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ON PERIODIC OSCILLATIONS FOR A CLASS OF FEEDBACK CONTROL SYSTEMS IN HILBERT SPACES

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Abstract

In this paper, by using the topological degree theory for multivalued maps and the method of guiding functions in Hilbert spaces we deal with the existence of periodic oscillations for a class of feedback control systems in Hilbert spaces.

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1. Introduction

The existence of periodic solutions for semilinear differential inclusions in Banach spaces was studied by a number of researchers (see, e.g., [12] and the references therein). The usual way for the investigation of this problem is to apply the method of integral multivalued operators or the method of the translation multivalued operator.

In the present paper, by combining the topological method and the method of guiding functions in Hilbert spaces (see [13, 14, 16]) we study the periodic oscillations in control systems governed by semilinear differential inclusions in Hilbert spaces. In comparison with the previous investigations, we consider the control function subject to a differential inclusion whose right-hand side depends on the state function, i.e., we consider the feedback control problem.

The paper is organized in the following way. In the next section we recall some basic facts from the theory of multivalued maps and the theory of linear Fredholm operators. The statement of the problem and the main result are given in Section 3.

2. Preliminaries

2.1. Multimaps

Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Z}, d_{\mathcal{Z}})$ be metric spaces. By the symbols $P(\mathcal{Z})$ $[K(\mathcal{Z})]$, we denote the collections of all nonempty [resp., nonempty compact] subsets of $\mathcal{Z}$. If $\mathcal{Z}$ is a normed space, the symbols $Kv(\mathcal{Z})$, $Cv(\mathcal{Z})$ denote the collections of nonempty compact convex and nonempty convex closed subsets of $\mathcal{Z}$, respectively.

A multivalued map (multimap) $\Sigma : \mathcal{X} \to P(\mathcal{Z})$ is said to be *upper semicontinuous (u.s.c.)* if for every open set $V \subset \mathcal{Z}$, the set $\Sigma_+^{-1}(V) = \{x \in \mathcal{X} : \Sigma(x) \subset V\}$ is open in $\mathcal{X}$.

An u.s.c. multimap $\Sigma : \mathcal{X} \to K(\mathcal{Z})$ is said to be *completely u.s.c.* if its restriction on each bounded subset $\Omega \subset \mathcal{X}$ is compact, i.e., the set $\Sigma(\Omega)$ is relatively compact in $\mathcal{Z}$.

Definition 1 (see, e.g., [3, 4, 8, 9, 15])**.** A set $\mathcal{M} \in K(\mathcal{Z})$ is said to be *aspheric* (or UV^∞, or ∞*-proximally connected*) if for every $\varepsilon > 0$ there exists $\delta, 0 < \delta < \varepsilon$, such that for each $n = 0, 1, 2, \ldots$ every continuous map $g : S^n \to O_\delta(\mathcal{M})$ can be extended to a continuous map $\widetilde{g} : B^{n+1} \to O_\varepsilon(\mathcal{M})$, where $S^n = \left\{x \in \mathbb{R}^{n+1} : \|x\| = 1\right\}$ and $B^{n+1} = \left\{x \in \mathbb{R}^{n+1} : \|x\| \leq 1\right\}$.

Definition 2 (see [11])**.** A nonempty compact space $\mathcal{A}$ is said to be an R_δ-set if it can be represented as the intersection of a decreasing sequence of compact, contractible spaces.

Definition 3 (see [8])**.** An u.s.c. multimap $\Sigma : \mathcal{X} \to K(\mathcal{Z})$ is said to be a J-multimap ($\Sigma \in J(\mathcal{X}, \mathcal{Z})$) if every value $\Sigma(x)$, $x \in \mathcal{X}$ is an aspheric set.

Let us recall (see, e.g., [5, 8]) that a metric space $\mathcal{X}$ is called *the absolute neighborhood retract* (*the ANR-space*) if for each homeomorphism h taking it onto a closed subset of a metric space $\mathcal{X}'$, the set $h(\mathcal{X})$ is a retract of its open neighborhood in $\mathcal{X}'$. If the set $h(\mathcal{X})$ is a retract of the whole space $\mathcal{X}'$, it is called *the absolute retract* (*the AR-space*).

Notice that the collection of the ANR-spaces is broad enough. In fact, the following assertion holds true (see [5]).

Proposition 4. *A compact subset of a finite-dimensional space is the ANR-space if and only if it is locally contractible.*

Remark 5. By the known Whitney embedding theorem, the previous proposition implies that each compact finite-dimensional manifold is the ANR-space.

Furthermore, the union of a finite number of convex closed subsets in a normed space is the ANR-space.

