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ON SOME TOPOLOGICAL METHODS IN THEORY OF NEUTRAL TYPE OPERATOR DIFFERENTIAL INCLUSIONS WITH APPLICATIONS TO CONTROL SYSTEMS

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Abstract

We consider a neutral type operator differential inclusion and apply the topological degree theory for condensing multivalued maps to justify the question of existence of its periodic solution. By using the averaging method,

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we apply the abstract result to an inclusion with a small parameter. As example, we consider a delay control system with the distributed control.

Keywords: operator differential inclusion, neutral type, periodic solution, fixed point, multivalued map, condensing map, topological degree, averaging method, control system, distributed control.

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1. Introduction

In this paper, we study the existence of periodic solutions for a neutral type operator differential inclusion of the form

$$\dot{x} \in \mathcal{F}(x, \dot{x}).$$

It is shown that under appropriate conditions this problem may be reduced to the existence of a fixed point for a condensing multivalued operator and the general existence principle (Theorem 3) can be formulated in terms of the topological degree theory for condensing multifields (see [1, 5]). As an example of application of this general principle, we consider, by using the averaging method, the solvability of a neutral type operator differential inclusion with a small parameter (Theorem 4). In the last section of the paper we present an example showing how the periodic problem for a system with the distributed control governed by the delay functional differential equation of neutral type can be reduced to the above inclusion.

2. Preliminaries

We describe some notions of the theory of multivalued maps that will be used in the sequel (details can be found, e.g., in [1]–[5]).

Let (X, d_X) and (Z, d_Z) be metric spaces. By the symbols $K(Z)$ we denote the collection of all nonempty compact subsets of Z. If Z is a normed space, the symbol $Kv(Z)$ denotes the collection of all nonempty compact convex subsets of Z.

Definition 1. A multivalued map (multimap) $\Phi : X \to K(Z)$ is said to be upper semicontinuous (u.s.c.) at a point $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_X(x_0, x) < \delta$ implies $\Phi(x) \subset V_\varepsilon(\Phi(x_0))$, where V_ε denotes the ε-neighborhood of a set. If Φ is u.s.c. at each point $x \in X$, it is called u.s.c.

Definition 2. A u.s.c. multimap $\Phi : X \to K(Z)$ is *completely u.s.c.* if its restriction on each bounded subset $\Omega \subset X$ is compact, i.e., the set $\Phi(\Omega)$ is relatively compact in Z.

Let us mention the following useful fact (see, e.g., [2]–[5]).

Proposition 1. If a multimap $F : X \to K(Y)$ is u.s.c. and $M \subset X$ is a compact set, then the set $F(M)$ is compact in Y.

Definition 3. A real-valued function χ that assigns to each bounded set $\Omega \subset X$ the number

$$\chi(\Omega) = \inf\{\epsilon > 0 : \Omega \text{ has a finite } \epsilon - \text{net}\}$$

is called the Hausdorff measure of noncompactness (MNC) in X.

In the sequel we will need the following assertion.

Lemma 1. *Let (X, d) be a metric space and Z a normed space; $A \subseteq X$ a closed subset; a multimap $\mathcal{G} : A \times X \to K(Z)$ satisfies the following conditions:*

($\mathcal{G}1$) *for each $y \in X$, the multimap $\mathcal{G}(\cdot, y) : A \to K(Z)$ is completely u.s.c.;*

($\mathcal{G}2$) *for each $x \in A$, the multimap $\mathcal{G}(x, \cdot) : X \to K(Z)$ is k-Lipschitz w.r.t. the Hausdorff metric h on $K(Z)$, where $k \geq 0$, i.e.,*

$$h\left(\mathcal{G}(x, y), \mathcal{G}(x, y')\right) \leq kd(y, y')$$

for all $y, y' \in X$.

Then the multimap $\mathcal{H} : A \to K(Z)$, $\mathcal{H}(x) = \mathcal{G}(x, x)$ is u.s.c. and (k, χ_0, χ_1)-bounded, where χ_0 and χ_1 are the Hausdorff MNCs in X and Z respectively, i.e.,

$$\chi_1(\mathcal{H}(\Omega)) \leq k\chi_0(\Omega)$$

for each bounded set $\Omega \subseteq A$.

