Discussiones Mathematicae
Probability and Statistics 33 (2013) 191–205
doi:10.7151/dmps.1152

MULTIVARIATE MULTIPLE COMPARISONS WITH A CONTROL IN ELLIPTICAL POPULATIONS

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Abstract

The approximate upper percentile of Hotelling's T^2-type statistic is derived in order to construct simultaneous confidence intervals for comparisons with a control under elliptical populations with unequal sample sizes. Accuracy and conservativeness of Bonferroni approximations are evaluated via a Monte Carlo simulation study. Finally, we explain the real data analysis using procedures derived in this paper.

Keywords: comparisons with a control, Bonferroni approximation and Monte Carlo simulation.

2010 Mathematics Subject Classification: 62H10, 60E05 and 65C05.

1. Introduction

Simultaneous confidence intervals for comparisons with a control among mean vectors are considered under k independent elliptical populations with unequal sample sizes. In order to construct them, it is necessary to obtain the upper percentile of $T^2_{\max\cdot c}$ which is Hotelling's T^2-type statistic. However, it is difficult to obtain upper percentiles exactly even when populations have the multivariate normal distribution. In order to obtain conservative approximate simultaneous confidence intervals, Bonferroni's inequality is applied to T^2-type statistic. Under elliptical populations with equal sample sizes, the first and the modified second order Bonferroni approximations for pairwise multiple comparisons are discussed by Seo [6]. Under elliptical populations with unequal sample sizes, these are discussed by Okamoto and Seo [5] and Okamoto [4]. This paper gives them for comparisons with a control, and their accuracy and conservativeness are evaluated via a Monte Carlo simulation study. Finally, an actual procedure is explained using the school-record data of the second-year student in a junior high school in Tokyo. Also, for graphical approaches using weighted Bonferroni, see e.g. Bretz *et al.* [1].

For the j-th population, a $p\times 1$ random vector $\boldsymbol{x}^{(j)}$ is said to have an elliptical distribution with parameters $\boldsymbol{\mu}^{(j)}$ $(p\times 1)$ and $\Lambda^{(j)}$ $(p\times p)$ if its density function is of the form

$$f(\boldsymbol{x}^{(j)}) = c_p^{(j)}|\Lambda^{(j)}|^{-\frac{1}{2}}g_j\left\{(\boldsymbol{x}^{(j)}-\boldsymbol{\mu}^{(j)})'{\Lambda^{(j)}}^{-1}(\boldsymbol{x}^{(j)}-\boldsymbol{\mu}^{(j)})\right\}$$

for some non-negative function g_j, where $c_p^{(j)}$ is a normalizing constant and $\Lambda^{(j)}$ is a positive definite. The characteristic function of the vector $\boldsymbol{x}^{(j)}$ is $\phi_j(\boldsymbol{t}) = \exp(i\boldsymbol{t}'\boldsymbol{\mu}^{(j)})\psi_j(\boldsymbol{t}'\Lambda^{(j)}\boldsymbol{t})$ for some function ψ_j, and $\mathrm{E}[\boldsymbol{x}^{(j)}] = \boldsymbol{\mu}^{(j)}$ and $\Sigma^{(j)} = \mathrm{Cov}[\boldsymbol{x}^{(j)}] = -2\psi_j'(0)\Lambda^{(j)}$, if they exist. Throughout this paper, we assume $\Sigma = \Sigma^{(1)} = \cdots = \Sigma^{(k)}$. We define the kurtosis parameter as $\kappa_j = \{\psi_j''(0)/(\psi_j'(0))^2\}-1$.

2. A first order Bonferroni approximation

Consider simultaneous confidence intervals for comparisons with a control among k independent p-dimensional mean vectors under elliptical populations. Let $\boldsymbol{x}_1^{(j)},\ldots,\boldsymbol{x}_{N_j}^{(j)}$ $(j=1,\ldots,k)$ be N_j independent observations on $\boldsymbol{x}^{(j)}$ that has an elliptical distribution with mean vector $\boldsymbol{\mu}^{(j)}$ and common covariance matrix Σ. Let the j-th sample mean vector, the j-th sample covariance matrix and the pooled sample covariance matrix be

$$\overline{\boldsymbol{x}}^{(j)} = \frac{1}{N_j}\sum_{i=1}^{N_j}\boldsymbol{x}_i^{(j)},$$

$$S^{(j)} = \frac{1}{N_j-1}\sum_{i=1}^{N_j}(\boldsymbol{x}_i^{(j)}-\overline{\boldsymbol{x}}^{(j)})(\boldsymbol{x}_i^{(j)}-\overline{\boldsymbol{x}}^{(j)})',$$

$$S = \frac{1}{\nu}\sum_{j=1}^{k}(N_j-1)S^{(j)},$$

respectively, where $\nu = \sum_{j=1}^{k} N_j - k$.

Letting the first population be a control, the simultaneous confidence intervals with the given confidence level $1-\alpha$ for comparisons with a control among mean vectors are given by

$$\begin{aligned} \boldsymbol{a}'(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(m)}) \in \left[\boldsymbol{a}'(\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(m)}) \pm t_\alpha \sqrt{d_{1m}\boldsymbol{a}'S\boldsymbol{a}}\right], \\ \forall \boldsymbol{a} \in \mathbf{R}^p - \{\mathbf{0}\},\ 2 \le m \le k, \end{aligned} \tag{1}$$

where $d_{1m} = 1/N_1 + 1/N_m$, $\mathbf{R}^p - \{\mathbf{0}\}$ is the set of any nonnull real p-dimensional vectors and the value t_α ($\equiv t > 0$) satisfies as follows:

$$\Pr\left\{T^2_{\max\cdot c} > t^2\right\} = \alpha,$$

where

$$\begin{aligned} T^2_{\max\cdot c} &= \max_{2\le m\le k}\left\{T^2_{1m}\right\}, \\ T^2_{1m} &= d_{1m}^{-1}\left(\boldsymbol{y}^{(1)} - \boldsymbol{y}^{(m)}\right)' S^{-1}\left(\boldsymbol{y}^{(1)} - \boldsymbol{y}^{(m)}\right), \\ \boldsymbol{y}^{(j)} &= \overline{\boldsymbol{x}}^{(j)} - \boldsymbol{\mu}^{(j)},\ j = 1, \ldots, k. \end{aligned}$$