Proposition 6 (see [8]). *Let $\mathcal{Z}$ be an ANR-space. In each of the following cases a u.s.c. multimap $\Sigma : \mathcal{X} \to K(\mathcal{Z})$ is a J-multimap: for each $x \in \mathcal{X}$ the value $\Sigma(x)$ is*

$(a1)$ *a convex set;*

$(a1)$ *a contractible set;*

$(a3)$ *an R_δ-set;*

$(a4)$ *an AR-space.*

In particular, every continuous map $\sigma\colon \mathcal{X} \to \mathcal{Z}$ is a J-multimap.

Definition 7. By $J^c(\mathcal{X}, \mathcal{Z})$ we will denote the collection of all multimaps $G\colon \mathcal{X} \to K(\mathcal{Z})$ of the form $G = \Sigma_q \circ \cdots \circ \Sigma_1$, $q \geq 1$, where $\Sigma_i \in J(\mathcal{X}_{i-1}, \mathcal{X}_i)$, $i = 1, 2, \dots, q$, $\mathcal{X}_0 = \mathcal{X}$, $\mathcal{X}_q = \mathcal{Z}$ and $\mathcal{X}_i$ for $1 < i < q$ are open subsets of normed spaces.

Let us recall (see, e.g., [1]) that if U is an open bounded subset of a Banach space E and $F\colon \overline{U} \to K(E)$ is a compact J^c-multimap such that $x \notin F(x)$ for all $x \in \partial U$, then for the corresponding multifield $i - F$, where i denotes the inclusion map, the topological degree $deg(i - F, \overline{U})$ is well-defined and has all usual properties.

2.2. Fredholm operators

Definition 8 (see, e.g., [7]). A linear operator $L\colon domL \subseteq X \to Y$ is called Fredholm of index zero if

(1) ImL is closed in Y;

(2) $KerL$ and $CokerL$ have the finite dimension and

$$dimKerL = dimCokerL.$$

For Banach spaces X and Y let us denote by $\mathcal{L}(X, Y)$ the set of all linear bounded operators from X to Y. If Y coincides with X, then instead of $\mathcal{L}(X, Y)$ we use the notation $\mathcal{L}(X)$.

Throughout this paper by H we denote a separable Hilbert space which is compactly embedded in a separable Banach space E with the relation of norms

$$\|w\|_E \leq q\|w\|_H, \quad \forall w \in H, \tag{2.1}$$

where $q > 0$. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of H. For every $n \in \mathbb{N}$, let H_n be an n-dimensional subspace of H with the basis $\{e_k\}_{k=1}^{n}$ and P_n be a projection of H onto H_n. By $\langle \cdot, \cdot \rangle_H$ we denote the inner product in H. In the sequel everywhere the symbol I denotes the interval $[0, T]$. By $C(I, H)$ $[L^2(I, H)]$

we denote the spaces of all continuous [respectively, square summable] functions $u\colon I \to H$ with usual norms

$$\|u\|_C = \max_{t\in I} \|u(t)\|_H \quad \text{and} \quad \|u\|_2 = \left(\int_0^T \|u(t)\|_H^2 dt\right)^{\frac{1}{2}}.$$

By $B_C(0,R)$ we denote the *open ball* of radius R centered at 0 in $C(I,H)$. The symbol $\langle\cdot,\cdot\rangle_{L^2}$ denotes the inner product in $L^2(I,H)$.

Consider the space of all absolutely continuous functions $u : I \to H$ whose generalized derivatives belong to $L^2(I,H)$. It is known (see, e.g., [2]) that this space can be identified with the Sobolev space $W^{1,2}(I,H)$ endowed with the norm

$$\|u\|_W = \left(\|u\|_2^2 + \|u'\|_2^2\right)^{1/2}.$$

The embedding $W^{1,2}(I,H) \hookrightarrow C(I,H)$ is continuous, and for every $n \geq 1$ the space $W^{1,2}(I,H_n)$ is compactly embedded in $C(I,H_n)$. The weak convergence in $W^{1,2}(I,H)$ $[L^2(I,H)]$ is denoted by $x_n \overset{W}{\rightharpoonup} x_0$ [respectively, $f_n \overset{L^2}{\rightharpoonup} f_0$].

By $W_T^{1,2}(I,H)$ we denote the subspace of all functions $x \in W^{1,2}(I,H)$ satisfying the boundary condition $x(0) = x(T)$.

Let $n \in \mathbb{N}$, and $\ell\colon W_T^{1,2}(I,H_n) \to L^2(I,H_n)$, $\ell x = x'$. Then ℓ is a linear Fredholm operator of index zero, and there exist the projections (see, e.g., [7]):

$$C_n\colon W_T^{1,2}(I,H_n) \to W_T^{1,2}(I,H_n)$$

and

$$Q_n\colon L^2(I,H_n) \to L^2(I,H_n)$$

such that $Im\ C_n = Ker\,\ell = H_n$ and $Ker\ Q_n = Im\,\ell$. If the operator

$$\ell_{C_n}\colon dom\,\ell \cap Ker\,C_n \to Im\,\ell$$

is defined as the restriction of ℓ on $dom\,\ell \cap Ker\,C_n$, then ℓ_{C_n} is a linear isomorphism and we can define the operator $K_{C_n}\colon Im\,\ell \to dom\,\ell$, $K_{C_n} = \ell_{C_n}^{-1}$. Now, set $Coker\,\ell = L^2(I,H_n)/Im\,\ell = H_n$; and let $\Pi_n\colon L^2(I,H_n) \to H_n$