Proof. Take any point $(x_0, y_0) \in A \times X$ and fix $\varepsilon > 0$. Denote by B_r an open ball in Z of radius $r > 0$ centered at the origin. Take convergent sequences $\{x_n\} \subset A$ and $\{y_n\} \subset X : x_n \to x_0$ and $y_n \to y_0$. Suppose that an integer N is such that for all $n \geq N$:

(i) $d(y_n, y_0) \leq \frac{\varepsilon}{2k}$;

(ii) $\mathcal{G}(x_n, y_0) \subset \mathcal{G}(x_0, y_0) + \frac{\varepsilon}{2} B_1$.

From condition $(\mathcal{G}2)$ it follows that for $n \geq N$ we have

$$\mathcal{G}(x_n, y_n) \subset \mathcal{G}(x_n, y_0) + kB_{d(y_n,y_0)} \subset \mathcal{G}(x_n, y_0) + \frac{\varepsilon}{2}B_1.$$

Applying (ii), we obtain

$$\mathcal{G}(x_n, y_n) \subset \mathcal{G}(x_0, y_0) + \varepsilon B_1$$

for $n \geq N$, demonstrating that $\mathcal{G}$ is u.s.c. at (x_0, y_0), the conclusion that $\mathcal{H}$ is u.s.c. immediately follows.

The fact that the multimap $\mathcal{H}$ is (k, χ_0, χ_1)-bounded, can be proved in the same way as Proposition 2.2.2 in [5]. ■

Now, suppose that U is an open bounded subset in a Banach space $\mathcal{E}$, its closure is denoted as $\overline{U}$ and ∂U denotes its boundary. Let $\Phi : \overline{U} \to Kv(\mathcal{E})$ be a u.s.c. multimap such that

$$Fix\Phi \cap \partial U = \emptyset,$$

where $Fix\Phi = \{x : x \in \Phi(x)\}$ is *the fixed point set.* If Φ is *k-condensing*, $0 \leq k < 1$, w.r.t. the Hausdorff MNC $\chi_{\mathcal{E}}$, i.e.,

$$\chi_{\mathcal{E}}(\Phi(\Omega)) \leq k\chi_{\mathcal{E}}(\Omega)$$

for each $\Omega \subset \overline{U}$, then *the topological degree* $deg(i - \Phi, \overline{U})$ of the corresponding *multivalued vector field (multifield)* $\Psi = i - \Phi$, $\Psi(x) = x - \Phi(x)$ is well defined and satisfies all usual properties. Let us mention some of them.

1) *Fixed point property.* If $deg(i - \Phi, \overline{U}) \neq 0$, then $Fix\Phi$ is a nonempty compact subset of U.

2) *Map restriction property.* If $\mathcal{E}'$ is a closed subspace of $\mathcal{E}$ and $\Phi(\overline{U}) \subset \mathcal{E}'$, then

$$deg(i - \Phi, \overline{U}) = deg_{\mathcal{E}'}(i - \Phi, \overline{U \cap \mathcal{E}'}),$$

where $deg_{\mathcal{E}'}$ denotes the degree evaluated in the subspace $\mathcal{E}'$.

3) *Homotopy invariance property.* If $\Upsilon : \overline{U} \times [0,1] \to Kv(\mathcal{E})$ is a family of k-condensing multimaps in the sense that

$$\chi_{\mathcal{E}}(\Upsilon(\Omega \times [0,1])) \leq k\chi_{\mathcal{E}}(\Omega)$$

for each $\Omega \subset \overline{U}$, and $Fix\Upsilon(\cdot, \lambda) \cap \partial U = \emptyset$ for all $\lambda \in [0,1]$, then

$$deg(i - \Upsilon(\cdot, 0), \overline{U}) = deg(i - \Upsilon(\cdot, 1), \overline{U}).$$

As the corollary of the last property, we can consider the following assertion.

Proposition 2. Suppose that u.s.c. k-condensing multimaps $\Phi_0, \Phi_1 : \overline{U} \to Kv(\mathcal{E})$ satisfy the following boundary condition: for each $x \in \partial U$, the sets $\Psi_0(x) = x - \Phi_0(x)$ and $\Psi_1(x) = x - \Phi_1(x)$ do not contain the vectors which have opposite directions, i.e.,

$$\frac{y_0}{\|y_0\|} \neq -\frac{y_1}{\|y_1\|} \text{ for all } y_0 \in \Psi_0(x),\ y_1 \in \Psi_1(x).$$