By using the first term of Bonferroni's inequality for $\Pr\left\{T^2_{\max\cdot c} > t^2\right\}$:

$$\Pr\left\{T^2_{\max\cdot c} > t^2\right\} < \sum_{m=2}^{k} \Pr\left\{T^2_{1m} > t^2\right\},$$

the approximate upper percentile t^2_{1c} of $T^2_{\max\cdot c}$ is given by

$$\sum_{m=2}^{k} \Pr\left\{T^2_{1m} > t^2_{1c}\right\} = \alpha.$$

Without loss of generality, we assume $\Sigma = I_p$ and $N = \max\{N_1, N_2, \ldots, N_k\}$. Put $r_j = N_j/N$ for $j = 1, \ldots, k$, $s = 1/(\sum_{j=1}^{k} r_j)$ and $w_{lm} = \sqrt{r_m/(r_l + r_m)}$.

Letting

$$\begin{aligned} \overline{\boldsymbol{x}}^{(j)} &= \boldsymbol{\mu}^{(j)} + \frac{1}{\sqrt{N_j}}\boldsymbol{z}^{(j)}, \\ W^{(j)} &= \frac{1}{N_j}\sum_{i=1}^{N_j}(\boldsymbol{x}_i^{(j)} - \boldsymbol{\mu}^{(j)})(\boldsymbol{x}_i^{(j)} - \boldsymbol{\mu}^{(j)})' \\ &= I_p + \frac{1}{\sqrt{N_j}}Z^{(j)}, \end{aligned}$$

we have

$$T_{1m}^2 = \boldsymbol{\tau}_{1m}' S^{-1} \boldsymbol{\tau}_{1m},$$

where

$$\begin{aligned}
\boldsymbol{\tau}_{1m} &= w_{1m}\boldsymbol{z}^{(1)} - w_{m1}\boldsymbol{z}^{(m)}, \\
S^{-1} &= I_p - \frac{1}{\sqrt{N}} s \sum_{j=1}^{k} \sqrt{r_j} Z^{(j)} + \frac{1}{N}\left[s\sum_{j=1}^{k} \boldsymbol{z}^{(j)}\boldsymbol{z}^{(j)\prime} + s^2 \sum_{j=1}^{k} r_j Z^{(j)2} \right. \\
&\quad \left. + s^2 \left\{ \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \sqrt{r_i r_j} \left(Z^{(i)} Z^{(j)} + Z^{(j)} Z^{(i)} \right) \right\} - skI_p \right] + o_p(N^{-1}).
\end{aligned}$$

Using the joint density function of $\boldsymbol{z}^{(j)}$ and $Z^{(j)}$ which is derived by Iwashita [2], the asymptotic expansion of the characteristic function of T_{1m}^2 can be written as

$$\mathrm{E}[\exp(itT_{1m}^2)] = u^{-\frac{p}{2}} \left[1 + \frac{1}{4N} \left(c_{1m}^{(0)} + c_{1m}^{(1)} u^{-1} + c_{1m}^{(2)} u^{-2} \right) \right] + o(N^{-1}),$$

where $u = 1 - 2it$, $i = \sqrt{-1}$ and

$$\begin{aligned}
c_{1m}^{(0)} &= -sp^2 + \frac{1}{2} p(p+2) \left[\left(\frac{1}{r_1} w_{1m}^4 - 2sw_{1m}^2 \right) \kappa_1 + \left(\frac{1}{r_m} w_{m1}^4 - 2sw_{m1}^2 \right) \kappa_m - s\kappa_r \right], \\
c_{1m}^{(1)} &= -2sp - p(p+2) \left[\left(\frac{1}{r_1} w_{1m}^4 - 4sw_{1m}^2 \right) \kappa_1 + \left(\frac{1}{r_m} w_{m1}^4 - 4sw_{m1}^2 \right) \kappa_m + s\kappa_r \right], \\
c_{1m}^{(2)} &= sp(p+2) \\
&\quad + \frac{1}{2} p(p+2) \left[\left(\frac{1}{r_1} w_{1m}^4 - 6sw_{1m}^2 \right) \kappa_1 + \left(\frac{1}{r_m} w_{m1}^4 - 6sw_{m1}^2 \right) \kappa_m + 3s\kappa_r \right], \\
\kappa_r &= s \sum_{j=1}^{k} r_j \kappa_j.
\end{aligned}$$

Using above result, the distribution of T_{1m}^2 can be expanded as

$$\Pr(T_{1m}^2 > t^2) = \Pr(\chi_p^2 > t^2) + \frac{1}{4N} \sum_{j=0}^{2} c_{1m}^{(j)} \Pr(\chi_{p+2j}^2 > t^2) + o(N^{-1}),$$

and its upper 100α percentile can be expanded as

$$t_{1m\cdot\chi^2}^2(\alpha) = \chi_p^2(\alpha) - \frac{1}{2N} \chi_p^2(\alpha) \left\{ \frac{1}{p} c_{1m}^{(0)} - \frac{1}{p(p+2)} c_{1m}^{(2)} \chi_p^2(\alpha) \right\} + o(N^{-1}),$$

where $\chi_p^2(\alpha)$ is the upper 100α percentile of the χ^2 distribution with p degrees of freedom. Therefore, we have the first order Bonferroni approximate upper 100α percentile of $T^2_{\max\cdot c}$ as follows:

$$\begin{aligned} t^2_{1\cdot\chi^2\cdot c}(\alpha) &= \chi_p^2\left(\frac{\alpha}{k-1}\right) - \frac{1}{2N(k-1)}\chi_p^2\left(\frac{\alpha}{k-1}\right) \\ &\quad\times\sum_{m=2}^{k}\left\{\frac{1}{p}c_{1m}^{(0)} - \frac{1}{p(p+2)}c_{1m}^{(2)}\chi_p^2\left(\frac{\alpha}{k-1}\right)\right\}. \end{aligned} \tag{2}$$