$$\Pi_n f = \frac{1}{T}\int_0^T f(s)ds,$$

and $\Lambda_n\colon Coker\,\ell \to Ker\,\ell$ be the identity map. Then the equation

$$\ell x = y,\ y \in L^2(I,H_n)$$

is equivalent to

$$(i - C_n)x = (\Pi_n + K_n)y,$$

where $K_n\colon L^2(I,H_n)\to W_T^{1,2}(I,H_n)$ is given as $K_n = K_{C_n}(i-Q_n)$.

The space $L^2(I,H_n)$ can be decomposed as:

$$L^2(I,H_n) = \mathcal{L}_0^{(n)} \oplus \mathcal{L}_1^{(n)},$$

where $\mathcal{L}_0^{(n)} = H_n$ and $\mathcal{L}_1^{(n)} = Im\ell$.

For every $f \in L^2(I,H_n)$ we denote its decomposition by

$$f = f_0^{(n)} + f_1^{(n)}.$$

3. Main result

Consider the following periodic problem for the feedback control system:

$$\begin{cases} x'(t) \in Ax(t) + F(t,x(t)) + By(t) \text{ for a.e. } t \in I, \\ y'(t) \in G(t,x(t),y(t)) \text{ for a.e. } t \in I, \\ x(0) = x(T),\ y(0) = 0, \end{cases} \tag{3.1}$$

where $F\colon I\times E \to P(E)$ and $G\colon I\times E\times E \to Kv(H)$ are given multimaps; $A\in\mathcal{L}(E)$ and $B\in\mathcal{L}(E,H)$; $y_0\in H$.

- (A) the restriction $A_{|_H}$ belongs to $\mathcal{L}(H)$ and it is positively definite, i.e.,there exists $a>0$ such that

  $$\langle Aw,w\rangle_H \geq a\langle w,w\rangle_H$$

  for all $w\in H$ and a.e. $t\in I$;

- $(F1)$ for a.e. $t\in I$ multimap $F(t,\cdot)\colon E\to P(E)$ is u.s.c.;
- $(F2)$ the restriction $F_{|_{I\times H}}$ takes values in $Kv(H)$ and is upper Carathéodory, i.e., for every $w\in H$ multimap $F(\cdot,w)\colon I\to Kv(H)$ is measurable and for a.e. $t\in I$ multimap $F(t,\cdot)\colon H\to Kv(H)$ is u.s.c.;
- $(F3)$ there is $\alpha>0$ such that

  $$\|F(t,w)\|_H = \max\{\|z\|_H\colon z\in F(t,w)\} \leq \alpha(1+\|w\|_H)$$

  for all $w\in H$ and a.e. $t\in I$.

- $(G1)$ for every $(w,z)\in E\times E$ multimap $G(\cdot,w,z)\colon I\to Kv(H)$ is measurable;
- $(G2)$ for every $t\in I$ multimap $G(t,\cdot,\cdot)\colon E\times E\to Kv(H)$ is u.s.c.;

$(G3)$ there exists $\beta > 0$ such that

$$\|G(t,w,z)\|_H \le \beta(1 + \|w\|_E + \|z\|_E),$$

for all $(t,w,z) \in I \times E \times E$.

By a solution to problem (3.1) we mean a pair of functions $(x,y) \in W_T^{1,2}(I,H) \times W^{1,2}(I,H)$ satisfying (3.1), or equivalently, by a solution to (3.1) we mean a function $x \in W_T^{1,2}(I,H)$ for which there exists $y \in W^{1,2}(I,H)$ such that the pair (x,y) satisfies (3.1).

In the sequel, we need using the following statements.

Lemma 9 (see Theorem 5.2.5 [12]). *Let $\mathcal{E}$ be a separable Banach space, Λ a metric space, and $\mathcal{F}\colon I \times \mathcal{E} \times \Lambda \to Kv(\mathcal{E})$ a multimap satisfying the following conditions:*

$(\mathcal{F}1)$ *multimap $\mathcal{F}(\cdot,w,\lambda)\colon I \to Kv(\mathcal{E})$ has a measurable selection for every $(w,\lambda) \in \mathcal{E} \times \Lambda$;*

$(\mathcal{F}2)$ *multimap $\mathcal{F}(t,\cdot,\cdot)\colon \mathcal{E} \times \Lambda \to Kv(\mathcal{E})$ is u.s.c. for a.e. $t \in I$;*

$(\mathcal{F}3)$ *there is $k > 0$ such that*

$$\|\mathcal{F}(t,w,\lambda)\|_{\mathcal{E}} \le k(1 + \|w\|_{\mathcal{E}} + \|\lambda\|_{\Lambda})$$

for all $(w,\lambda) \in \mathcal{E} \times \Lambda$ and for a.e. $t \in I$;

$(\mathcal{F}4)$ *there exists a function $\omega \in L^1_+[0,T]$ such that*

$$\chi\big(\mathcal{F}(t,\Omega,\Lambda)\big) \le \omega(t)\chi(\Omega)$$

for every nonempty bounded subset $\Omega \subset \mathcal{E}$, where χ denotes the Hausdoff measuare of noncompactness.