Then

$$deg(i - \Phi_0, \overline{U}) = deg(i - \Phi_1, \overline{U}).$$

3. General existence principle

For a given $T > 0$, by C_T and L^p_T we will denote, respectively, the space of $T-$periodic continuous functions $x : \mathbb{R} \to \mathbb{R}^n$ and the space of $T-$periodic $p-$integrable functions $y : \mathbb{R} \to \mathbb{R}^n$ $(p > 1)$ with norms:

$$\|x\|_C = sup_{t\in[0,T]}\|x(t)\|$$

and

$$\|y\|_{L^p} = \Big(\int_0^T \|y(s)\|^p ds\Big)^{\frac{1}{p}}.$$

We will consider the neutral type operator differential inclusion of the following form

$$\dot{x} \in \mathcal{F}(x, \dot{x}), \tag{3.1}$$

where the multimap $\mathcal{F} : C_T \times L^p_T \to Kv(L^p_T)$ satisfies the following conditions:

$\mathcal{F}1)$ for each $y \in L^p_T$, the multimap $\mathcal{F}(\cdot, y) : C_T \to Kv(L^p_T)$ is u.s.c.;

$\mathcal{F}2)$ for each $x \in C_T$, the multimap $\mathcal{F}(x, \cdot) : L^p_T \to Kv(L^p_T)$ is $k-$Lipschitz with respect to the Hausdorff metric h on $Kv(L^p_T)$ with $0 \leq k < 1$, i.e.,

$$h\big(\mathcal{F}(x, y_0), \mathcal{F}(x, y_1)\big) \leq k\|y_0 - y_1\|_{L^p} \tag{3.2}$$

for all $y_0, y_1 \in L^p_T$.

By *a solution of problem* (3.1) we mean a function x belonging to the Sobolev space $H^{1,p}_T$ of T-periodic functions satisfying inclusion (3.1).

Let us demonstrate that problem (3.1) can be reduced to a fixed point problem for an appropriate multimap. Denote $\mathcal{Y} = \mathbb{R}^n \times L^p_T$ and define the multimap $\Phi : \mathcal{Y} \to Kv(\mathcal{Y})$ as

$$\Phi(\lambda, v) = \big(\Phi_0(\lambda, v), \Phi_1(\lambda, v)\big), \tag{3.3}$$

where

$$\Phi_0(\lambda, v) = \lambda + m(v)$$

with

$$m(v) = \frac{1}{T}\int_0^T v(s)ds$$

and

$$\Phi_1(\lambda, v) = \mathcal{F}(\lambda + Jv, v)$$

with

$$(Jv)(t) = \int_0^t v(s)ds - tm(v).$$

The multimap Φ is said to be *associated with inclusion* (3.1).

One can verify the following assertion.

Proposition 3. If a function $x \in H^{1,p}_T$ is a solution of problem (3.1), then $(x(0), \dot{x}) \in Fix\Phi$. Conversely, if $(\lambda, v) \in Fix\Phi$, then $x = \lambda + Jv$ is the solution of (3.1).

Let the space $\mathcal{Y}$ be endowed with the following norm:

$$\|(\lambda, v)\|_{\mathcal{Y}} = \max\big\{\|\lambda\|_{\mathbb{R}^n}, \|v\|_{L^p_T}\big\}.$$

The corresponding Hausdorff MNC in $\mathcal{Y}$ will be denoted by $\chi_{\mathcal{Y}}$.

We will need the following auxiliary statement which can be easily verified.

Proposition 4. For each bounded set $\Delta \subset \mathcal{Y}$ we have

$$\chi_{\mathcal{Y}}(\Delta) = \chi_L(pr_L(\Delta)), \tag{3.4}$$

where $pr_L(\Delta)$ denotes the natural projection of Δ on L^p_T and χ_L is the Hausdorff MNC in the space L^p_T.

Let us mention the next important property of the multioperator Φ.

Theorem 2. *Under conditions* ($\mathcal{F}1$), ($\mathcal{F}2$) *the multioperator* Φ *is u.s.c. and* k*-condensing with respect to* $\chi_{\mathcal{Y}}$.