Also, since Hotelling's T^2-statistic under normality is an F-statistic, we obtain another approximate upper 100α percentile of $T^2_{\max\cdot c}$ as follows:

$$\begin{aligned} t^2_{1\cdot F\cdot c}(\alpha) &= \frac{\nu p}{\nu-p+1}F_{p,\nu-p+1}\left(\frac{\alpha}{k-1}\right) - \frac{1}{2N(k-1)}\chi_p^2\left(\frac{\alpha}{k-1}\right) \\ &\quad\times\sum_{m=2}^{k}\left\{\left(\frac{1}{p}c_{1m}^{(0)} + sp\right) - \left(\frac{1}{p(p+2)}c_{1m}^{(2)} - s\right)\chi_p^2\left(\frac{\alpha}{k-1}\right)\right\}, \end{aligned} \tag{3}$$

where $F_{p,\nu-p+1}(\alpha/(k-1))$ is the upper $100(\alpha/(k-1))$ percentile of the F-distribution with p and $\nu-p+1$ degrees of freedom.

3. A modified second order Bonferroni approximation

The first order Bonferroni approximation becomes conservative too much when the number of populations or the kurtosis parameter is large. In this section, a modified second order Bonferroni procedure, which uses the first and the second terms of Bonferroni's inequality, is described to improve conservativeness of the first order Bonferroni approximation.

Let $\boldsymbol{y}_1 = w_{12}\boldsymbol{z}^{(1)} - w_{21}\boldsymbol{z}^{(2)}$, $\boldsymbol{y}_2 = w_{13}\boldsymbol{z}^{(1)} - w_{31}\boldsymbol{z}^{(3)}, \ldots, \boldsymbol{y}_{k-1} = w_{1k}\boldsymbol{z}^{(1)} - w_{k1}\boldsymbol{z}^{(k)}$. Bonferroni's inequality for $\Pr\{T^2_{\max\cdot c} > t^2\}$ is given by

$$\sum_{i=1}^{k-1}\Pr\left\{\boldsymbol{y}_i'S^{-1}\boldsymbol{y}_i > t^2\right\} - \beta_c(t^2) < \Pr\{T^2_{\max\cdot c} > t^2\} < \sum_{i=1}^{k-1}\Pr\left\{\boldsymbol{y}_i'S^{-1}\boldsymbol{y}_i > t^2\right\},$$

where

$$\beta_c(t^2) = \sum_{i=1}^{k-2}\sum_{j=i+1}^{k-1}\Pr\left\{\boldsymbol{y}_i'S^{-1}\boldsymbol{y}_i > t^2,\ \boldsymbol{y}_j'S^{-1}\boldsymbol{y}_j > t^2\right\}.$$

The first order Bonferroni approximation t_{1c}^2 is defined as a critical value that satisfies the equality

$$\sum_{i=1}^{k-1} \Pr\left\{\boldsymbol{y}_i' S^{-1} \boldsymbol{y}_i > t_{1c}^2\right\} = \alpha.$$

The second order Bonferroni approximation t_{2c}^2 is defined as a critical value that satisfies the equality

$$\sum_{i=1}^{k-1} \Pr\left\{\boldsymbol{y}_i' S^{-1} \boldsymbol{y}_i > t_{2c}^2\right\} - \beta_c(t_{2c}^2) = \alpha.$$

The modified second order Bonferroni approximation t_{Mc}^2 is defined as a critical value that satisfies the equality

$$\sum_{i=1}^{k-1} \Pr\left\{\boldsymbol{y}_i' S^{-1} \boldsymbol{y}_i > t_{Mc}^2\right\} = \alpha + \beta_c(t_{1c}^2),$$

where

$$\beta_c(t_{1c}^2) = \sum_{j=2}^{k-1} \sum_{h=j+1}^{k} \Pr\{T_{1j}^2 > t_{1c}^2, T_{1h}^2 > t_{1c}^2\}. \tag{4}$$

In order to obtain the modified second order Bonferroni approximation t_{Mc}^2, it is necessary to evaluate $\Pr\{T_{1j}^2 > t_{1c}^2, T_{1h}^2 > t_{1c}^2\}$. For convenience, we discuss the joint characteristic function of T_{12}^2 and T_{13}^2: $\mathrm{E}[\exp(it_1 T_{12}^2 + it_2 T_{13}^2)]$ as follows.

