariate exponential-power generalization of the usual Normal distribution, constructed to play the role of the usual Normal distribution for the generalized Fisher's entropy type information measure, see [7] for details. Recall that the density function f_X of a γ-order normally distributed random variable $X \sim \mathcal{N}_\gamma^p(\mu, \Sigma)$, with location vector $\mu \in \mathbb{R}^{1\times p}$, positive definite scale matrix $\Sigma \in \mathbb{R}^{p\times p}$ and shape parameter $\gamma \in \mathbb{R} \setminus [0, 1]$ is given by, [7],

$$(1)\quad f_X(x) = f_X(x;\ \mu, \Sigma, \gamma) := C_\gamma^p \left|\det \Sigma\right|^{-\frac{1}{2}} \exp\left\{-\tfrac{\gamma-1}{\gamma} Q_\theta(x)^{\frac{\gamma}{2(\gamma-1)}}\right\},\ x \in \mathbb{R}^{1\times p},$$

where the quadratic form $Q_\theta(x) = (x-\mu)^{\mathrm{T}}\,\Sigma^{-1}(x-\mu)$, $\theta = (\mu, \Sigma)$ while C^p_γ being the normalizing factor

$$C^p_\gamma := \pi^{-p/2}\frac{\Gamma(\frac{p}{2}+1)}{\Gamma(p\frac{\gamma-1}{\gamma})}(\tfrac{\gamma-1}{\gamma})^{p\frac{\gamma-1}{\gamma}-1}. \tag{2}$$

The location parameter $\mu \in \mathbb{R}^{1\times p}$ is in fact the mean vector of X_γ, i.e. $\mu = \mathrm{E}(X)$. Notice also that the second-ordered Normal is the known multivariate normal distribution, i.e., $\mathcal{N}^p_2(\mu, \Sigma) = \mathcal{N}^p(\mu, \Sigma)$. Moreover, for $\gamma \to 1^+, \pm\infty$ or $\gamma \to \pm\infty$ the $\mathcal{N}^1_\gamma(\mu, \sigma^2)$ converges, respectively, to the Uniform $\mathcal{U}(\mu - \sigma, \mu + \sigma)$ and the Laplace $\mathcal{L}(\mu, \sigma)$ distribution, while for $\gamma \to 0^-$, $\mathcal{N}^1_\gamma(\mu, \sigma^2)$ converges to the degenerate Dirac $\mathcal{D}(\mu)$ distribution with pole at $\mu \in \mathbb{R}$. Therefore, the shape parameter γ can be extended to be $\gamma \in \mathbb{R} \cup \{\pm\infty\} \setminus [0, 1]$ and thus the γ-order Normal family of distributions include four significant type of distributions such as the Uniform, Normal, Laplace and Dirac. For a comprehensive study of the $\mathcal{N}_\gamma$ family see [9, 8].

Now, the Lognormal distribution has been widely applied in many different aspects of life sciences, including Biology, Ecology, Geology and Meteorology as well as in Economics, Finance and Risk Analysis, see [4]. Also, it plays an important role in Astrophysics and Cosmology, see [2, 3] among others.

In principle, the Lognormal distribution is defined as the distribution of a random variable whose logarithm is normally distributed, and usually is formulated with two parameters. Furthermore, Log-Uniform and Log-Laplace distributions can be similarly defined with applications in Finance, see [11]. Especially, the power-tail phenomenon of the Log-Laplace distributions [10] attracts attention quite often in Environmental Sciences, Physics, Economics.

The Lognormal distribution can be easily extended to the γ-order Lognormal distribution, denoted here by $\mathcal{LN}_\gamma(\mu, \sigma)$, in the sense that if $X \sim \mathcal{N}^1_\gamma(\mu, \sigma^2)$ then $Y = e^X$ will follow the $\mathcal{LN}_\gamma(\mu, \sigma)$, and the p.d.f. of X_γ is then given by

$$f_Y(y) := \tfrac{1}{y} f_X(\log y) = C^1_\gamma \sigma y^{-1} \exp\left\{-\tfrac{\gamma-1}{\gamma}\left|\tfrac{\log y-\mu}{\sigma}\right|^{\frac{\gamma}{\gamma-1}}\right\}, \quad y \in \mathbb{R}^*_+, \tag{3}$$

while $\log Y \sim \mathcal{N}_\gamma(\mu, \sigma^2)$.