For each $\lambda \in \Lambda$ denote by $\Sigma_{u_0}^{\mathcal{F}(\cdot,\cdot,\lambda)}$ the solution set of the Cauchy problem

$$\begin{cases} u'(t) \in \mathcal{F}(t,u(t),\lambda) \text{ for a.e. } t \in I, \\ u(0) = u_0 \in \mathcal{E}. \end{cases}$$

Then the multimap $\lambda \to \Sigma_{u_0}^{\mathcal{F}(\cdot,\cdot,\lambda)}$ is u.s.c.

Lemma 10 (see Theorem 70.12 [8]). *Let $\mathcal{E}$ be a separable Banach space and $\mathcal{F}\colon I \times \mathcal{E} \to Kv(\mathcal{E})$ be a multimap such that*

($\mathcal{F}1$) *multimap* $\mathcal{F}(\cdot, w)\colon I \to Kv(\mathcal{E})$ *has a measurable selection for every* $w \in \mathcal{E}$*;*

($\mathcal{F}2$) *multimap* $\mathcal{F}(t, \cdot)\colon \mathcal{E} \to Kv(\mathcal{E})$ *is completely u.s.c. for every* $t \in I$*;*

($\mathcal{F}3$) *the set* $\mathcal{F}(\Omega)$ *is compact for every compact subset* $\Omega \subset I \times \mathcal{E}$*;*

($\mathcal{F}4$) *there is* $b > 0$ *such that*

$$\|\mathcal{F}(t,w)\|_{\mathcal{E}} \le b(1 + \|w\|_{\mathcal{E}})$$

for all $(t,w) \in I \times \mathcal{E}$*.*

Then the solution set of the Cauchy problem

$$\begin{cases} u'(t) \in \mathcal{F}(t, u(t)), \ \text{for a.e. } t \in I, \\ u(0) = u_0 \in \mathcal{E}, \end{cases}$$

is an R_δ*-set in* $C(I, \mathcal{E})$*.*

Now we are in position to present the main result of this paper.

Theorem 11. *Let conditions* (A)*,* $(F1)-(F3)$ *and* $(G1)-(G3)$ *hold. In addition, assume that*

$$a > \alpha + q^2 \beta T e^{q\beta T} \|B\|.$$

Then problem (3.1) *has a solution.*

Proof. For a given function $x \in C(I, H)$ consider the following multimap

$$G_x\colon I \times E \to Kv(E), \ G_x(t,z) = G(t, x(t), z).$$

It is easy to verify that G_x satisfies all conditions in Lemma 10. So we obtain that for every $x \in C(I, H)$ the set Ψ_x of all solutions to the following problem

$$\begin{cases} y'(t) \in G(t, y(t), x(t)) \text{ for a.e. } t \in I \\ y(0) = 0 \end{cases}$$

is an R_δ-set in $C(I, E)$.

Moreover, for a chosen number $r > 0$ let $\Lambda = B_C(0, r)$, and consider the multimap

$$\Pi\colon I \times E \times \Lambda \to Kv(E), \ \Pi(t, w, x) = G(t, w, x(t)).$$

It is clear that multimap Π satisfies conditions $(\mathcal{F}1)-(\mathcal{F}3)$ in Lemma 9 and for every bounded subset $\Omega \subset E$ the set $\Pi(t,\Omega,\Lambda)$ is bounded in H, and hence, it is a relatively compact subset in E. Therefore, Π satisfies all conditions in Lemma 9. By virtue of Lemma 9 the multimap $x \to \Psi_x$ is u.s.c. at all points $x \in \Lambda$. Since we can choose arbitrarily $r > 0$, so if we define the multimap $\Psi\colon C(I,H) \to K(C(I,E))$, $\Psi(x) = \Psi_x$, then it is upper semicontinuous, too.