$$\begin{aligned} &\mathrm{E}[\exp(it_1 T_{12}^2 + it_2 T_{13}^2)] \\ &\quad = \mathrm{E}\left[\exp(it_1 T_{12}^{(1)} + it_2 T_{13}^{(1)}) \left(1 + \frac{1}{\sqrt{N}} D_1 + \frac{1}{N} D_2\right)\right] + o(N^{-1}), \end{aligned}$$

where

$$\begin{aligned} D_1 &= it_1 T_{12}^{(2)} + it_2 T_{13}^{(2)}, \\ D_2 &= it_1 T_{12}^{(3)} + \frac{(it_1)^2}{2} (T_{12}^{(2)})^2 + it_2 T_{13}^{(3)} + \frac{(it_2)^2}{2} (T_{13}^{(2)})^2 + (it_1)(it_2) T_{12}^{(2)} T_{13}^{(2)}, \end{aligned}$$

and

$$\begin{aligned} T_{12}^{(1)} &= \boldsymbol{\tau}_{12}' \boldsymbol{\tau}_{12}, \quad T_{13}^{(1)} = \boldsymbol{\tau}_{13}' \boldsymbol{\tau}_{13}, \\ T_{12}^{(2)} &= -\boldsymbol{\tau}_{12}' \left(s \sum_{j=1}^{k} \sqrt{r_j} Z^{(j)} \right) \boldsymbol{\tau}_{12}, \quad T_{13}^{(2)} = -\boldsymbol{\tau}_{13}' \left(s \sum_{j=1}^{k} \sqrt{r_j} Z^{(j)} \right) \boldsymbol{\tau}_{13}, \end{aligned}$$

$$T_{12}^{(3)} = \boldsymbol{\tau}_{12}' \left(s \sum_{j=1}^{k} \boldsymbol{z}^{(j)} \boldsymbol{z}^{(j)\prime} + s^2 \sum_{i=1}^{k} \sum_{j=1}^{k} \sqrt{r_i r_j} Z^{(i)} Z^{(j)} - skI_p \right) \boldsymbol{\tau}_{12},$$

$$T_{13}^{(3)} = \boldsymbol{\tau}_{13}' \left(s \sum_{j=1}^{k} \boldsymbol{z}^{(j)} \boldsymbol{z}^{(j)\prime} + s^2 \sum_{i=1}^{k} \sum_{j=1}^{k} \sqrt{r_i r_j} Z^{(i)} Z^{(j)} - skI_p \right) \boldsymbol{\tau}_{13},$$

and

$$\boldsymbol{\tau}_{12} = w_1 \boldsymbol{z}^{(1)} - w_2 \boldsymbol{z}^{(2)}, \quad w_1 \equiv w_{12} = \sqrt{\frac{r_2}{r_1 + r_2}}, \quad w_2 \equiv w_{21} = \sqrt{\frac{r_1}{r_1 + r_2}},$$

$$\boldsymbol{\tau}_{13} = w_3 \boldsymbol{z}^{(1)} - w_4 \boldsymbol{z}^{(3)}, \quad w_3 \equiv w_{13} = \sqrt{\frac{r_3}{r_1 + r_3}}, \quad w_4 \equiv w_{31} = \sqrt{\frac{r_1}{r_1 + r_3}}.$$

Using the joint density function of $\boldsymbol{z}^{(j)}$ and $Z^{(j)}$, we obtain an asymptotic expansion for the expectation of $\exp(it_1 T_{12}^{(1)} + it_2 T_{13}^{(1)})$ in elliptical distributions as follows.

$$\begin{aligned}
&\mathrm{E}[\exp(it_1 T_{12}^{(1)} + it_2 T_{13}^{(1)})] \\
&\quad = U^{-\frac{p}{2}} + \frac{1}{8N} p(p+2) U^{-\frac{p}{2}-2} \\
&\qquad \times \left[\frac{1}{r_1} \{(u_1 - 1) u_2 w_1^2 + (u_2 - 1) u_1 w_3^2 - 2(u_1 - 1)(u_2 - 1) v_0\}^2 \kappa_1 \right. \\
&\qquad \left. + \frac{1}{r_2} (u_1 - 1)^2 u_2^2 w_2^4 \kappa_2 + \frac{1}{r_3} u_1^2 (u_2 - 1)^2 w_4^4 \kappa_3 \right] + o(N^{-1}),
\end{aligned}$$

where $U = u_1 u_2 - (u_1 - 1)(u_2 - 1) v_0$, $u_1 = 1 - 2it_1$, $u_2 = 1 - 2it_2$, $v_0 = w_1^2 w_3^2$.

Let $\lambda_1 = 1 - 2(1 - v_0) it_1$, $\lambda_2 = 1 - 2(1 - v_0) it_2$, then $u_1 = (\lambda_1 - v_0)/(1 - v_0)$, $u_2 = (\lambda_2 - v_0)/(1 - v_0)$ and

$$\begin{aligned}
U^{-\frac{p}{2}} &= \left(\frac{\lambda_1 \lambda_2 - v_0}{1 - v_0} \right)^{-\frac{p}{2}} \\
&= (1 - v_0)^{\frac{p}{2}} \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}p\right)_m}{m!} v_0^m \lambda_1^{-\frac{p}{2}-m} \lambda_2^{-\frac{p}{2}-m},
\end{aligned}$$

where

$$\left(\frac{1}{2} p \right)_m = \frac{\Gamma\left(\frac{p}{2} + m\right)}{\Gamma\left(\frac{p}{2}\right)} = \frac{p}{2} \left(\frac{p}{2} + 1 \right) \cdots \left(\frac{p}{2} + m - 1 \right).$$

Repeating such calculations about expectation of $\boldsymbol{z}^{(j)}$ and $Z^{(j)}$, an asymptotic expansion for the joint probability $\Pr\left\{T_{1j}^2 > t_{1c}^2, T_{1h}^2 > t_{1c}^2\right\}$ is given by

$$\begin{aligned}
&\Pr\left\{T_{1j}^2 > t_{1c}^2, T_{1h}^2 > t_{1c}^2\right\} \\
&\quad = (1-v_0)^{\frac{p}{2}} \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}p\right)_m}{m!} v_0^m \\
&\qquad \times \left[G_{\frac{p}{2}+m}^2(\eta_2) + \frac{1}{N}\left\{ d_1 g_{\frac{p}{2}+m}(\eta_2) G_{\frac{p}{2}+m}(\eta_2) + d_2 g_{\frac{p}{2}+m}^2(\eta_2) \right\} \right] + o(N^{-1}),
\end{aligned}$$