Notice that, for $\gamma = 2$, $\mathcal{LN}_2(\mu, \sigma)$ is reduced to the well known Lognormal distribution. Moreover, for the extended shape parameter $\gamma \in \mathbb{R} \cup \{\pm\infty\} \setminus [0, 1]$ the first-ordered $\mathcal{LN}_1(\mu, \sigma)$ coincides with the Log-Uniform distribution $\mathcal{LU}(e^{\mu-\sigma}, e^{\mu+\sigma})$, while the infinity-ordered $\mathcal{LN}_{\pm\infty}(\mu, \sigma)$ coincides with the known (symmetric) Log-Laplace distribution $\mathcal{LL}(e^\mu, 1/\sigma, 1\sigma)$, see [13].

In this paper the cumulative distribution function (c.d.f) of the γ-order lognormally distributed $e^X \sim \mathcal{LN}_\gamma(\mu, \sigma)$, with $X \sim \mathcal{N}_\gamma(\mu, \sigma^2)$, is derived, uniformly approximated and bounded.

2. THE C.D.F. OF THE $\mathcal{LN}_\gamma$ DISTRIBUTION

The generalized error function that briefly discussed here, plays an important role to the development of c.d.f. of the $\mathcal{LN}_\gamma$. The generalized error function, denoted by Erf_a, [6], is defined as

$$\mathrm{Erf}_a(x) := \tfrac{\Gamma(a+1)}{\sqrt{\pi}} \int_0^x e^{-t^a}\,dt, \quad x \in \mathbb{R}, \quad a \geq 0, \tag{4}$$

while the generalized complementary error function $\mathrm{Erfc}_a = 1 - \mathrm{Erf}_a$, $a \geq 0$. The generalized error function, can be expressed (by changing to variable t^a) through the lower incomplete gamma function $\gamma(a, x)$ or the upper (complementary) incomplete gamma function $\Gamma(a, x) = \Gamma(a) - \gamma(a, x)$, as

$$\mathrm{Erf}_a(x) = \tfrac{\Gamma(a)}{\sqrt{\pi}}\gamma\left(\tfrac{1}{a}, x^a\right) = \tfrac{\Gamma(a)}{\sqrt{\pi}}\left[\Gamma\left(\tfrac{1}{a}\right) - \Gamma\left(\tfrac{1}{a}, x^a\right)\right], \quad x \in \mathbb{R}, \quad a \geq 0, \tag{5}$$

see [6]. Moreover, adopting the series expansion form of the lower incomplete gamma function,

$$\gamma(a, x) := \int_0^x t^{a-1}e^{-t}dt = \sum_{k=0}^{\infty} \tfrac{(-1)^k}{k!(a+k)} x^{a+k}, \quad x, a \in \mathbb{R}_+, \tag{6}$$

a series expansion form of the generalized error function can be extracted, i.e.

$$\mathrm{Erf}_a(x) = \tfrac{\Gamma(a+1)}{\sqrt{\pi}} \sum_{k=0}^{\infty} \tfrac{(-1)^k}{k!(ka+1)} x^{ka+1}, \quad x, a \in \mathbb{R}_+. \tag{7}$$

Notice that, Erf_2 is the known error function erf, i.e., $\mathrm{Erf}_2(x) = \mathrm{erf}(x)$, while Erf_0 is the function of a straight line through the origin with slope $(e\sqrt{\pi})^{-1}$. Applying $a = 2$, the known incomplete gamma function identities such as $\gamma(1/2, x) = \sqrt{\pi}\,\mathrm{erf}\,\sqrt{x}$, and $\Gamma(1/2, x) = \sqrt{\pi}(1 - \mathrm{erf}\,\sqrt{x}) = \sqrt{\pi}\,\mathrm{erfc}\,\sqrt{x}$, $x \geq 0$ is obtained. Moreover, while $\mathrm{Erf}_a\, 0 = 0$ for all $a \in \mathbb{R}_+$. and

$$\lim_{x\to\pm\infty} \mathrm{Erf}_a\, x = \pm\tfrac{1}{\sqrt{\pi}}\,\Gamma(a)\,\Gamma\left(\tfrac{1}{a}\right), \quad a \in \mathbb{R}_+,$$

as $\gamma(a, x) \to \Gamma(a)$ when $x \to +\infty$.

For the evaluation of the cumulative distribution function of the generalized Lognormal distribution, we state and prove the following.