Now define the following maps and multimaps

$$\widetilde{\Psi}\colon C(I,H) \to K\big(C(I,H) \times C(I,E)\big),\ \widetilde{\Psi}(x) = \{x\} \times \Psi(x),$$

$$B^*\colon C(I,E) \to L^2(I,H),\ (B^*y)(t) = By(t),$$

$$\widetilde{B}\colon C(I,H) \times C(I,E) \to C(I,H) \times L^2(I,H),\ \widetilde{B}(x,y) = \{x\} \times \{B^*y\},$$

$$\widetilde{A}\colon C(I,H) \to L^2(I,H),\ (\widetilde{A}x)(t) = Ax(t),$$

$$\widetilde{\mathcal{P}}_F\colon C(I,H) \times L^2(I,H) \to Cv\big(L^2(I,H) \times L^2(I,H)\big),$$

$$\widetilde{\mathcal{P}}_F(x,y) = \{\widetilde{A}x + \mathcal{P}_F(x)\} \times \{y\},$$

and $\sigma\colon L^2(I,H) \times L^2(I,H) \to L^2(I,H)$,

$$\sigma(x,y) = x + y.$$

Then problem (3.1) can be written in the form

$$\ell x \in \sigma \circ \widetilde{\mathcal{P}}_F \circ \widetilde{B} \circ \widetilde{\Psi}(x), \tag{3.2}$$

where $\ell\colon W_T^{1,2}(I,H) \to L^2(I,H)$ is the differentiation operator.

Let us show that the solutions of (3.2) are a priori bounded. In fact, assume that $x_* \in W_T^{1,2}(I,H)$ is a solution to (3.2). Then there exist $y_* \in \Psi(x_*)$ and $f_* \in \mathcal{P}_F(x_*)$ such that

$$x_*^{'}(t) = Ax_*(t) + f_*(t) + By_*(t) \text{ for a.e. } t \in I.$$

Therefore,

$$\int_0^T \big\langle Ax_*(t) + f(t) + By(t), x_*(t)\big\rangle_H dt = \int_0^T \big\langle x_*^{'}(t), x_*(t)\big\rangle_H dt = 0.$$

On the other hand, for every $x \in W_T^{1,2}(I,H)$ and for all $y \in \Psi(x)$, $f \in \mathcal{P}_F(x)$ the following estimation holds:

$$\int_0^T \big\langle Ax(t) + f(t) + By(t), x(t)\big\rangle_H dt$$
$$\geq a\|x\|_2^2 - \int_0^T \|x(t)\|_H \|f(t)\|_H dt - \int_0^T \|x(t)\|_H \|By(t)\|_H dt$$
$$\geq a\|x\|_2^2 - \alpha \int_0^T \|x(t)\|_H (1 + \|x(t)\|_H) dt - \int_0^T \|B\| \|x(t)\|_H \|y(t)\|_E dt$$
$$\geq (a-\alpha)\|x\|_2^2 - \alpha\sqrt{T}\|x\|_2 - q\|B\| \int_0^T \|x(t)\|_H \|y(t)\|_H dt,$$

where q is the constant from (2.1).

From $y \in \Psi(x)$ it follows that there is $g \in L^2(I,H)$ such that

$$g(t) \in G(t, x(t), y(t)) \text{ for a.e. } t \in I$$

and

$$y(t) = \int_0^t g(s)ds, \ t \in I.$$

Hence,

$$\|y(t)\|_H \leq \int_0^t \|g(t)\|_H \, dt \leq \beta \int_0^t \big(1 + \|y(t)\|_E + \|x(t)\|_E\big) dt$$
$$\leq \ \beta T + q\beta\sqrt{T}\|x\|_2 + q\beta \int_0^t \|y(t)\|_H \, dt, \ t \in I.$$

Applying the Gronwall Lemma (see, e.g., [10]) we obtain

$$\|y(t)\|_H \leq \beta(T + q\sqrt{T}\|x\|_2)e^{q\beta T} \text{ for all } t \in I. \tag{3.3}$$

Consequently,

$$\int_0^T \big\langle Ax(t) + f(t) + By(t), x(t)\big\rangle_H \, dt$$
$$\geq \big(a - \alpha - q^2\beta T e^{q\beta T}\|B\|\big)\|x\|_2^2 - \big(\alpha\sqrt{T} + q\beta\|B\|T\sqrt{T}e^{q\beta T}\big)\|x\|_2 > 0$$

provided

$$\|x\|_2 > \frac{\alpha\sqrt{T} + q\beta\|B\|T\sqrt{T}e^{q\beta T}}{a - \alpha - q^2\beta T e^{q\beta T}\|B\|}.$$

Therefore,

$$\|x_*\|_2 \leq \frac{\alpha\sqrt{T} + q\beta\|B\|T\sqrt{T}e^{q\beta T}}{a - \alpha - q^2\beta T e^{q\beta T}\|B\|}. \tag{3.4}$$

By virtue of $(F3)$ and (3.4) there is $K > 0$ such that $\|x_*^{'}\|_2 \leq K$. So, the solution set of inclusion (3.2) is a priori bounded in $W_T^{1,2}(I,H)$. From the continuous embedding $W_T^{1,2}(I,H) \hookrightarrow C(I,H)$ it follows that there exists $M > 0$ such that $\|x_*\|_C \leq M$.