Proof. It is clear that, to prove the upper semicontinuity of Φ, it is sufficient to demonstrate the same property for the multimap Φ_1. Consider the multimap $\mathcal{G} : \mathcal{Y} \times \mathcal{Y} \to Kv(L^p_T)$ defined as

$$\mathcal{G}\big((\lambda, v), (\mu, w)\big) = \mathcal{F}(\lambda + Jv, w)$$

Notice that the map $\psi : \mathcal{Y} \to C_T$, $\psi(\lambda, v) = \lambda + Jv$ transforms each bounded set $\Delta \subset \mathcal{Y}$ into a relatively compact set in C_T. Then, by using Proposition 1, it is easy to see that the multimap $\mathcal{G}$ satisfies conditions of Lemma 2 and hence the multimap $\Phi_1(\lambda, v) = \mathcal{G}\big((\lambda, v), (\lambda, v)\big)$ is u.s.c. and, moreover, applying Proposition 4, for each bounded set $\Delta \subset \mathcal{Y}$ we have:

$$\chi_{\mathcal{Y}}(\Phi(\Delta)) = \chi_L(\Phi_1(\Delta)) \leq k\chi_{\mathcal{Y}}(\Delta).$$

■

As the direct consequence, it follows that we may formulate the next general existence principle for inclusion (3.1).

Theorem 3. *Let the multimap $\mathcal{F} : C_T \times L^p_T \to Kv(L^p_T)$ satisfy conditions ($\mathcal{F}1$), ($\mathcal{F}2$) and there exists a bounded open set $U \subset \mathcal{Y}$ such that inclusion (3.1) has no solutions $x(\cdot)$ such that $(x(0), \dot{x}) \in \partial U$ and*

$$deg(i - \Phi, \overline{U}) \neq 0.$$

Then inclusion (3.1) has a solution $x(\cdot)$ such that $(x(0), \dot{x}) \in U$.

We will demonstrate the application of this principle to the solvability of the following parametrized inclusion

$$\dot{x} \in \varepsilon\mathcal{F}(x, \dot{x}), \tag{3.5}$$

where the multimap $\mathcal{F}$ satisfies conditions ($\mathcal{F}1$), ($\mathcal{F}2$) and the parameter $\varepsilon > 0$ is sufficiently small.

Towards this goal, introduce the multioperator $\mathcal{F}_0 : \mathbb{R}^n \to Kv(\mathbb{R}^n)$ defined in the following way

$$\mathcal{F}_0(\lambda) = \frac{1}{T}\int_0^T \mathcal{F}(\overline{\lambda}, 0)(s)ds := \left\{\frac{1}{T}\int_0^T f(s)ds : f \in \mathcal{F}(\overline{\lambda}, 0)\right\}, \tag{3.6}$$

where $\overline{\lambda} \in C_T$ is the constant function that is equal identically to $\lambda \in \mathbb{R}^n$.

Theorem 4. *Suppose that inclusion*

$$0 \in \mathcal{F}_0(\lambda) \tag{3.7}$$

has a solution $\lambda^* \in \mathbb{R}^n$ *with a bounded neighborhood* $U(\lambda^*)$ *such that*

$$0 \notin \mathcal{F}_0(\lambda), \ \ \forall \lambda \in \partial U(\lambda^*) \tag{3.8}$$

and

$$deg\,(-\mathcal{F}_0, U(\lambda^*)) \neq 0. \tag{3.9}$$

Then for all sufficiently small $\varepsilon > 0$ *inclusion* (3.5) *has a solution* x_ε *such that* $x_\varepsilon(t) \in U(\lambda^*)$ *for all* $t \in [0,T]$ *and* $\|\dot{x}_\varepsilon\|_{L^p} \to 0$ *while* $\varepsilon \to 0$.

Proof. Consider the set $\mathcal{V} = U(\lambda^*) \times B_{L^p_T}(0,r) \subset \mathcal{Y}$, where $r > 0$ and define the multimap $\Gamma_\varepsilon : \overline{\mathcal{V}} \to Kv(\mathbb{R}^n \times \mathbb{R}^n) \subset Kv(\mathcal{Y})$ as

$$\Gamma_\varepsilon(\lambda, y) = (\lambda + m(y),\, m(y) + \varepsilon\mathcal{F}_0(\lambda))\,.$$

Notice that for each $(\lambda, y) \in \partial\mathcal{V}$ we have

$$(\lambda, y) \notin \Gamma_\varepsilon(\lambda, y).$$

Indeed, supposing the contrary, we will have for some $(\lambda_0, y_0) \in \partial\mathcal{V}$:

$$\lambda_0 = \lambda_0 + m(y_0),$$

$$y_0 \in m(y_0) + \varepsilon\mathcal{F}_0(\lambda_0).$$

Since y_0 is a constant function, from the first equality it follows that $y_0 \equiv 0$. Hence $\lambda_0 \in \partial U(\lambda^*)$ and the last inclusion yields

$$0 \in \varepsilon\mathcal{F}_0(\lambda_0).$$

Since $\varepsilon > 0$ we get

$$0 \in \mathcal{F}_0(\lambda_0).$$

contrary to assumption (3.8).