Theorem 1. *The c.d.f.* F_{X_γ} *of a* γ*-order Lognormal random variable* $X_\gamma \sim \mathcal{LN}_\gamma(\mu, \sigma)$ *is given by*

$$F_{X_\gamma}(x) = \tfrac{1}{2} + \frac{\sqrt{\pi}}{2\,\Gamma(\frac{\gamma-1}{\gamma})\,\Gamma(\frac{\gamma}{\gamma-1})}\,\mathrm{Erf}_{\frac{\gamma}{\gamma-1}}\left\{\left(\tfrac{\gamma-1}{\gamma}\right)^{\frac{\gamma-1}{\gamma}}\tfrac{\log x-\mu}{\sigma}\right\} \tag{8}$$

$$= 1 - \frac{1}{2\,\Gamma(\frac{\gamma-1}{\gamma})}\,\Gamma\left(\tfrac{\gamma-1}{\gamma}, \tfrac{\gamma-1}{\gamma}\left(\tfrac{\log x-\mu}{\sigma}\right)^{\frac{\gamma}{\gamma-1}}\right), \quad x \in \mathbb{R}_+^*. \tag{9}$$

Proof. From density function $f_{X\gamma}$, as in (3), we have

$$F_{X_\gamma}(x) = \int_0^x f_{X_\gamma}(t)dt = \sigma^{-1}C_\gamma^1 \int_0^x t^{-1}\exp\left\{-\tfrac{\gamma-1}{\gamma}\left|\tfrac{\log t-\mu}{\sigma}\right|^{\frac{\gamma}{\gamma-1}}\right\}dt.$$

Applying the transformation $w = \frac{\log t-\mu}{\sigma}$, $t>0$, the above c.d.f. is reduced to

$$F_{X_\gamma}(x) = C_\gamma^1 \int_{-\infty}^{\frac{\log x-\mu}{\sigma}} \exp\left\{-\tfrac{\gamma-1}{\gamma}|w|^{\frac{\gamma}{\gamma-1}}\right\}dw = \Phi_{Z_\gamma}\left(\tfrac{\log x-\mu}{\sigma}\right), \tag{10}$$

where Φ_{Z_γ} is the c.d.f. of the standardized γ-order Normal distribution $Z_\gamma = \frac{1}{\sigma}(\log X_\gamma - \mu) \sim \mathcal{N}_\gamma(0,1)$. Moreover, Φ_{Z_γ} can be expressed in terms of the generalized error function. In particular

$$\Phi_{Z_\gamma}(z) = C_\gamma^1 \int_{-\infty}^{z} \exp\left\{-\tfrac{\gamma-1}{\gamma}|w|^{\frac{\gamma}{\gamma-1}}\right\}dw = \Phi_{Z_\gamma}(0) + C_\gamma^1 \int_0^z \exp\left\{-\tfrac{\gamma-1}{\gamma}|w|^{\frac{\gamma}{\gamma-1}}\right\}dw,$$

and as f_{Z_γ} is a symmetric density function around zero, we have

$$\Phi_{Z_\gamma}(z) = \tfrac{1}{2}+C_\gamma^1\int_0^z \exp\left\{-\tfrac{\gamma-1}{\gamma}|w|^{\frac{\gamma}{\gamma-1}}\right\}dw = \tfrac{1}{2}+C_\gamma^1\int_0^z \exp\left\{-\left|\left(\tfrac{\gamma-1}{\gamma}\right)^{\frac{\gamma-1}{\gamma}} w\right|^{\frac{\gamma}{\gamma-1}}\right\}dw,$$

and thus

$$\Phi_{Z_\gamma}(z) = \tfrac{1}{2} + C_\gamma^1 \left(\tfrac{\gamma}{\gamma-1}\right)^{\frac{\gamma-1}{\gamma}} \int_0^{\left(\frac{\gamma-1}{\gamma}\right)^{\frac{\gamma-1}{\gamma}} z} \exp\left\{-u^{\frac{\gamma}{\gamma-1}}\right\}du. \tag{11}$$

Substituting the normalizing factor, as in (2), and using (4) we obtain

$$\Phi_{Z_\gamma}(z) = \tfrac{1}{2} + \frac{\sqrt{\pi}}{2\,\Gamma(\frac{\gamma-1}{\gamma}+1)\,\Gamma(\frac{2\gamma-1}{\gamma-1})}\operatorname{Erf}_{\frac{\gamma}{\gamma-1}}\left\{\left(\tfrac{\gamma-1}{\gamma}\right)^{\frac{\gamma-1}{\gamma}} z\right\}, \quad z\in\mathbb{R}, \tag{12}$$

and finally, through (10), we derive (8), which forms (9) through (5). ∎

Notice that the (non log-scaled) location parameter e^μ is in fact the median for all generalized lognormally distributed $X_\gamma \sim \mathcal{LN}_\gamma(\mu,\sigma)$. Specifically, through (8) and the fact that $\operatorname{Erf}_a 0 = 0$, $a\in\mathbb{R}_+^*$, it holds that $\operatorname{Med} X_\gamma = F_{X_\gamma}^{-1}(1/2) = e^\mu$, i.e., $\operatorname{Med} X_\gamma$ is a γ-invariant location measure.