Now, let $R > M$. Then inclusion (3.2) has no solutions x provided $\|x\|_C \geq R$. For each $n \in \mathbb{N}$ denote by the same symbol ℓ the restriction $\ell_{|_{W_T^{1,2}(I,H_n)}}$ and consider the inclusion

$$\ell x \in \mathbb{P}_n \circ \sigma \circ \widetilde{\mathcal{P}}_F \circ \widetilde{B} \circ \widetilde{\Psi}(x),$$

or equivalently,

$$x \in \Sigma_n(x), \tag{3.5}$$

where $\Sigma_n \colon C(I,H_n) \to P\big(C(I,H_n)\big)$,

$$\Sigma_n(x) = C_n x + (\Pi_n + K_n) \circ \mathbb{P}_n \circ \sigma \circ \widetilde{\mathcal{P}}_F \circ \widetilde{B} \circ \widetilde{\Psi}(x).$$

Let us show that Σ_n is a completely u.s.c. J^c-multimap. Towards this goal, we define the following maps and multimaps

$$\widehat{\Psi} \colon C(I,H_n) \to K\big(C(I,H_n) \times C(I,H) \times C(I,E)\big),$$
$$\widehat{\Psi}(x) = \{x\} \times \{x\} \times \{\Psi(x)\},$$
$$\widehat{B} \colon C(I,H_n) \times C(I,H) \times C(I,E) \to C(I,H_n) \times C(I,H) \times L^2(I,H),$$
$$\widehat{B}(x,u,v) = \{x\} \times \{u\} \times \{B^* v\},$$
$$\widehat{H} \colon C(I,H_n) \times C(I,H) \times L^2(I,H) \to P\big(C(I,H_n) \times C(I,H_n)\big),$$
$$\widehat{H}(x,u,v) = \{x\} \times \{(\Pi_n + K_n)\mathbb{P}_n \circ \sigma \circ \widetilde{\mathcal{P}}_F(u,v)\},$$
$$\widehat{\sigma} \colon C(I,H_n) \times C(I,H_n) \to C(I,H_n),$$
$$\widehat{\sigma}(x,y) = C_n x + y.$$

It is clear that $\widehat{\Psi}$ is a J-multimap, $\widehat{B}$ and $\widehat{\sigma}$ are continuous maps and

$$\Sigma_n(x) = \widehat{\sigma} \circ \widehat{H} \circ \widehat{B} \circ \widehat{\Psi}(x).$$

The multimap $(\Pi_n + K_n)\mathbb{P}_n \circ \sigma \circ \widetilde{\mathcal{P}}_F \colon C(I,H) \times L^2(I,H) \to P\big(C(I,H_n)\big)$ is a completely u.s.c. multimap with compact convex values, so it is a J-multimap. Therefore, Σ_n is a completely u.s.c. J^c-multimap.

Assume that there is $x \in \partial B_C^{(n)}(0,R)$ satisfying (3.5), where $B_C^{(n)}(0,R) = B_C(0,R) \cap C(I,H_n)$. Then there exist $y \in \Psi(x)$ and $f \in \mathcal{P}_F(x)$ such that

$$x'(t) = \mathbb{P}_n\big(Ax(t) + f(t) + By(t)\big), \text{ for a.e. } t \in I.$$

Since $x(t) \in H_n$ for all $t \in I$ we have

$$\begin{aligned}
&\int_0^T \big\langle Ax(t) + f(t) + By(t), x(t)\big\rangle_H dt \\
&= \int_0^T \big\langle \mathbb{P}_n Ax(t) + \mathbb{P}_n f(t) + \mathbb{P}_n By(t), x(t)\big\rangle_H dt \\
&= \int_0^T \big\langle x'(t), x(t)\big\rangle_H dt = 0.
\end{aligned}$$

Consequently, x satisfies relation (3.4), and hence, $\|x\|_C < R$ giving a contradiction.

So, the topological degree $\gamma_n = deg\big(i - \Sigma_n, B_C^{(n)}(0,R)\big)$ is well defined. To evaluate this characteristic, we consider the following multimap

$$\begin{gathered}
G_n \colon B_C^{(n)}(0,R) \times [0,1] \to Kv(C(I,H_n)), \\
G_n(x,\eta) = C_n x + (\Pi_n + K_n) \circ \varphi(\mathbb{P}_n Q(x), \eta),
\end{gathered}$$

where $Q \colon B_C^{(n)}(0,R) \to L^2(I,H)$,

$$Q = \sigma \circ \widetilde{\mathcal{P}}_F \circ \widetilde{B} \circ \widetilde{\Psi},$$

and $\varphi \colon L^2(I,H_n) \times [0,1] \to L^2(I,H_n)$,

$$\varphi(h,\eta) = h_0 + \eta h_1,\ h = h_0 + h_1; h_0 \in \mathcal{L}_0^{(n)}, h_1 \in \mathcal{L}_1^{(n)}.$$

It is easy to verify that G_n is a compact J^c-multimap. Assume that there is $(x_*, \eta_*) \in \partial B_C^{(n)}(0,R) \times [0,1]$ such that $x_* \in G_n(x_*, \eta_*)$. Then there exist $f_* \in \mathcal{P}_F(x_*)$ and $y_* \in \Psi(x_*)$ such that

$$\begin{cases} x_*' = \eta_* g_{*_1}^{(n)} \\ 0 = g_{*_0}^{(n)}, \end{cases}$$

where $g_*^{(n)}(t) = P_n A x_*(t) + P_n f_*(t) + P_n B y_*(t)$, $t \in I$.