Now, let us evaluate

$$deg(i - \Gamma_\varepsilon, \overline{\mathcal{V}}).$$

Since the image of the multimap Γ_ε is contained in the space $\mathbb{R}^n \times \mathbb{R}^n$, by applying the map restriction property of the topological degree, we get

$$deg(i - \Gamma_\varepsilon, \overline{\mathcal{V}}) = deg_{\mathbb{R}^n \times \mathbb{R}^n}(i - \Gamma_\varepsilon, \overline{\mathcal{V}_n}),$$

where $\mathcal{V}_n = U(\lambda^*) \times B_{\mathbb{R}^n}(0, r)$.

Further, considering ε as the homotopy parameter, we obtain

$$deg_{\mathbb{R}^n\times\mathbb{R}^n}(i-\Gamma_\varepsilon,\overline{\mathcal{V}_n}) = deg_{\mathbb{R}^n\times\mathbb{R}^n}(i-\Gamma_1,\overline{\mathcal{V}_n}).$$

The multimap Γ_1 on $\overline{\mathcal{V}_n}$ is defined by the formula

$$\Gamma_1(u_1,u_2) = (u_1+u_2, u_2+\mathcal{F}_0(u_1))$$

and hence the corresponding multifield $i-\Gamma_1$ on the same set has the form

$$(i-\Gamma_1)(u_1,u_2) = A(-\mathcal{F}_0(u_1), -u_2),$$

where $A(v_1,v_2)=(v_2,v_1)$.

Applying the multiplicative property of the topological degree (see, e.g., [1, 6]), we get

$$deg_{\mathbb{R}^n\times\mathbb{R}^n}(i-\Gamma_1,\overline{\mathcal{V}_n}) = (-1)^n deg\,(-\mathcal{F}_0, U(\lambda^*))$$

yielding

$$deg(i-\Gamma_\varepsilon,\overline{\mathcal{V}}) = (-1)^n deg\,(-\mathcal{F}_0, U(\lambda^*)) \neq 0$$

for each $\varepsilon > 0$.

Let $\Phi_\varepsilon : \mathcal{Y}\to Kv(\mathcal{Y})$ be the multimap associated with inclusion (3.5). We will show that, for sufficiently small $\varepsilon>0$, the multifields $i-\Phi_\varepsilon$ and $i-\Gamma_\varepsilon$ do not admit opposite directions on $\partial\mathcal{V}$.

Supposing the contrary, we will have sequences $\varepsilon_k\to 0$, and

$$(\lambda_k, y_k)\in\partial\mathcal{V} \tag{3.10}$$

such that the sets $(\lambda_k,y_k)-\Phi_\varepsilon(\lambda_k,y_k)$ and $(\lambda_k,y_k)-\Gamma_\varepsilon(\lambda_k,y_k)$ contain opposite vectors. Since the first components of both sets are equal to $m(y_k)$, we have $m(y_k)=0$ for all $k\geq 1$. Considering the second component, we conclude that there exists a sequence $\{\alpha_k\}\subset[0,1]$ such that

$$y_k\in\alpha_k\varepsilon_k\mathcal{F}(\lambda_k+Jy_k,y_k)+(1-\alpha_k)\varepsilon_k\mathcal{F}_0(\lambda_k),\ \forall k\geq 1. \tag{3.11}$$

From Theorem 2 it follows that the set $\bigcup_k\mathcal{F}(\lambda_k+Jy_k,y_k)$ is bounded, so there exists a constant $C\geq 0$ such that

$$\|y_k\|_{L^p}\leq C\varepsilon_k,\ \forall k\geq 1. \tag{3.12}$$

Hence

$$\|y_k\|_{L^p}\to 0 \text{ as } k\to\infty.$$

Further, we can assume, w.l.o.g., that $\lambda_k \to \lambda_0$. Passing to the limit in inclusions (3.10), we obtain

$$(\lambda_0, 0) \in \partial \Big(U(\lambda^*) \times B_{L^p_T}(0, r) \Big)$$

that yields $\lambda_0 \in \partial U(\lambda^*)$.