It is essential for numeric calculations to express (8) considering positive arguments for Erf. Indeed, through (11), we obtain

$$F_{X_\gamma}(x) = \tfrac{1}{2} + \frac{\operatorname{sgn}(\log x - \mu)\sqrt{\pi}}{2\,\Gamma(\frac{\gamma-1}{\gamma})\,\Gamma(\frac{\gamma}{\gamma-1})}\operatorname{Erf}_{\frac{\gamma}{\gamma-1}}\left\{(\tfrac{\gamma-1}{\gamma})^{\frac{\gamma-1}{\gamma}}\left|\tfrac{\log x-\mu}{\sigma}\right|\right\}, \tag{13}$$

while applying (5) into (13) we obtain

$$F_{X_\gamma}(x) = \tfrac{1+\operatorname{sgn}(\log x-\mu)}{2} - \frac{\operatorname{sgn}(\log x-\mu)}{2\,\Gamma(\frac{\gamma-1}{\gamma})}\,\Gamma\left(\tfrac{\gamma-1}{\gamma}, \tfrac{\gamma-1}{\gamma}\left|\tfrac{\log x-\mu}{\sigma}\right|^{\frac{\gamma}{\gamma-1}}\right). \tag{14}$$

Letting $Z_\gamma := \log X_\gamma \sim \mathcal{N}_\gamma(\mu, \sigma^2)$ where $X_\gamma \sim \mathcal{LN}_\gamma(\mu, \sigma)$, we have, through (10), that

$$F_{Z_\gamma}(z) = F_{\log X_\gamma}(z) = F_{X_\gamma}(e^z).$$

Therefore, through Theorem 1 the following holds.

Corollary 2. *The c.d.f.* F_{Z_γ} *of a* γ*-order normally distributed random variable* $Z_\gamma \sim \mathcal{N}_\gamma(\mu, \sigma^2)$ *is given by*

$$F_{Z_\gamma}(z) = \tfrac{1}{2} + \frac{\sqrt{\pi}}{2\,\Gamma(\frac{\gamma-1}{\gamma})\,\Gamma(\frac{\gamma}{\gamma-1})}\operatorname{Erf}_{\frac{\gamma}{\gamma-1}}\left\{(\tfrac{\gamma-1}{\gamma})^{\frac{\gamma-1}{\gamma}}\tfrac{z-\mu}{\sigma}\right\} \tag{15}$$

$$= 1 - \frac{1}{2\,\Gamma(\frac{\gamma-1}{\gamma})}\,\Gamma\left(\tfrac{\gamma-1}{\gamma}, \tfrac{\gamma-1}{\gamma}(\tfrac{z-\mu}{\sigma})^{\frac{\gamma}{\gamma-1}}\right), \quad x \in \mathbb{R}, \tag{16}$$

while considering positive arguments for Erf *and* $\Gamma(\cdot,\cdot)$,

$$F_{Z_\gamma}(z) = \tfrac{1}{2} + \frac{\operatorname{sgn}(x-\mu)\sqrt{\pi}}{2\,\Gamma(\frac{\gamma-1}{\gamma})\,\Gamma(\frac{\gamma}{\gamma-1})}\operatorname{Erf}_{\frac{\gamma}{\gamma-1}}\left\{(\tfrac{\gamma-1}{\gamma})^{\frac{\gamma-1}{\gamma}}\left|\tfrac{z-\mu}{\sigma}\right|\right\} \tag{17}$$

$$= \tfrac{1+\operatorname{sgn}(x-\mu)}{2} - \frac{\operatorname{sgn}(x-\mu)}{2\,\Gamma(\frac{\gamma-1}{\gamma})}\,\Gamma\left(\tfrac{\gamma-1}{\gamma}, \tfrac{\gamma-1}{\gamma}\left|\tfrac{z-\mu}{\sigma}\right|^{\frac{\gamma}{\gamma-1}}\right), \quad x \in \mathbb{R}. \tag{18}$$