If $\eta_* \neq 0$, then

$$\int_0^T \big\langle x_*'(t), x_*(t)\big\rangle_H dt = \frac{1}{\eta_*}\int_0^T \big\langle P_n A x_*(t) + P_n f_*(t) + P_n B y_*(t), x_*(t)\big\rangle_H dt > 0,$$

giving a contradiction.

If $\eta_* = 0$, then $x_*(t) = w \in H_n$ with $\|w\|_H = R$ for all $t \in I$. Since $\|w\|_H = R$ we have

$$\int_0^T \langle P_n Aw + P_n f(t) + P_n By(t), w\rangle_H dt > 0,$$

for all $f \in \mathcal{P}_F(w)$ and all $y \in \Psi(w)$.
or equivalently,

$$\langle \Pi_n g^{(n)}, w\rangle_{H_n} > 0 \text{ for all } g^{(n)} \in \mathbb{P}_n Q(w). \tag{3.6}$$

In particular, $\langle \Pi_n g_*^{(n)}, w\rangle_{H_n} > 0$, giving the contradiction.

Hence, G_n is a homotopy connecting the multifields $G_n(\cdot, 1) = \Sigma_n$ and $G_n(\cdot, 0) = C_n + \Pi_n \circ \mathbb{P}_n Q$. From relation (3.6) and the fact that multimap $G_n(\cdot, 0)$ takes values in H_n we obtain

$$deg\big(i - \Sigma_n, B_C^{(n)}(0, R)\big) = deg\big(-\Pi_n \mathbb{P}_n Q, B_H^{(n)}(0, R)\big) = (-1)^n,$$

where $B_H^{(n)}(0, R) = B_C^{(n)}(0, R) \cap H_n$.

Thus, for every $n \in \mathbb{N}$ there is a solution $x_n \in B_C^{(n)}(0, R)$ to (3.5).

Now let us show that inclusion (3.2), and therefore problem (3.1), has a solution. Towards this goal, let us mention that the set $\{x_n\}_{n=1}^\infty$ is bounded in $W_T^{1,2}(I, H)$, and hence, it is weakly relatively compact in $W_T^{1,2}(I, H)$. W.l.o.g. assume that $x_n \overset{W}{\rightharpoonup} x_0$. Therefore, $\ell x_n \overset{L^2}{\rightharpoonup} \ell x_0$ and $x(t) \overset{H}{\rightharpoonup} x_0(t)$ for $t \in I$. From the compact embedding $H \hookrightarrow E$ we obtain

$$x_n(t) \overset{E}{\to} x_0(t), \ t \in I. \tag{3.7}$$

Let $y_n \in \Psi(x_n)$ and $f_n \in \mathcal{P}_F(x_n)$ be such that

$$\ell x_n = \mathbb{P}_n\big(\widetilde{A} x_n + f_n + B^* y_n\big).$$

The sequence $\{f_n\}_{n=1}^\infty$ is bounded in $L^2(I, H)$, and therefore it is weakly relatively compact. W.l.o.g. assume that

$$f_n \overset{L^2}{\rightharpoonup} f_0 \in L^2(I, H).$$

The sequence $\{y_n\}_{n=1}^\infty$ is bounded in $W^{1,2}(I, H)$, and therefore it is weakly compact. W.l.o.g. assume that $y_n \overset{W}{\rightharpoonup} y_0$. Therefore,

$$y_n' \overset{L^2}{\rightharpoonup} y_0' \quad and \quad y_n(t) \overset{E}{\to} y_0(t), \quad for \quad t \in I. \tag{3.8}$$