Averaging both the parts of inclusion (3.11), and dividing on $\varepsilon_k > 0$ we get

$$0 \in \alpha_k m \left(\mathcal{F}(\lambda_k + Jy_k, y_k) \right) + (1 - \alpha_k)\mathcal{F}_0(\lambda_k), \ \forall k \geq 1. \tag{3.13}$$

Taking into consideration that

$$\|Jy_k\|_C \leq T^{1-\frac{1}{p}} 2 \|y_k\|_{L^p_T} \to 0$$

and passing to the limit in inclusion (3.13) we obtain

$$0 \in \mathcal{F}_0(\lambda_0)$$

that is the contradiction.

So, applying Proposition 2, we come to the conclusion that

$$deg(i - \Phi_\varepsilon, \overline{\mathcal{V}}) = deg(i - \Gamma_\varepsilon, \overline{\mathcal{V}}) \neq 0$$

for each $\varepsilon > 0$ sufficiently small, yielding the existence of pairs $(\lambda_\varepsilon, y_\varepsilon) \in \mathcal{V}$ satisfying

$$(\lambda_\varepsilon, y_\varepsilon) \in \Phi_\varepsilon(\lambda_\varepsilon, y_\varepsilon). \tag{3.14}$$

But then functions

$$x_\varepsilon = \lambda_\varepsilon + J(y_\varepsilon)$$

are solutions of (3.5).

Notice that inclusions (3.14) imply the existence of a constant $C_1 \geq 0$ such that

$$\|y_\varepsilon\|_{L^p} \leq C_1 \varepsilon,$$

yielding

$$\|\dot{x}_\varepsilon\|_{L^p} = \|y_\varepsilon\|_{L^p} \to 0 \text{ as } \varepsilon \to 0.$$

■

4. Example: periodic control problem

Consider a periodic control system governed by the following neutral type functional differential equation:

$$x'(t) = f\left(t, x(t-h), x'(t-h), \frac{1}{\delta}\int_{t-\delta}^{t} u(s)ds\right), \tag{4.1}$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a Carathéodory type function, T-periodic in the first argument, $h > 0$ and $\delta > 0$ are given numbers. It is supposed that f is locally integrably bounded in the sense that for each $r > 0$ there exists a function $\nu_r \in L^p_+[0,T]$, $p > 1$ such that

$$\|f(t,x,y,v)\| \leq \nu_r(t) \quad \text{for a.e.} \quad t \in [0,T],$$

provided $\|x\|, \|y\|, \|v\| \leq r$.

We will assume also that f is k-Lipschitz in the third argument with $0 \leq k < 1$:

$$\|f(t,x,y_0,v) - f(t,x,y_1,v)\| \leq k\|y_0 - y_1\|$$

for a.e. $t \in [0,T]$ and $x, y_0, y_1 \in \mathbb{R}^n; v \in \mathbb{R}^m$.

We will suppose that the set $f(t,x,y,M)$, $t \in \mathbb{R}$; $x, y \in \mathbb{R}^n$ is convex for each convex $M \subset \mathbb{R}^m$.

At last, we assume that control functions u belong to the space of periodic functions $L^q_T(\mathbb{R}^m)$, $q > 1$ and they fill the set of admissible controls $\mathfrak{M}$ defined as

$$\mathfrak{M} = \{u \in L^q_T(\mathbb{R}^m) : u(t) \in V \quad \text{for a.e.} \quad t \in [0,T]\},$$

where $V \subset \mathbb{R}^m$ is a given compact convex set.

Notice that the corresponding set

$$\mathfrak{N} = \left\{v : v(t) = \frac{1}{\delta}\int_{t-\delta}^{t} u(s)ds\right\}$$

is compact and convex in $C_T(\mathbb{R}^m)$.

Define the multimap $\mathcal{F} : C_T \times L^p_T \multimap L^p_T$ by

$$\mathcal{F}(x,y) = \{z \in L^p_T : z(t) = f(t, x(t-h), y(t-h), v(t)), \ v \in \mathfrak{N}\}.$$

By using [7], page 384, one can show that each set $\mathcal{F}(x,y)$ is convex compact. It is easy to verify also that the multimap $\mathcal{F}$ satisfies conditions ($\mathcal{F}1$) and ($\mathcal{F}2$) of the previous section. So, the study of system (4.1) can be reduced to the investigation of inclusion (3.1) where we can apply the topological degree theory to its analysis.

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