Corollary 3. *The c.d.f. F_X of $X \sim \mathcal{LN}_\gamma(\mu,\sigma)$ can be expressed in the series expansion form*

$$F_X(x) = \tfrac{1}{2} + \frac{\left(\frac{\gamma-1}{\gamma}\right)^{\frac{\gamma-1}{\gamma}}}{\frac{2}{\gamma}\Gamma\left(\frac{\gamma-1}{\gamma}\right)}\left(\tfrac{\log x-\mu}{\sigma}\right)\sum_{k=0}^{\infty}\frac{\left(\frac{1-\gamma}{\gamma}\left|\frac{\log x-\mu}{\sigma}\right|^{\frac{\gamma}{\gamma-1}}\right)^k}{k![(k+1)\gamma-1]}, \quad x \in \mathbb{R}^*_+. \tag{19}$$

Proof. Substituting the series expansion form of (7) into (13) we get

$$F_X(x) = \tfrac{1}{2} + (\gamma-1)C^1_\gamma \sum_{k=0}^{\infty}\tfrac{(-1)^k}{k!}\cdot\tfrac{(\frac{\gamma-1}{\gamma})^k}{\gamma(k+1)-1}\left|\tfrac{\log x-\mu}{\sigma}\right|^{\frac{k\gamma}{\gamma-1}+1}, \quad x \in \mathbb{R}^*_+,$$

and expressing the infinite series using the integer powers k, and the fact that $\operatorname{sgn}(x)x = |x|$, $x \in \mathbb{R}$, we finally derive the series expansions as in (19) respectively. ∎

3. Global approximation for the $\mathcal{LN}_\gamma$

For the c.d.f. evaluation of a $X_\gamma \sim \mathcal{LN}_\gamma(\mu,\sigma)$ or $\log X_\gamma \sim \mathcal{N}_\gamma(\mu,\sigma^2)$ over all defined parameters $\gamma \in \mathbb{R}\setminus[1,0]$, a heuristic method is developed that allow us to construct uniform approximations of these functions. This can be achieved through a generalized Hermite-Padé approximation applied on the generalized error function $\operatorname{Erf}_{\gamma/(\gamma-1)}(x)$ at $x=0$ and in infinity.

In particular, we need a finite approximation $f(x)$ of $\operatorname{Erf}_{\gamma/(\gamma-1)}(x)$ at $x=0$ (polynomial approx.) and at $x=+\infty$ (asymptotic approx.), i.e.

$$f(x) = \sum_{k=0}^{m-1} a_k x^k + O(x^m) \approx \sum_{k=0}^{n-1} k_k x^{-n} + O(x^{-n}), \quad x \in (0,+\infty). \tag{20}$$

Then, we construct a uniform approximation of the rational form

$$f(x) \approx \frac{p_0 + p_1 x + x^2}{q_0 + q_1 x + x^2}, \quad x \in (0,+\infty), \tag{21}$$

which is similar to the Hermite-Padé interpolation problem with two anchor points, one for the zero point and the other at infinity, see [5] and [12]. The coefficients p_i's and q_i's $i=0,1$ are obtained through an inhomogeneous linear system derived from (20). Therefore, the F_X and F_Y cumulative functions can be uniformly approximated through rational expressions as in (21). Several examples are given and evaluations are provided.

The upper incomplete gamma function admits the following asymptotic series expansion

$$\Gamma(a,x) = \frac{x^{a-1}}{e^x}\sum_{k=0}^{\infty}\frac{\Gamma(a)}{\Gamma(a-k)}x^{-k} := \frac{x^{a-1}}{e^x}g_a(x), \quad x,a \in \mathbb{R}_+^*, \tag{22}$$

while its series expansion around $x = 0$ is given, through (6), by

$$\Gamma(a,x) = \Gamma(a) - \gamma(a,x) = \Gamma(a) - \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(a+k)}x^{a+k}, \quad x,a \in \mathbb{R}_+. \tag{23}$$

Therefore, the asymptotic series $g_a(x)$, as in (22), can be expressed as a series expansion around $x = 0$ of the form

$$g_a(x) = x^{1-a}e^x\,\Gamma(a) - G_a(x), \quad x,a \in \mathbb{R}_+, \tag{24}$$

where

$$G_a(x) := e^x\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(a+k)}x^{k+1}, \quad x,a \in \mathbb{R}_+,$$

or, using the exponential series expansion $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$,

$$G_a(x) = \sum_{m=1}^{\infty}G_{a;m}x^m\left\{= -g_a(x) + \Gamma(a)x^{1-a}e^x\right\}, \quad x,a \in \mathbb{R}_+, \tag{25}$$

with coefficients $G_{a;m}$ being

$$G_{a;m} = \sum_{k=0}^{m-1}\frac{(-1)^k}{k!(a+k)(m-k)!}, \quad a \in \mathbb{R}_+, \quad m \in \mathbb{N}^*. \tag{26}$$