From the compact embedding $H \hookrightarrow E$ it follows that the embedding $W^{1,2}(I,H) \hookrightarrow C(I,E)$ is compact. Consequently, $y_n \to y_0$ in $C(I,E)$, and therefore, $B^* y_n \to B^* y_0$ in $L^2(I,H)$. Let us show that $\mathbb{P}_n f_n \overset{L^2}{\rightharpoonup} f_0$. Towards this goal, let us mention that since

$$L^2(I,H) = \overline{\bigcup_{n=1}^{\infty} L^2(I,H_n)},$$

for every element $g \in L^2(I,H)$ we have:

$$\mathbb{P}_n g \overset{L^2(I,H)}{\longrightarrow} g.$$

Now for every $g \in L^2(I,H)$ we obtain

$$\begin{aligned}
\langle \mathbb{P}_n f_n - f_0, g \rangle_{L^2} &= \langle \mathbb{P}_n f_n - \mathbb{P}_n f_0, g \rangle_{L^2} + \langle \mathbb{P}_n f_0 - f_0, g \rangle_{L^2} \\
&= \langle f_n - f_0, \mathbb{P}_n g \rangle_{L^2} + \langle \mathbb{P}_n f_0 - f_0, g \rangle_{L^2} \\
&= \langle f_n - f_0, g \rangle_{L^2} + \langle f_n - f_0, \mathbb{P}_n g - g \rangle_{L^2} + \langle \mathbb{P}_n f_0 - f_0, g \rangle_{L^2}.
\end{aligned}$$

Thus,

$$\lim_{n \to \infty} \langle \mathbb{P}_n f_n - f_0, g \rangle_{L^2} = 0,$$

or equivalently, $\mathbb{P}_n f_n \overset{L^2}{\rightharpoonup} f_0$.

From the diagram

$$\ell x_0 \overset{L^2}{\leftharpoondown} \ell x_n = \mathbb{P}_n \big(\widetilde{A} x_n + f_n + B^* y_n \big) \overset{L^2}{\rightharpoonup} \widetilde{A} x_0 + f_0 + B^* y_0$$

it follows that $\ell x_0 = \widetilde{A} x_0 + f_0 + B^* y_0$.

Let us show that $y_0 \in \Psi(x_0)$. Notice that from $y_n \in \Psi(x_n)$ it follows that there is $\{g_n\}_{n=1}^{\infty} \subset L^2(I,H)$ such that $g_n(t) \in G(t, x_n(t), y_n(t))$ for a.e. $t \in I$, and $y_n'(t) = g_n(t)$ for a.e. $t \in I$. So, $g_n \overset{L^2}{\rightharpoonup} y_0'$. By the Mazur Lemma (see, e.g., [6]) there is a sequence of convex combinations $\{\widehat{g}_m\}$

$$\widehat{g}_m = \sum_{k=m}^{\infty} \lambda_{mk} g_k, \;\; \lambda_{mk} \geq 0 \;\text{ and } \sum_{k=m}^{\infty} \lambda_{mk} = 1,$$

which converges in $L^2(I,H)$ to y_0'. Applying Theorem 38 ([17]) we again can assume w.l.o.g. that

$$\widehat{g}_m(t) \overset{H}{\to} y_0'(t) \tag{3.9}$$

for a.e. $t \in I$.

From (3.7) and (3.8) it follows that for every $t \in I$ and $\varepsilon > 0$ there is $i_0 = i_0(\varepsilon, t)$ such that

$$G(t, x_i(t), y_i(t)) \subset O^H_\varepsilon\Big(G\big(t, x_0(t), y_0(t)\big)\Big), \quad for\ all\ i \geq i_0.$$

Then $g_i(t) \in O^H_\varepsilon\Big(G\big(t, x_0(t), y_0(t)\big)\Big)$ for all $i \geq i_0$, and hence, from the convexity of the set $O^H_\varepsilon\Big(G\big(t, x_0(t), y_0(t)\big)\Big)$ we have

$$\hat{g}_m(t) \in O^H_\varepsilon\Big(G\big(t, x_0(t), y_0(t)\big)\Big), \ for\ all\ m \geq i_0.$$

Thus, $y_0^{'}(t) \in G\big(t, x_0(t), y_0(t)\big)$ for a.e. $t \in I$, i.e., $y_0 \in \Psi(x_0)$.

To complete the proof we need to prove that $f_0 \in \mathcal{P}_F(x_0)$. Since $f_n \overset{L^2}{\rightharpoonup} f_0$, then w.l.o.g. we can assume that there is a sequence of convex combinations $\{\overline{f}_m\}$,

$$\overline{f}_m = \sum_{k=m}^{\infty} \lambda_{mk} f_k, \ \lambda_{mk} \geq 0 \ \text{ and } \sum_{k=m}^{\infty} \lambda_{mk} = 1,$$

which converges to f_0 for a.e. $t \in I$.

From (3.7) and $(F2)$ it follows that for a.e. $t \in I$ and for a given $\varepsilon > 0$ there is an integer $i_0 = i_0(\varepsilon, t)$ such that

$$F(t, x_i(t)) \subset O^E_\varepsilon\Big(F\big(t, x_0(t)\big)\Big) \ \text{ for all } i \geq i_0.$$

Therefore,

$$\overline{f}_m(t) \in O^E_\varepsilon\Big(F\big(t, x_0(t)\big)\Big), \text{ for all } m \geq i_0.$$

Hence, $f_0 \in \mathcal{P}_F(x_0)$. ■

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