where

$$\eta_2 = \frac{1}{2(1-v_0)} t_{1c}^2,$$

$$G_{\frac{p}{2}+m}(\eta_2) = \int_{\eta_2}^{\infty} g_{\frac{p}{2}+m}(t)dt, \quad g_{\frac{p}{2}+m}(t) = \frac{1}{\Gamma\left(\frac{p}{2}+m\right)} t^{\frac{p}{2}+m-1} e^{-t},$$

and

$$\begin{aligned}
d_1 &= \frac{\eta_2}{32 v_1^2}\left\{ 32 s v_1^2 (p - 2m + 2\eta_2) + 8 s v_1 d_{11} + d_{12} \right\}, \\
d_2 &= \frac{\eta_2^2}{16 q v_1^2 (p+2m)}\left\{ 32 s q v_1^2 (2m+1) + 8 s v_1 d_{21} + d_{22} \right\}, \\
d_{11} &= 2\left[3(m - \eta_2 v_0) + v_1 v_2 \left\{2\eta_2(2v_1 - 1) + q\right\}\right]\kappa_1 \\
&\quad + \left[2 v_1 w_2^2 (4 v_1 \eta_2 + q) + 9m + \eta_2 \left\{ v_1(4 w_1^2 - 13) - 9 \right\}\right]\kappa_j \\
&\quad + \left[2 v_1 w_4^2 (4 v_1 \eta_2 + q) + 9m + \eta_2 \left\{ v_1(4 w_3^2 - 13) - 9 \right\}\right]\kappa_h \\
&\quad + \left[2 v_1 \left\{ p + 6m - 6\eta_2(2v_1 + 1) + 2 \right\}\right]\kappa_r, \\
d_{12} &= 8\left[\frac{1}{r_1}(2\eta_2 - q) v_1^2 (v_2^2 - 2v_0) + m - \eta_2(v_1 + 1)\right]\kappa_1 \\
&\quad + \left[\frac{8}{r_j}(2\eta_2 - q) v_1^2 w_2^4 + 5m - 5\eta_2(v_1+1)\right]\kappa_j \\
&\quad + \left[\frac{8}{r_h}(2\eta_2 - q) v_1^2 w_4^4 + 5m - 5\eta_2(v_1+1)\right]\kappa_h, \\
d_{21} &= \left[4 v_0 \eta_2^2 \left\{4 v_0 (v_2 - 4) + 4 v_2 - 1\right\} + \left\{-8 v_0 + 2(v_0+1) v_2 + 1\right\} q^2 \right. \\
&\qquad \left. - \left\{ p - 2 v_0 \eta_2 (4 v_2 (v_0 - 4) + 21) + 2 \right\} q \right]\kappa_1 \\
&\quad + \left[2 v_0 \eta_2^2 \left\{ -8(v_0+1) w_1^2 + 8 v_0 + 3 \right\} \right. \\
&\qquad + v_0 \eta_2 \left\{ -8(v_0 - 4) w_1^2 + 8 v_0 - 41 \right\} q + 5m^2 + 2(p+2)^2 (v_0+1) w_2^2
\end{aligned}$$

$$
\begin{aligned}
&\quad + (p+m+2)m\left\{-8(v_0+1)w_1^2+8v_0+13\right\}\Big]\,\kappa_j \\
&+ \Big[2v_0\eta_2^2\left\{-8(v_0+1)w_3^2+8v_0+3\right\} \\
&\quad + v_0\eta_2\left\{-8(v_0-4)w_3^2+8v_0-41\right\}q+5m^2+2(p+2)^2(v_0+1)w_4^2 \\
&\quad + (p+m+2)m\left\{-8(v_0+1)w_3^2+8v_0+13\right\}\Big]\,\kappa_h \\
&+ \left[2v_1(p+6m-12v_0\eta_2+2)q\right]\kappa_r,
\end{aligned}
$$

$$
\begin{aligned}
d_{22} \;=\; &\Big[4\left\{(m-2v_0\eta_2)q-2v_0\eta_2^2\right\} \\
&\quad + \frac{8v_0}{r_1}\left[\left\{(v_2-2)^2+v_1\left(2v_1-v_2^2+4\right)\right\}q^2\right. \\
&\quad \left. + 4\eta_2(2v_1-v_2+2)(v_2-2)q+4v_0\eta_2^2(v_2-2)^2\;\right]\Big]\kappa_1 \\
&+ \left[\frac{8v_0w_2^4}{r_j}(2\eta_2-q)\left\{2v_0\eta_2+(v_1-1)q\right\}+(m-5v_0\eta_2)q-2v_0\eta_2^2\right]\kappa_j \\
&+ \left[\frac{8v_0w_4^4}{r_h}(2\eta_2-q)\left\{2v_0\eta_2+(v_1-1)q\right\}+(m-5v_0\eta_2)q-2v_0\eta_2^2\right]\kappa_h,
\end{aligned}
$$

$q = p+2m+2$, $w_1 \equiv w_{1j}$, $w_2 \equiv w_{j1}$, $w_3 \equiv w_{1h}$, $w_4 \equiv w_{h1}$, $v_1 = v_0 - 1$, $v_2 = w_1^2 + w_3^2$.

Therefore, the modified second order Bonferroni approximate upper 100α percentiles of $T^2_{\max\cdot c}$ are obtained as follows:

$$
\begin{aligned}
t^2_{M\cdot\chi^2\cdot c}(\alpha) \;=\; &\chi^2_p(\gamma_c)-\frac{1}{2N(k-1)}\chi^2_p(\gamma_c) \\
&\times\sum_{m=2}^{k}\left\{\frac{1}{p}c^{(0)}_{1m}-\frac{1}{p(p+2)}c^{(2)}_{1m}\chi^2_p(\gamma_c)\right\}, \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
t^2_{M\cdot F\cdot c}(\alpha) \;=\; &\frac{\nu p}{\nu-p+1}F_{p,\nu-p+1}(\gamma_c)-\frac{1}{2N(k-1)}\chi^2_p(\gamma_c) \\
&\times\sum_{m=2}^{k}\left\{\left(\frac{1}{p}c^{(0)}_{1m}+sp\right)-\left(\frac{1}{p(p+2)}c^{(2)}_{1m}-s\right)\chi^2_p(\gamma_c)\right\}, \qquad (6)
\end{aligned}
$$

where $\gamma_c = \{\alpha+\beta_c(t^2_{1c})\}/(k-1)$.