A uniform approximation of $\Gamma(a,x)$ can be obtained through a uniform approximation of the asymptotic series expansion $g_a(x)$ which is also admits a series expansion at $x = 0$ due to (24). We can then apply the global Padé approximation method for $g_a(x)$. In particular, $g_a(x)$ admits a rational approximation of the form

$$g_a(x) \approx \frac{p_0 + p_1x + x^2}{q_0 + q_1x + x^2}, \quad x,a \in \mathbb{R}_+. \tag{27}$$

Utilizing the series expansion form of $g_a(x)$ as in (24), (27) implies

$$p_0 + p_1x + x^2 \approx \Gamma(a)\frac{e^x}{x^{a-1}}(q_0 + q_1x + x^2) - G_{a;1}q_0x - (G_{a;1}q_1 + G_{a;2}q_0)x^2 - G_{a;1}x^3,$$

and thus

$$p_0 = 0, \tag{28}$$

$$p_1 = -q_0 G_{a;1}, \tag{29}$$

$$1 = -q_1 G_{a;1} - q_0 G_{a;2}. \tag{30}$$

Letting $g_a(x) := \sum_{k=0}^{\infty} g_{a;k} x^{-k}$, (27), through (24), (27) also implies

$$1 + \frac{p_1}{x} + \frac{p_0}{x^2} \approx \left(1 + \frac{g_{a;1}}{x} + \frac{g_{a;2}}{x^2}\right)\left(1 + \frac{q_1}{x} + \frac{q_0}{x^2}\right), \quad x \in \mathbb{R}_+^*,$$

hence

$$p_1 = q_1 + g_{a;1}. \tag{31}$$

Applying (29) to (31) we get $q_1 = -q_0 G_{a;1} - g_{a;1}$ and hence, through (30), we obtain

$$q_0 = \frac{g_{a;1} G_{a;1} - 1}{G_{a;2} - G_{a;1}^2}. \tag{32}$$

Moreover, (32) through (29) yields

$$p_1 = \frac{G_{a;1} - g_{a;1} G_{a;1}^2}{G_{a;2} - G_{a;1}^2}, \tag{33}$$

while (33) through (31) yields

$$q_1 = \frac{G_{a;1} - g_{a;1} G_{a;2}}{G_{a;2} - G_{a;1}^2}. \tag{34}$$

Considering now (22) and (26), we evaluate

$$g_{a;1} = \Gamma(a)/\,\Gamma(a-1) = a - 1, \quad a \in \mathbb{R}_+,$$

$$G_{a;1} = 1/a, \quad a \in \mathbb{R}_+, \quad \text{and}$$

$$G_{a;2} = \tfrac{1}{a} - \tfrac{1}{a+1} = \tfrac{1}{a(a+1)}, \quad a \in \mathbb{R}_+,$$

and substituting the above coefficients to (32), (34) and (33) we obtain respectively

$$q_0 = a(a+1), \quad q_1 = -2a \quad \text{and} \quad p_1 = -(a+1),$$

and hence g_a, as in (27), adopts a global approximation of the form

$$g_a(x) \approx \frac{x^2-(a+1)x}{x^2-2ax+a(a+1)}, \quad x,a\in\mathbb{R}_+. \tag{35}$$

The above methodology is formed into the following Theorem.

Corollary 4. *The c.d.f.* F_{X_γ} *of the generalized lognormally distributed* $X_\gamma \sim \mathcal{LN}_\gamma(\mu,\sigma)$ *admits a uniform approximation of the form*

$$\begin{aligned} F_{X_\gamma}(x) \;&\approx\; \tfrac{1}{2}+\tfrac{1}{2}\operatorname{sgn}(\log x-\mu)-\frac{\operatorname{sgn}(\log x-\mu)(\frac{\gamma-1}{\gamma})^{\frac{\gamma-1}{\gamma}}}{2\,\Gamma(\frac{\gamma-1}{\gamma}+1)}e^{\frac{1-\gamma}{\gamma}k(x)} \\ &\times\frac{k(x)-2\frac{\gamma-1}{\gamma}}{k^2(x)-2(\frac{\gamma}{\gamma-1})^2k(x)+\frac{\gamma^3(2\gamma-1)}{(\gamma-1)^4}}, \quad x\in\mathbb{R}_+, \end{aligned} \tag{36}$$

where $k(x)=|\frac{\log x-\mu}{\sigma}|^{\gamma/(\gamma-1)}$, $x\in\mathbb{R}_+$.

Proof. From g_a as in (22) and the the fact that $\Gamma(a,x)=\Gamma(a)-\gamma(a,x)$, $x,a\in\mathbb{R}_+$, we have

$$g_a(x)=\frac{e^x}{x^{a-1}}[\Gamma(a,x)-\gamma(a,x)], \quad x,a\in\mathbb{R}_+,$$

while substituting the lower incomplete gamma function from the above relation to (5), we readily get

$$\operatorname{Erf}_a(x)=\pi^{-1/2}\,\Gamma(a)\,\Gamma(1/a)-\frac{\Gamma(a)}{\sqrt{\pi}x^{a-1}e^{x^a}}g_{1/a}(x^a), \quad x,a\in\mathbb{R}_+, \tag{37}$$

and therefore, through (35), we obtain

$$\operatorname{Erf}_a(x)\approx\pi^{-1/2}\,\Gamma(a)\,\Gamma(\tfrac{1}{a})-\pi^{-1/2}\,\Gamma(a)xe^{-x^a}\frac{x^a-a-1}{x^{2a}-2ax^a+a(a+1)}. \tag{38}$$

Applying the uniform approximation of the generalized error function as in (38) into (13) we obtain (36). ∎

Table 1 provides the probability values $F_{X_\gamma}(x)=\Pr\{X_\gamma\le x\}$ for $x=0.5,2,3,4,5$ for various $X_\gamma\sim\mathcal{LN}_\gamma(0,1)$. Notice that $F_{X_\gamma}(1)=1/2$ for all γ values due to the fact that $1=e^\mu|_{\mu=0}=\operatorname{Med}X_\gamma$, i.e., the point $x=1$ coincides with the γ-invariant median of the $\mathcal{LN}_\gamma(0,1)$ family. Moreover, the last two columns provide the 1st and 3rd quartile points $Q_{X_\gamma}(1/4)$ and $Q_{X_\gamma}(3/4)$ of X_γ, i.e. $\Pr\{X_\gamma\le Q_{X_\gamma}(k/4)\}=k/4$, $k=1,3$, for various γ values. These quartiles evaluated using the quantile function of X_γ,

$$\begin{aligned} Q_{X_\gamma}(P) &:= \inf\left\{x\in\mathbb{R}_+ |\ F_{X_\gamma}(x)\geq P\right\} = F^{-1}_{X_\gamma}(P) \\ &= \exp\left\{\operatorname{sgn}(2P-1)\sigma\left[\tfrac{\gamma}{\gamma-1}\,\Gamma^{-1}\left(\tfrac{\gamma-1}{\gamma}, |2P-1|\right)\right]^{\frac{\gamma-1}{\gamma}}\right\}, \quad P\in(0,1), \end{aligned}$$

for $P=1/4, 3/4$, that derived through (14). The values of $Q_{X_\gamma}(P)$ were numerically calculated through the roots of the function $\phi(x)=F_{X_\gamma}(x)-P$ with $P=1/4,3/4$.

Table 1. Probability values $F_{X_\gamma}(x)$ for various $x\in\mathbb{R}_+$ as well as the 1st and 3rd quartile points $Q_{X_\gamma}(1/4)$, $Q_{X_\gamma}(3/4)$, for certain generalized lognormally distributed $X_\gamma\sim\mathcal{LN}_\gamma(0,1)$.

γ	$F_{X_\gamma}(\frac{1}{2})$	$F_{X_\gamma}(2)$	$F_{X_\gamma}(3)$	$F_{X_\gamma}(4)$	$F_{X_\gamma}(5)$	$Q_{X_\gamma}(\frac{1}{4})$	$Q_{X_\gamma}(\frac{3}{4})$
-50	0.2501	0.7499	0.8326	0.8739	0.8987	0.4998	2.0008
-10	0.2505	0.7495	0.8297	0.8698	0.8940	0.4990	2.0038
-5	0.2508	0.7492	0.8264	0.8652	0.8887	0.4982	2.0071
-2	0.2515	0.7485	0.8187	0.8539	0.8756	0.4964	2.0145
-1	0.2521	0.7479	0.8097	0.8408	0.8601	0.4945	2.0223
$-1/2$	0.2524	0.7476	0.7989	0.8248	0.8410	0.4925	2.0303
$-1/10$	0.2528	0.7482	0.7757	0.7895	0.7984	0.4986	2.0426
1	0.1534	0.8466	1.0000	1.0000	1.0000	0.6065	1.6487
3/2	0.2381	0.7619	0.8848	0.9437	0.9721	0.5172	1.9334
2	0.2441	0.7559	0.8640	0.9172	0.9462	0.5094	1.9630
3	0.2472	0.7528	0.8505	0.8989	0.9267	0.5049	1.9804
4	0.2481	0.7519	0.8452	0.8917	0.9188	0.5034	1.9867
5	0.2486	0.7514	0.8425	0.8878	0.9145	0.5025	1.9899
10	0.2494	0.7506	0.8375	0.8810	0.9068	0.5011	1.9954
50	0.2499	0.7501	0.8341	0.8761	0.9013	0.5002	1.9992
$\pm\infty$	0.2500	0.7500	0.8333	0.8750	0.9000	0.5000	2.0000