4. Accuracy and conservativeness of approximations

In order to evaluate accuracy and conservativeness of the first and the modified second order Bonferroni approximations for comparisons with a control, the Monte Carlo simulation for the upper percentiles of $T^2_{\max\cdot c}$ is implemented for varied parameters. In the simulation, the k populations have the same distributions, and consider three types of distributions: the multivariate normal ($\kappa = 0$), the ε-contaminated normal ($\kappa = 1.78$ with $\varepsilon = 0.1$ & $\sigma = 3$) and the ε-contaminated normal ($\kappa = 3.24$ with $\varepsilon = 0.1$ & $\sigma = 4$) (see Muirhead [3] p.32).

Table 1 gives the simulated and approximate values of the upper percentile of $T_{\max\cdot c}$ ($= \sqrt{T^2_{\max\cdot c}}$) and lower tail probabilities for the following parameters: $p = 5$, $k = 10$, N_j ($= N$) $= 10, 20, 40, 80$ ($j = 1, \ldots, k$), $r = 1$ and $\alpha = 0.05$. Values $t_{1\cdot\chi^2}$, $t_{1\cdot F}$, $t_{M\cdot\chi^2}$ and $t_{M\cdot F}$ stand for approximations $\sqrt{t^2_{1\cdot\chi^2\cdot c}(\alpha)}$, $\sqrt{t^2_{1\cdot F\cdot c}(\alpha)}$, $\sqrt{t^2_{M\cdot\chi^2\cdot c}(\alpha)}$ and $\sqrt{t^2_{M\cdot F\cdot c}(\alpha)}$ found in (2), (3), (6) and (6), respectively. $P_{1\cdot\chi^2}$, $P_{1\cdot F}$, $P_{M\cdot\chi^2}$ and $P_{M\cdot F}$ stand for lower tail probabilities $\Pr\{T^2_{\max\cdot c} < t^2_{1\cdot\chi^2}\}$, $\Pr\{T^2_{\max\cdot c} < t^2_{1\cdot F}\}$, $\Pr\{T^2_{\max\cdot c} < t^2_{M\cdot\chi^2}\}$ and $\Pr\{T^2_{\max\cdot c} < t^2_{M\cdot F}\}$, respectively. t^* is a simulated value and $\Pr\{T^2_{\max\cdot c} < t^{*2}\} = 1 - \alpha$.

If lower tail probability is larger than $1 - \alpha = 0.95$, we can construct conservative simultaneous confidence intervals from (1). For large N, the first order approximation always constructs conservative simultaneous confidence intervals because the effect of the asymptotic expansion is ignored. However, there is no guarantee to which the modified second order approximation always constructs them. Also, $t_{1\cdot\chi^2}$ and $t_{M\cdot\chi^2}$ tend to be influenced of the asymptotic expansion. In Table 1, for large N, lower tail probabilities of the first order approximations are about 0.96 regardless of κ, and those of the modified second order approximations are just 0.95.

Table 2 gives the simulated and approximate values of the upper percentile of $T_{\max\cdot c}$ and lower tail probabilities for the following parameters: $p = 5$, $k = 10$, $r = 0.5$, $\alpha = 0.05$ and $N = 10, 20, 40, 80$; the sample sizes of the first 5 populations are N and the rest of them are rN, that is, the sample size of the first population which is control is N. Table 3 gives them for the following parameters: $p = 5$, $k = 10$, $r = 0.5$, $\alpha = 0.05$ and $N = 10, 20, 40, 80$; the sample sizes of the first 5 populations are rN and the rest of them are N, that is, the sample size of the first population which is control is rN.

In Table 2, lower tail probabilities are smaller than those in Table 1 on the whole. For $\kappa = 0$, lower tail probability of $P_{1\cdot\chi^2}$ is less than 0.95 even when it is used the first order approximation because of the asymptotic expansion. Note that the first order approximation leads to conservative simultaneous confidence intervals for large N. Although the modified second order approximation may not lead to them, lower tail probabilities are actually equal to 0.95 or almost close to 0.95.

In Table 3, although lower tail probabilities of first order approximations are quite large, the modified second order approximations have rectified conservativeness considerably as N increases. There is a case that lower tail probabilities of the modified second order approximations are less than 0.95; however, these errors are few. The modified second order Bonferroni approximations construct conservative simultaneous confidence intervals with good accuracy for many parameters.

5. EXAMPLE

We explain the real data analysis using the procedures proposed in this paper. We use the school-record data of the second-year student in a junior high school in Tokyo which appears in the website of the Institute of Statistical Science (http://www.statistics.co.jp/). We divided into three populations according to the score of physical education. The first population (Π_1) consists of students of 80 or more points. The second population (Π_2) consists of students of 60 or more points. The third population (Π_3) consists of students of 40 or more points. Let the first population be a control. We compare the score of main 5 subjects (Japanese, Social studies, Mathematics, Science and English) of the second and the third populations with that of the first population. Table 4 is these data. We assume that these data are distributed as elliptical populations. Parameters are as follows: $p = 5, k = 3, N_1 = 46, N_2 = 37, N_3 = 32, r_1 = 1, r_2 = 37/46, r_3 = 32/46$ and $\alpha = 0.05$. Kurtosis parameters are calculated as $\kappa_1 = -0.0933, \kappa_2 = -0.0443, \kappa_3 = -0.1458$ using $\hat{\kappa}^{**}$ derived by Seo and Toyama [7]. The sample mean vectors are

$$\begin{aligned}\overline{\boldsymbol{x}}^{(1)} &= (65.6739, 46.7609, 51.3261, 52.6957, 50.6739)',\\ \overline{\boldsymbol{x}}^{(2)} &= (49.8378, 33.1892, 39.7297, 43.7297, 34.6486)',\\ \overline{\boldsymbol{x}}^{(3)} &= (52.8125, 43.6563, 51.3750, 56.3438, 43.3125)'.\end{aligned}$$