Proposition 5. *The c.d.f. of the positive-ordered lognormally distributed $X_\gamma\sim\mathcal{LN}_{\gamma>1}(\mu,\sigma)$ admits the following bounds,*

$$B(x;\ \tfrac{\gamma-1}{\gamma}) < F_{X_\gamma}(x) < B\left(x;\ \left[(\tfrac{\gamma-1}{\gamma})^{\frac{1}{\gamma}}\,\Gamma(\tfrac{\gamma-1}{\gamma})\right]^{\frac{\gamma-1}{\gamma}}\right), \quad x\in\mathbb{R}_+, \tag{39}$$

where, for $k\in\mathbb{R}_+$,

$$B(x;\ k) = \tfrac{1}{2}+\tfrac{1}{2}\operatorname{sgn}(\log x-\mu)\left(1-\exp\left\{-k\left|\tfrac{\log x-\mu}{\sigma}\right|^{\frac{\gamma}{\gamma-1}}\right\}\right)^{\frac{\gamma-1}{\gamma}}. \tag{40}$$

The inverted inequalities hold for the negative-ordered $X_\gamma\sim\mathcal{LN}_{\gamma<0}(\mu,\sigma)$.

Proof. Applying the inequalities in [1], for $x \in \mathbb{R}_+$,

$$\Gamma(1+\tfrac{1}{a})\left[1-e^{-u(a)x^a}\right]^{1/a} < \int_0^x e^{-t^a}\,dt < \Gamma(1+\tfrac{1}{a})\left[1-e^{-v(a)x^a}\right]^{1/a}, \tag{41}$$

where

$$u(a)=\begin{cases}\Gamma^{-a}(1+\frac{1}{a}), & 0<a<1,\\ 1, & a>1,\end{cases} \quad \text{and} \quad v(a)=\begin{cases}1, & 0<a<1,\\ \Gamma^{-a}(1+\frac{1}{a}), & a>1,\end{cases}$$

into the definition of the generalized error function in (4) we obtain, through the additive identity of the gamma function, that
(42)

$$\tfrac{1}{\sqrt{\pi}}\,\Gamma(a)\,\Gamma(\tfrac{1}{a})\left[1-e^{-u(a)x^a}\right]^{1/a} < \operatorname{Erf}_a(x) < \tfrac{1}{\sqrt{\pi}}\,\Gamma(a)\,\Gamma(\tfrac{1}{a})\left[1-e^{-v(a)x^a}\right]^{1/a}.$$

Consider now the generalized lognormally distributed $X_\gamma \sim \mathcal{LN}_\gamma(\mu,\sigma)$ with $\gamma \in \mathbb{R}\setminus[0,1]$ and let $a=\frac{\gamma}{\gamma-1}$. Then, for the positive-ordered X_γ, i.e. for $\gamma>1$, it is $a>1$ while for the negative-ordered X_γ it is $0<a<1$. Therefore, setting $B(x;\,\cdot)$ as in (40), the bounds (39) for $\gamma>1$ hold true as (42) is applied to (13), while for $\gamma<0$ the inverted bounds of (39) hold. ∎

Example 6. *The c.d.f. of the lognormally distributed $X \sim \mathcal{LN}(\mu,\sigma)$ admits the following bounds,*

$$F_X(x) > \tfrac{1}{2}+\tfrac{1}{2}\operatorname{sgn}(\log x-\mu)\sqrt{1-e^{-\frac{1}{2}(\frac{\log x-\mu}{\sigma})^2}}, \quad \textit{and}$$

$$F_X(x) < \tfrac{1}{2}+\tfrac{1}{2}\operatorname{sgn}(\log x-\mu)\sqrt{1-e^{-\frac{2}{\pi}(\frac{\log x-\mu}{\sigma})^2}}.$$

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