The pooled covariance matrix S is

$$S = \begin{pmatrix} 361.898 & 322.611 & 323.133 & 301.460 & 379.691\\ 322.611 & 437.395 & 369.877 & 361.251 & 442.165\\ 323.133 & 369.877 & 529.256 & 400.391 & 505.424\\ 301.460 & 361.251 & 400.391 & 433.421 & 449.286\\ 379.691 & 442.165 & 505.424 & 449.286 & 753.499 \end{pmatrix}$$

and T^2_{1m} is calculated as $T^2_{12} = 17.0499$ and $T^2_{13} = 32.6876$. The first order Bonferroni approximate upper 95 percentiles of $T^2_{\max\cdot c}$ (2) and (3) are calculated as $t^2_{1\cdot\chi^2\cdot c}(0.05) = 3.722$ and $t^2_{1\cdot\chi^2\cdot F}(0.05) = 3.735$. Also, $\beta_c(t^2_{1c})$ in (4) is calculated as $\beta_c(t^2_{1\cdot\chi^2\cdot c}) = 0.0020$ and $\beta_c(t^2_{1\cdot\chi^2\cdot F}) = 0.0019$. Therefore, the modified second order approximations (6) and (6) are calculated as $t^2_{M\cdot\chi^2\cdot c}(0.05) = 3.707$ and $t^2_{M\cdot F\cdot c}(0.05) = 3.721$. For example, let $\boldsymbol{a} = (1, 0, 0, 0, 0)'$, then the simultaneous confidence intervals for comparisons with a control (1) are constructed as

$$\boldsymbol{a}'(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}) \in [0.265, 31.407]\,,$$
$$\boldsymbol{a}'(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(3)}) \in [-3.370, 29.093]$$

using the modified second order approximation $t^2_{M\cdot\chi^2\cdot c}(0.05)$. This shows a significant difference between scores of Japanese for the first and the second population. As other examples, let $\boldsymbol{a} = (1/5, 1/5, 1/5, 1/5, 1/5)'$, then the simultaneous confidence intervals for comparisons with a control (1) are constructed as

$$\boldsymbol{a}'(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}) \in [-3.484, 29.883]\,,$$
$$\boldsymbol{a}'(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(3)}) \in [-13.465, 21.317]$$

using the modified second order approximation $t^2_{M\cdot\chi^2\cdot c}(0.05)$. Therefore, there is no significant difference in scores of main 5 subject by physical education group.

Although it becomes the same conclusion at the first and the modified second order Bonferroni approximations in this example, a different result may come out when sample size is small and kurtosis parameter is large.

$k = 10, p = 5, \alpha = 0.05, r = 1$										
κ	N	$t_{1\cdot\chi^2}$	$t_{1\cdot F}$	$P_{1\cdot\chi^2}$	$P_{1\cdot F}$	$t_{M\cdot\chi^2}$	$t_{M\cdot F}$	$P_{M\cdot\chi^2}$	$P_{M\cdot F}$	t^*
0	10	4.27	4.32	.955	.960	4.22	4.26	.947	.953	4.24
	20	4.17	4.18	.958	.959	4.11	4.12	.949	.951	4.11
	40	4.12	4.12	.958	.959	4.05	4.06	.950	.950	4.06
	80	4.09	4.09	.958	.958	4.03	4.03	.950	.950	4.03
1.78	10	4.43	4.48	.970	.974	4.38	4.43	.966	.970	4.23
	20	4.25	4.26	.960	.961	4.19	4.20	.952	.953	4.17
	40	4.16	4.16	.959	.959	4.09	4.09	.950	.951	4.09
	80	4.11	4.11	.959	.959	4.05	4.05	.950	.950	4.05
3.24	10	4.56	4.61	.970	.973	4.52	4.56	.966	.970	4.37
	20	4.32	4.33	.963	.964	4.25	4.26	.956	.957	4.21
	40	4.19	4.20	.960	.960	4.12	4.13	.951	.952	4.11
	80	4.13	4.13	.959	.959	4.06	4.06	.950	.950	4.06

Table 1. Simulated and approximate values and lower tail probabilities for equal sample sizes.

$k = 10, p = 5, \alpha = 0.05, N_1 = N, r = 0.5$										
κ	N	$t_{1\cdot\chi^2}$	$t_{1\cdot F}$	$P_{1\cdot\chi^2}$	$P_{1\cdot F}$	$t_{M\cdot\chi^2}$	$t_{M\cdot F}$	$P_{M\cdot\chi^2}$	$P_{M\cdot F}$	t^*
0	10	4.34	4.43	.946	.957	4.31	4.40	.941	.953	4.38
	20	4.20	4.23	.953	.955	4.17	4.19	.948	.950	4.18
	40	4.13	4.14	.955	.956	4.09	4.10	.949	.950	4.10
	80	4.10	4.10	.955	.955	4.06	4.06	.950	.950	4.06
1.78	10	4.64	4.72	.959	.966	4.62	4.71	.957	.965	4.56
	20	4.36	4.38	.957	.959	4.33	4.35	.954	.957	4.30
	40	4.21	4.22	.956	.956	4.18	4.19	.952	.952	4.17
	80	4.14	4.14	.955	.956	4.10	4.10	.950	.951	4.10
3.24	10	4.86	4.94	.968	.973	4.86	4.94	.967	.973	4.66
	20	4.48	4.50	.962	.963	4.46	4.48	.960	.962	4.38
	40	4.28	4.28	.957	.958	4.25	4.25	.954	.954	4.22
	80	4.17	4.17	.956	.956	4.14	4.14	.951	.951	4.13

Table 2. Simulated and approximate values and lower tail probabilities for unequal sample sizes ($N_i = N$ $(i = 1, \ldots, 5)$, $N_j = rN$ $(j = 6, \ldots, 10)$).

$k = 10, p = 5, \alpha = 0.05, N_1 = rN, r = 0.5$										
κ	N	$t_{1\cdot\chi^2}$	$t_{1\cdot F}$	$P_{1\cdot\chi^2}$	$P_{1\cdot F}$	$t_{M\cdot\chi^2}$	$t_{M\cdot F}$	$P_{M\cdot\chi^2}$	$P_{M\cdot F}$	t^*
0	10	4.34	4.43	.954	.964	4.26	4.34	.944	.954	4.31
	20	4.20	4.23	.961	.963	4.11	4.13	.948	.951	4.12
	40	4.13	4.14	.962	.962	4.04	4.04	.949	.950	4.04
	80	4.10	4.10	.962	.962	4.00	4.00	.949	.949	4.01
1.78	10	4.80	4.88	.980	.983	4.74	4.82	.977	.981	4.38
	20	4.45	4.47	.970	.971	4.34	4.35	.960	.962	4.24
	40	4.26	4.26	.966	.966	4.14	4.14	.953	.953	4.11
	80	4.16	4.16	.964	.964	4.05	4.05	.949	.949	4.05
3.24	10	5.15	5.22	.987	.989	5.13	5.20	.987	.989	4.47
	20	4.64	4.65	.976	.977	4.55	4.56	.970	.971	4.32
	40	4.36	4.36	.969	.970	4.23	4.24	.957	.958	4.17
	80	4.21	4.21	.965	.965	4.09	4.09	.951	.951	4.08

Table 3. Simulated and approximate values and lower tail probabilities for unequal sample sizes ($N_i = rN$ $(i = 1, \ldots, 5)$, $N_j = N$ $(j = 6, \ldots, 10)$).

	Π_1					Π_2					Π_3				
No.	Ja	So	Ma	Sc	En	Ja	So	Ma	Sc	En	Ja	So	Ma	Sc	En
1	64	36	20	31	26	42	31	47	44	32	29	21	26	30	6
2	68	59	60	63	63	95	87	77	100	87	77	54	58	84	57
3	68	53	41	57	71	74	60	57	78	71	42	16	29	43	2
4	81	53	78	81	80	66	43	54	72	53	39	19	5	32	10
5	58	62	40	66	46	27	29	34	40	40	31	14	8	16	2
6	72	41	36	44	31	50	28	29	31	7	39	21	49	56	16
7	32	28	32	41	10	36	4	15	43	32	48	63	72	83	67
8	89	83	73	81	68	54	21	43	28	14	55	42	85	86	80
9	68	28	60	69	36	49	18	27	20	28	82	86	75	84	61
10	81	50	61	63	49	26	15	13	19	8	41	34	28	43	18
11	63	32	43	52	56	76	63	74	72	72	84	90	87	100	90
12	77	64	87	71	71	69	49	30	34	40	83	83	58	71	92
13	91	69	100	83	91	70	57	60	61	76	56	54	37	59	7
14	64	40	30	49	41	43	32	74	55	39	32	7	14	18	7
15	58	17	31	23	25	35	23	17	38	36	73	81	94	85	95
16	42	16	27	22	20	46	53	34	30	27	74	66	65	78	76
17	87	86	77	78	76	87	85	84	93	77	36	27	62	58	66
18	73	74	74	61	78	59	25	45	48	32	71	38	80	61	84
19	34	27	48	52	39	49	26	62	50	82	29	15	37	19	12
20	50	31	53	44	58	28	4	37	29	14	70	71	78	67	44
21	33	33	32	42	11	16	16	25	22	40	66	34	57	50	27
22	47	21	26	25	23	70	23	43	50	46	78	43	59	55	71
23	76	42	58	51	39	57	53	75	58	69	82	84	86	88	88
24	62	39	42	40	34	67	53	56	61	40	74	63	79	69	45
25	39	24	26	24	15	23	9	5	26	3	51	41	63	60	42
26	96	76	90	77	91	35	18	28	25	4	23	24	20	36	10
27	42	29	33	48	38	70	41	44	34	6	8	12	0	20	1
28	70	56	76	62	88	45	26	29	24	27	80	63	59	73	82
29	62	43	14	48	15	43	24	29	47	18	50	43	80	73	10
30	65	43	36	49	30	52	29	32	36	27	0	8	2	9	1
31	78	53	45	54	45	32	21	24	28	1	50	44	47	56	75
32	83	54	44	45	81	53	23	28	30	18	37	36	45	41	42
33	60	48	67	29	54	22	9	6	26	1					
34	58	17	34	19	14	61	49	71	71	71					
35	73	60	59	54	65	40	38	7	21	23					
36	74	62	44	58	66	19	12	9	28	5					
37	82	78	80	88	90	58	31	46	46	16					
38	84	51	76	84	95										
39	84	57	45	64	79										
40	57	40	43	28	32										
41	86	62	78	54	64										
42	65	43	38	70	26										
43	73	31	43	32	56										
44	44	32	31	28	37										
45	60	85	89	80	85										
46	48	23	41	40	23										

Table 4. The school-record data of the second-year student in a junior high school in Tokyo.

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Received 18 September 2013
Revised 18 November 2013